

A FAMILY VERSION OF LEFSCHETZ-NIELSEN FIXED POINT THEORY

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Abstract

by

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In classical Lefschetz-Nielsen theory, one defines the Lefschetz invariant  $L(f)$  of an endomorphism  $f$  of a manifold  $M$ . The definition depends on the fundamental group of  $M$ , and hence on choosing a base point  $* \in M$  and a base path from  $*$  to  $f(*)$ . Our goal is to develop a family version of Lefschetz-Nielsen theory, i.e., for a smooth fiber bundle  $p : E \rightarrow B$  and a fiber bundle endomorphism  $f : E \rightarrow E$ . A family version of the classical approach involves choosing a section  $s : B \rightarrow E$  of  $p$  and a path of sections from  $s$  to  $f \circ s$ . Not only is this artificial, but such a path does not always exist.

To avoid this difficulty, we replace the fundamental group with the fundamental groupoid. This gives us a base point free version of the Lefschetz invariant. In the family setting, we define the Lefschetz invariant using a bordism theoretic construction, and prove a Hopf-Lefschetz theorem. We then describe our ideas for extending the algebraic base point free invariant to get an algebraic version of the Lefschetz invariant in the family setting.

To Paul.

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## CHAPTER 1

### CLASSICAL RESULTS

#### 1.1 Introduction

In classical Lefschetz fixed point theory [5], one considers an endomorphism  $f : M \rightarrow M$  of a compact, connected polyhedra  $M$ . Lefschetz used an elementary trace construction to define the Lefschetz invariant  $L(f) \in \mathbb{Z}$ . The Hopf-Lefschetz theorem states that if  $L(f) \neq 0$ , then every map homotopic to  $f$  has a fixed point. The converse is false. However, a converse can be achieved by strengthening the invariant. To begin, one chooses a base point  $*$  of  $M$  and a base path  $\tau$  from  $*$  to  $f(*)$ . Then, using the fundamental group and an advanced trace construction one defines a Lefschetz-Nielsen invariant  $L(f, *, \tau)$ , which is an element of a zero dimensional Hochschild homology group [12]. Wecken proved that when  $M$  is a compact manifold of dimension  $n > 2$ ,  $L(f, *, \tau) = 0$  if and only if  $f$  is homotopic to a map with no fixed points.

We wish to extend Lefschetz-Nielsen theory to a family of manifolds and endomorphisms, i.e., a smooth fiber bundle  $p : E \rightarrow B$  together with a map  $f : E \rightarrow E$  such that  $p = p \circ f$ . One problem with extending the definitions comes from choosing base points in the fibers, i.e., a section  $s$  of  $p$ , and the fact that  $f$  is not necessarily fiber homotopic to a map which fixes the base points (as is the case for a single space and a single endomorphism.) To avoid this difficulty, we reformulate the classical definitions of the Lefschetz-Nielsen invariant by employing a trace construction over

the fundamental groupoid, rather than the fundamental group.

In this chapter, we describe the classical Lefschetz-Nielsen theory following the treatment given by Geoghegan [12] (see also Jiang [19], Brown [5] and Lück [22]). Given an endomorphism  $f$  of a suitable space  $M$ , one defines a Lefschetz invariant in three ways: geometrically, algebraically and via bordism. We show that these definitions are equivalent, and that there is a Hopf-Lefschetz Theorem with a converse, i.e., for suitable  $M$  the invariant is trivial if and only if  $f$  is homotopic to a map with no fixed points. We also introduce the Hattori-Stallings trace, which will replace the usual trace in the construction of the algebraic invariant.

In Chapter 2, we develop the background necessary to explain our base point free definitions. This includes the general theory of groupoids and modules over ringoids, as well as our version of the Hattori-Stallings trace. In Chapter 3, we present our base point free definitions of the Lefschetz-Nielsen invariant, and show that they are equivalent to the classical definitions. In Chapter 4 we give an outline of how one would extend our definitions to the family setting. We also describe a bordism theoretic Lefschetz invariant for the family setting, and prove a Hopf-Lefschetz theorem with a converse, i.e., that under certain dimension restrictions, the invariant is zero if and only if the endomorphism is fiber homotopic to a map with no fixed points.

## 1.2 The Geometric Invariant

In this section,  $M^n$  is a compact, connected manifold of dimension  $n$ , and  $f : M \rightarrow M$  is a continuous endomorphism.

The concatenation of two paths  $\alpha : I \rightarrow X$  and  $\beta : I \rightarrow X$  such that  $\alpha(1) = \beta(0)$

is defined by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \beta(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The fixed point set of  $f$  is

$$\text{Fix}(f) = \{x \in M \mid f(x) = x\}.$$

Note that  $\text{Fix}(f)$  is compact. Define an equivalence relation  $\sim$  on  $\text{Fix}(f)$  by letting  $x \sim y$  if there is a path  $\nu$  in  $M$  from  $x$  to  $y$  such that  $\nu \cdot (f \circ \nu)^{-1}$  is homotopic to a constant path.

Choose a base point  $*$  in  $M$  and a base path  $\tau$  from  $*$  to  $f(*)$ . Let  $\pi = \pi_1(M, *)$ . Given these choices,  $f$  induces a homomorphism

$$\phi : \pi \rightarrow \pi$$

defined by

$$\phi([w]) = [\tau \cdot (f \circ w) \cdot \tau^{-1}],$$

where  $[w]$  is the homotopy class of a path  $w$  rel endpoints. Define an equivalence relation on  $\pi$  by saying  $g, h \in \pi$  are equivalent if there is some  $w \in \pi$  such that  $h = wg\phi(w)^{-1}$ . The equivalence classes are called semiconjugacy classes, and the set of semiconjugacy classes is denoted  $\pi_\phi$ . Let  $\mathbb{Z}\pi_\phi$  be the free abelian group generated by the set  $\pi_\phi$ . This will be the home of the geometric Lefschetz invariant.

Define a map

$$\Phi : \text{Fix}(f) \rightarrow \pi_\phi$$

by

$$x \mapsto [\mu \cdot (f \circ \mu)^{-1} \cdot \tau^{-1}]$$

where  $x \in \text{Fix}(f)$  and  $\mu$  is a path in  $M$  from  $*$  to  $x$ . Note that this map is well-defined and induces an injection

$$\Phi : \text{Fix}(f)/\sim \rightarrow \pi_\phi.$$

It follows that  $\text{Fix}(f)/\sim$  is compact and discrete, and hence finite. Denote the fixed point classes by  $F_1, \dots, F_s$ .

Next, we assume that the fixed point set of  $f$  is finite. Let  $x$  be a fixed point. Let  $U$  be an open neighborhood of  $x$  in  $M$  and  $h : U \rightarrow \mathbb{R}^n$  a chart. Let  $V$  be an open  $n$ -ball neighborhood of  $x$  in  $U$  such that  $f(V) \subset U$ . Then the fixed point index of  $f$  at  $x$ ,  $i(f, x)$  is the degree of the map of pairs

$$(\text{id} - fh^{-1}) : (h(V), h(V) - \{h(x)\}) \rightarrow (\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

For a fixed point class  $F_k$ , define

$$i(f, F_k) = \sum_{x \in F_k} i(f, x) \in \mathbb{Z}.$$

**Definition 1.2.1.** *The classical geometric Lefschetz invariant of  $f$  with respect to the base point  $*$  and the base path  $\tau$  is*

$$L^{geo}(f, *, \tau) = \sum_{k=1}^s i(f, F_k) \Phi(F_k) \in \mathbb{Z}\pi_\phi.$$

We note that  $L^{geo}$  is an homotopy invariant and can be extended to endomorphisms of path connected, compact Euclidean neighborhood retracts (ENR's).

### 1.3 The Algebraic Invariant

To construct the classical algebraic Lefschetz invariant, we let  $M$  be a finite connected CW complex and  $f : M \rightarrow M$  a cellular map. Again, we choose a base point  $*$   $\in M$  (a vertex of  $M$ ) and a base path  $\tau$  from  $*$  to  $f(*)$ . Also, choose an orientation on each cell in  $M$ .

Let  $p : \widetilde{M} \rightarrow M$  be the universal cover of  $M$ . The CW structure on  $M$  lifts to a CW structure on  $\widetilde{M}$ . Choose a lift of the base point  $*$  to a base point  $\tilde{*} \in \widetilde{M}$ , and lift the base path  $\tau$  to a path  $\tilde{\tau}$  such that  $\tilde{\tau}(0) = \tilde{*}$ . Then  $f$  lifts to a cellular map  $\tilde{f} : \widetilde{M} \rightarrow \widetilde{M}$  such that  $\tilde{f}(\tilde{*}) = \tilde{\tau}(1)$ .

The group  $\pi = \pi_1(M, *)$  acts on  $\widetilde{M}$  on the left by covering transformations. For each cell  $\sigma$  in  $M$ , choose a lift  $\tilde{\sigma}$  in  $\widetilde{M}$  and orient it compatibly with  $\sigma$ . Take the cellular chain complex  $C(\widetilde{M})$  of  $\widetilde{M}$ . The action of  $\pi$  on  $\widetilde{M}$  makes  $C_k(\widetilde{M})$  into a finitely generated free left  $\mathbb{Z}\pi$ -module with basis given by the chosen lifts of the oriented  $k$ -cells of  $M$ .

As in the geometric construction,  $f$  and  $\tau$  induce a homomorphism  $\phi : \pi \rightarrow \pi$ . Since  $\tilde{f}$  is cellular, it induces a chain map  $\tilde{f}_k : C_k(\widetilde{M}) \rightarrow C_k(\widetilde{M})$  which is  $\phi$ -linear, namely if  $\tilde{\sigma}$  is a  $k$ -cell of  $\widetilde{M}$  and  $g \in \pi$  then  $\tilde{f}_k(g\tilde{\sigma}) = \phi(g)\tilde{f}_k(\tilde{\sigma})$ . Classically, one represents  $\tilde{f}_k$  by a matrix over  $\mathbb{Z}\pi$  whose  $(i, j)$  entry is the coefficient of  $\tilde{\sigma}_j$  in the chain  $\tilde{f}_k(\tilde{\sigma}_i)$ , where  $\tilde{\sigma}_i$  and  $\tilde{\sigma}_j$  are  $k$ -cells. For each  $k$ , one can now take the trace of  $\tilde{f}_k$ , i.e., the sum of the diagonal entries of the matrix which represents  $\tilde{f}_k$ .

**Definition 1.3.1.** *The classical algebraic Lefschetz invariant of  $f$  with respect to the base point  $*$  and the base path  $\tau$  is*

$$L^{alg}(f, *, \tau) = \sum_{k \geq 0} (-1)^k q(\text{trace}(\tilde{f}_k)) \in \mathbb{Z}\pi_\phi$$

where  $q : \mathbb{Z}\pi \rightarrow \mathbb{Z}\pi_\phi$  is the map sending  $g \in \pi$  to its semiconjugacy class.

As with the geometric construction,  $L^{alg}$  is homotopy invariant and can be extended to endomorphisms of path connected, compact ENR's.

#### 1.4 The Bordism Invariant

The normal bundle of an immersion  $h : Y \rightarrow X$  will be denoted  $\nu(h)$ . The stable normal bundle of a smooth manifold  $X$  will be denoted  $\nu(X)$ .

**Definition 1.4.1.** Given a space  $X$  and a bundle  $\eta$  over  $X$ , the  $n$ -dimensional framed bordism group of  $X$  with coefficients in  $\eta$ ,  $\Omega_n^{fr}(X; \eta)$ , is the bordism group of  $n$ -dimensional manifolds mapping to  $X$  together with stable bundle isomorphisms of the normal bundle to the pullback of  $\eta$ . More precisely, elements are represented by triples  $(Y^n, h, \bar{h})$ , where  $Y$  is a closed  $n$ -dimensional manifold,  $h : Y \rightarrow X$ , and  $\bar{h} : \nu(Y) \rightarrow h^*\eta$  is a stable bundle isomorphism. Two elements  $(Y, h, \bar{h})$  and  $(Y', h', \bar{h}')$  are framed bordant if there exists a triple  $(W, H, \bar{H})$  such that  $W$  is a compact  $(n + 1)$ -dimensional manifold with  $\partial W = Y \amalg Y'$ ,  $H : W \rightarrow X$  is a map extending  $h$  and  $h'$ , and  $\bar{H} : \nu(W) \rightarrow H^*\eta$  is a stable bundle isomorphism extending  $\bar{h}$  and  $\bar{h}'$ .

If  $\eta$  is the trivial bundle, we will write  $\Omega_n^{fr}(X)$  for  $\Omega_n^{fr}(X; \eta)$ .

**Definition 1.4.2.** The homotopy pullback of the diagram

$$A \xrightarrow{g} C \xleftarrow{h} B \tag{1.4.1}$$

of topological spaces and continuous maps is the space

$$E(g, h) = \{(a, \theta, b) \in A \times C^I \times B \mid \theta \text{ is a path in } C \text{ from } g(a) \text{ to } h(b)\}.$$

Under the projection maps, the “homotopy pullback diagram”

$$\begin{array}{ccc} E(g, h) & \xrightarrow{pr_3} & B \\ pr_1 \downarrow & & \downarrow h \\ A & \xrightarrow{g} & C \end{array}$$

commutes up to homotopy, i.e.,  $g \circ pr_1 \simeq h \circ pr_3$ .

**Lemma 1.4.3 (Universal property).** Consider the homotopy pullback of 1.4.1. Given a space  $D$ , continuous maps  $i : D \rightarrow A$  and  $j : D \rightarrow B$ , and a homotopy  $H$  from  $g \circ i$  to  $h \circ j$  (written  $H : g \circ i \simeq h \circ j$ ),

$$\begin{array}{ccc} D & \xrightarrow{j} & B \\ i \downarrow & & \downarrow h \\ A & \xrightarrow{g} & C \end{array}$$

there exists a map  $k : D \rightarrow E(g, h)$  such that  $pr_1 \circ k \simeq i$  and  $pr_3 \circ k \simeq j$ .

*Proof.* Define  $k : D \rightarrow E(g, h)$  by  $k(d) = (i(d), H(d, -), j(d))$ , where  $d \in D$ .  $\square$

**Lemma 1.4.4 (Homotopy invariance).** *Consider the homotopy pullback of (1.4.1), and suppose  $H$  is a homotopy from  $g$  to  $g'$ ,  $H : g \simeq g'$ . Then  $H$  induces a homotopy  $H' : E(g, h) \simeq E(g', h)$ .*

*Proof.* Using  $H$  and the universal property of the homotopy pullback of  $g$  and  $h$  one gets a map  $k : E(g', h) \rightarrow E(g, h)$ . Similarly, one gets a map  $k' : E(g, h) \rightarrow E(g', h)$ . It can be seen that  $k' \circ k \simeq \text{id}_{E(g', h)}$  and  $k \circ k' \simeq \text{id}_{E(g, h)}$ .  $\square$

In this section,  $M^n$  is a closed, connected manifold embedded in  $S^{n+k}$ ,  $k \gg n$ , and  $f : M \rightarrow M$  is a continuous map. Let  $\Delta : M \rightarrow M \times M$  be the diagonal map, and let  $\Delta_f : M \rightarrow M \times M$  be the graph of  $f$  (or the “ $f$ -twisted diagonal”) given by

$$\Delta_f(x) = (x, f(x))$$

for  $x \in M$ .

We will use  $f$  in order to produce an element of the framed bordism group  $\Omega_0^{fr}(\mathcal{L}_f)$ , where  $\mathcal{L}_f = E(\Delta, \Delta_f)$  is the “twisted loop space”,

$$\begin{array}{ccc} \mathcal{L}_f & \longrightarrow & M \\ \downarrow & & \downarrow \Delta \\ M & \xrightarrow{\Delta_f} & M \times M \end{array} \tag{1.4.2}$$

Since  $\text{Fix}(f)$  is naturally homeomorphic to the (strict) pullback of  $\Delta$  and  $\Delta_f$ , the universal property of homotopy pullbacks give us a map  $\text{Fix}(f) \rightarrow \mathcal{L}_f$ . However, in order to produce an element of  $\Omega_0^{fr}(\mathcal{L}_f)$  we need  $\Delta_f$  to be transversal to  $\Delta$ . The idea, then, is to replace  $f$  with a map  $f_1$  which is homotopic to  $f$  and for which  $\Delta_{f_1}$  is transversal to  $\Delta$ . Transversality will give us the bundle data. The following argument makes this idea explicit.

**Proposition 1.4.5.** *There exists a homotopy  $H$  from  $f$  to a map  $f_1$  such that  $\Delta_{f_1}$  is transversal to  $\Delta$ .*

The proof relies on the following result of Kozniowski [20, Proposition 6] concerning  $B$ -manifolds. Given a smooth manifold  $B$ , a  $B$ -manifold  $X$  is a manifold  $X$  together with a locally trivial submersion  $p : X \rightarrow B$ . A  $B$ -map is a smooth fiber-preserving map.

**Lemma 1.4.6.** *Let  $X$  and  $Y$  be  $B$ -manifolds, let  $Z$  be a  $B$ -submanifold of  $Y$ , and let  $g : X \rightarrow Y$  be a  $B$ -map. Then there is a  $B$ -map  $g_1 : X \rightarrow Y$  such that  $g_1$  is smoothly  $B$ -homotopic to  $g$  and  $g_1$  is transversal to  $Z$ .*

*Proof of Proposition 1.4.5.* Consider the  $M$ -manifolds  $\text{id} : M \rightarrow M$  and  $p : M \times M \rightarrow M$ , where  $p$  is projection on the first factor. Abusing notation, let  $\Delta$  denote the image of the diagonal map  $\Delta : M \rightarrow M \times M$ . Then  $\Delta$  is a  $M$ -submanifold of  $M \times M$ . The map  $\Delta_f : M \rightarrow M \times M$  is an  $M$ -map, and by Lemma 1.4.6,  $\Delta_f$  is smoothly  $M$ -homotopic to an  $M$ -map  $\Gamma : M \rightarrow M \times M$  which is transversal to  $\Delta$ . Since  $\Gamma$  is an  $M$ -map, it is of the form  $\Gamma = \Delta_{f_1}$  for some map  $f_1 : M \rightarrow M$ . The homotopy from  $\Delta_f$  to  $\Delta_{f_1}$  induces a homotopy  $H$  from  $f$  to  $f_1$ .  $\square$

We can view this data as a transversal (pullback) square

$$\begin{array}{ccc} \text{Fix}(f_1) & \xrightarrow{i} & M \\ i \downarrow & & \downarrow \Delta \\ M & \xrightarrow{\Delta_{f_1}} & M \times M \end{array}$$

where  $i$  is the inclusion map. Transversality implies that  $\nu(i) \cong i^*\nu(\Delta) \cong i^*\tau M$ . Combining this with the inclusions  $\text{Fix}(f_1) \subset M \subset S^{n+k}$  gives

$$\nu_{\text{Fix}(f_1)} \cong \nu(i) \oplus i^*\nu_M \cong i^*\tau M \oplus i^*\nu_M \cong \epsilon.$$

Denote this bundle isomorphism by

$$\bar{g} : \nu_{\text{Fix}(f_1)} \rightarrow \epsilon.$$

As noted above for  $\text{Fix}(f)$ , by universality we get a map  $g' : \text{Fix}(f_1) \rightarrow \mathcal{L}_{f_1}$ . Furthermore, since  $\Delta_f$  is homotopic to  $\Delta_{f_1}$ , Lemma 1.4.4 gives a homotopy equivalence  $\mathcal{L}_{f_1} \simeq \mathcal{L}_f$ . Composing this homotopy equivalence with  $g'$ , gives a map

$$g : \text{Fix}(f_1) \rightarrow \mathcal{L}_f.$$

It follows that the triple  $(\text{Fix}(f_1), g, \bar{g})$  represents an element of the bordism group  $\Omega_0^{fr}(\mathcal{L}_f)$ .

**Definition 1.4.7.** *The bordism theoretic Lefschetz invariant of  $f$  is given by*

$$L^{bord}(f) = [\text{Fix}(f_1), g, \bar{g}] \in \Omega_0^{fr}(\mathcal{L}_f).$$

The fact that this Lefschetz invariant is well-defined follows from [17, Proposition 2.1]. In essence, an  $M$ -homotopy of  $\Delta_{f_1}$  to  $\Delta_{f_2}$  will give a bordism from  $\text{Fix}(f_1)$  to  $\text{Fix}(f_2)$ . It also follows that  $L^{bord}(f)$  is a homotopy invariant of  $f$ .

**Theorem 1.4.8 (Hopf-Lefschetz Theorem with Converse).** *If  $f$  is homotopic to a map with no fixed points, then  $L^{bord}(f) = 0$ . If  $n > 2$  and  $L^{bord}(f) = 0$ , then  $f$  is homotopic to a map with no fixed points.*

This is a special case of Theorem 4.2.1.

## 1.5 Equivalence of the Invariants

There are three things to show: the geometric and algebraic definitions are equivalent; the homes of the invariants are isomorphic, i.e.,  $\mathbb{Z}\pi_\phi \cong \Omega_0^{fr}(\mathcal{L})$ ; and  $L^{geo}$  maps to  $L^{bord}$  under this isomorphism.

**Proposition 1.5.1.**  $L^{geo}(f) = L^{alg}(f)$ .

We sketch the proof of Proposition 1.5.1 following [12]. First, assume  $M$  is a path connected manifold. Let  $x \in \text{Fix}(f)$  and let  $\mu$  be a path from  $*$  to  $x$ . Lift  $\mu$  to

a path  $\tilde{\mu}$  in  $\tilde{M}$  with  $\tilde{\mu}(0) = \tilde{*}$  and let  $\tilde{x} = \tilde{\mu}(1)$ . Then for some  $g_x \in \pi$ ,  $\tilde{f}(\tilde{x}) = g_x \tilde{x}$ ; in fact, one can show that  $g_x = [\mu \cdot (f \circ \mu)^{-1} \cdot \tau^{-1}]$ . Thus if  $F_k$  is the fixed point class of  $x$ , then

$$\Phi(F_k) = q(g_x),$$

where  $q : \pi \rightarrow \pi_\phi$  is the quotient map.

Next, we consider the special case where  $J$  triangulates  $M^n$ , a compact manifold. Let  $K$  be a subdivision of  $J$ . Let  $f' : K \rightarrow J$  be a simplicial map with finite fixed point set, such that each fixed point lies in the interior of an  $n$ -simplex of  $K$ . Note that there is then at most one fixed point in the interior of each cell  $\sigma^n$ . Let  $f : K \rightarrow K$  be the map  $f'$  considered as a cellular map. Let  $x \in \text{Fix}(f) \cap \overset{\circ}{\sigma}^n$ , and define

$$\varepsilon_x = \begin{cases} 0, & \text{when } f \text{ is orientation preserving near } x, \\ 1, & \text{when } f \text{ is orientation reversing near } x. \end{cases}$$

One can show that

$$i(f, x) = (-1)^{n+\varepsilon_x}.$$

Furthermore, choosing a lift  $\tilde{\sigma}$  in  $\tilde{K}$  for each cell  $\sigma$  of  $K$ , and giving it a compatible orientation, we get a basis for the cellular chains  $C_k(\tilde{K})$ . We can write  $\tilde{f}_k : C_k(\tilde{K}) \rightarrow C_k(\tilde{K})$  as a matrix with entries in  $\pm\pi$ . A fixed point  $x$  will correspond to having an entry of  $\pm g_x$  on the diagonal of  $\tilde{f}_n$ .

Combining these observations, we see that in this special case,

$$\begin{aligned}
L^{geo}(f) &= \sum_{k=1}^s i(f, F_k) \Phi(F_k) \\
&= \sum_{x \in \text{Fix}(f)} i(f, x) [g_x] \\
&= \sum_{x \in \text{Fix}(f)} (-1)^{n+\varepsilon_x} [g_x] \\
&= \sum_{k \geq 0} (-1)^k q(\text{trace}(\tilde{f}_k)) \\
&= L^{alg}(f).
\end{aligned}$$

In the general case, one can reduce to this special case by triangulating  $M$  and replacing  $f$  with a homotopic map  $f'$  with suitable properties.

**Proposition 1.5.2.** *There is an isomorphism*

$$\Omega_0^{fr}(\mathcal{L}_f) \cong \mathbb{Z}\pi_\phi.$$

The proof will use the following lemma.

**Lemma 1.5.3.** *Let  $p : E \rightarrow B$  be a fibration, with  $E$  a space and  $B$  a path connected space. Let  $F$  be the fiber of some fixed  $b \in B$ , and let  $\iota : F \rightarrow E$  be the inclusion of the fiber into the total space. Then  $\iota_\# : \pi_0 F \rightarrow \pi_0 E$  is a surjection. Furthermore,  $\pi_1(B, b)$  acts on  $\pi_0 F$  and  $\iota_\#$  factors through a bijection  $\pi_1(B, b) \backslash \pi_0 F \rightarrow \pi_0 E$ .*

*Proof.* The first statement follows from the homotopy lifting property for fibrations.

The action of  $\pi_1(B, b)$  on  $\pi_0 F$  is given by  $[g] \bullet [x] = \tilde{g}(1)$  where  $[g] \in \pi_1(B, b)$ ,  $x \in F$ , and  $\tilde{g}$  is a lift of  $g$  such that  $g\tilde{g}(0) = x$ . It is straightforward to show that  $\iota_\#$  induces an injection  $\pi_1(B, b) \backslash \pi_0 F \rightarrow \pi_0 E$ .  $\square$

Note that an inverse to the map  $\pi_1(B, b) \backslash \pi_0 F \rightarrow \pi_0 E$  is given as follows. Let  $x \in E$ . There exists a path  $g$  in  $B$  from  $p(x)$  to  $b$ . Let  $\tilde{g}$  be a lift of  $g$  such that  $\tilde{g}(0) = x$ . Then the inverse  $\pi_0 E \rightarrow \pi_1(B, b) \backslash \pi_0 F$  is given by  $[x] \mapsto [\tilde{g}(1)]$ .

*Proof of Proposition 1.5.2.* For a space  $X$ , let  $X_+$  denote  $X$  with a disjoint base point. By the Pontryagin-Thom isomorphism [7]

$$\Omega_0^{fr}(\mathcal{L}_f) \cong H_0(\mathcal{L}_f; \mathbf{S}) = \pi_0^s(\mathcal{L}_{f+})$$

where  $\mathbf{S}$  is the sphere spectrum and the equality is by definition. Since

$$\pi_0^s(\mathcal{L}_{f+}) \cong H_0(\mathcal{L}_f) \cong \mathbb{Z}\pi_0\mathcal{L}_f,$$

it follows that

$$\Omega_0^{fr}(\mathcal{L}_f) \cong \mathbb{Z}\pi_0\mathcal{L}_f.$$

To show that  $\mathbb{Z}\pi_0\mathcal{L}_f \cong \mathbb{Z}\pi_\phi$ , we apply Lemma 1.5.3 to the following fibration. Let  $p : \mathcal{L}_f M \rightarrow M$  be given by  $p(\sigma) = \sigma(0)$ , where  $\sigma \in \mathcal{L}_f M$  is a path in  $M$  such that  $\sigma(1) = f \circ \sigma(0)$ . The fiber over the base point  $* \in M$  is the space of paths in  $M$  from  $*$  to  $f(*)$ , denoted  $P(M, *, f(*))$ . By Lemma 1.5.3, there is a bijection  $\pi_0\mathcal{L}_f M \rightarrow \pi \backslash \pi_0 P(M, *, f(*))$  given by  $[\sigma] \mapsto [\mu \cdot \sigma \cdot (f \circ \mu)^{-1}]$ , where  $\mu$  is a choice of path from  $*$  to  $\sigma(0)$ . This bijection is independent of the choice of  $\mu$ . The action of  $\pi$  on  $P(M, *, f(*))$  is given by  $[g] \bullet [\alpha] = [g \cdot \alpha \cdot (f \circ g)^{-1}]$ , where  $[g] \in \pi$  and  $\alpha$  is a path from  $*$  to  $f(*)$ .

The base path  $\tau$  from  $*$  to  $f(*)$  induces an bijection  $\pi_0 P(M, *, f(*)) \rightarrow \pi$  given by  $[\alpha] \mapsto [\alpha \cdot \tau^{-1}]$ . Under this bijection, we get an action of  $\pi$  on  $\pi$  given by  $[g] \bullet [h] = [g \cdot h \cdot \tau \cdot (f \circ g)^{-1} \cdot \tau^{-1}]$ , which is precisely the equivalence relation induced by the map  $\phi : \pi \rightarrow \pi$ . Hence, we have a bijection  $\pi_0\mathcal{L}_f M \rightarrow \pi_\phi$  given by  $[\sigma] \mapsto [\mu \cdot \sigma \cdot (f \circ \mu)^{-1} \cdot \tau^{-1}]$ , where  $\mu$  is a path from  $*$  to  $\sigma(0)$ .

Therefore,

$$\Omega_0^{fr}(\mathcal{L}_f) \cong \mathbb{Z}\pi_0\mathcal{L}_f \cong \mathbb{Z}\pi_\phi.$$

□

**Proposition 1.5.4.** *Under the above isomorphism,  $L^{geo}(f)$  and  $L^{bord}(f)$  correspond.*

The ideas for the proof of Proposition 1.5.4 can be found in [4] and [15], for instance. We sketch the proof here.

Consider a  $n$ -dimensional vector space  $V$  and a linear map  $L : V \rightarrow V$ . Zero is an isolated fixed point of  $L$  if and only if the determinant of  $I - L$  is nonzero, where  $I$  is the identity matrix. Let  $i(L)$  be the sign of this determinant. If  $V$  is also oriented, then the isomorphism  $\Delta_L(V) \oplus \Delta(V) \cong V \times V$  is either orientation preserving or reversing. It will be orientation preserving when  $i(L) = 1$  and orientation reversing if  $i(L) = -1$ . Applying this to the derivative  $d_x f : T_x M \rightarrow T_x M$  of  $f$  at a fixed point  $x$ , we get that  $i(d_x f) = i(f, x)$ .

A fixed point  $x \in \text{Fix}(f)$  is represented in  $\mathcal{L}_f M$  by the constant path  $c_x$  at  $x$ . Thus, under the bijection  $\pi_0 \mathcal{L}_f M \cong \pi_\phi$  in the proof of Proposition 1.5.2,  $x$  corresponds to  $[\mu \cdot (f \circ \mu)^{-1} \cdot \tau^{-1}] = \Phi(x)$  in  $\pi_\phi$ . Hence,

$$L^{bord}(f) \mapsto \sum_{x \in \text{Fix}(f)} i(d_x f) \Phi(x) = \sum_{k=1}^s i(f, F_k) \Phi(F_k) = L^{geo}(f).$$

## 1.6 Hattori-Stallings Trace

Recall that in the classical algebraic construction of the Lefschetz invariant, one views  $\tilde{f}_k$  as a matrix and takes its trace. This was used by Reidemeister to define  $L^{alg}(f)$ . In our generalizations, we will need to use a more sophisticated trace map, namely the Hattori-Stallings trace. Since on finitely generated free modules, the Hattori-Stallings trace agrees with the usual trace of a matrix, we could use it in the classical case as well. We will introduce the classical Hattori-Stallings trace here. (For the special case when  $M = R$ , see [25], [2] and [1].)

Let  $R$  be a ring,  $M$  an  $R$ -bimodule, and  $P$  a finitely generated projective left  $R$ -module. Let  $P^* = \text{Hom}_R(P, R)$  be the dual of  $P$ . Let  $[R, M]$  denote the abelian subgroup of  $M$  generated by elements of the form  $rm - mr$ , for  $r \in R$  and  $m \in M$ .

The Hattori-Stallings trace map,  $tr$  is given by the following composition.

$$\begin{array}{ccc} \mathrm{Hom}_R(P, M \otimes_R P) & \xleftarrow{\cong} P^* \otimes_R M \otimes_R P & \longrightarrow M/[R, M] \\ & \searrow tr & \parallel \\ & & HH_0(R; M) \end{array}$$

The map

$$P^* \otimes_R M \otimes_R P \rightarrow \mathrm{Hom}_R(P, M \otimes_R P)$$

is given by

$$\alpha \otimes m \otimes p \mapsto (p_1 \mapsto \alpha(p_1)(m \otimes p)),$$

where  $\alpha \in P^*$ ,  $m \in M$  and  $p, p_1 \in P$ . The map

$$P^* \otimes_R M \otimes_R P \rightarrow M/[R, M]$$

is given by

$$\alpha \otimes m \otimes p \mapsto \alpha(p)m.$$

The fact that the first map is an isomorphism is an application of the following lemma.

**Lemma 1.6.1.** *Let  $R$  be a ring,  $P$  a finitely generated projective right  $R$ -module and  $N$  a left  $R$ -module. Define*

$$f_P : P^* \otimes_R N \rightarrow \mathrm{Hom}_R(P, N)$$

by  $f_P(\alpha, n)(p) = \alpha(p)n$ . Then  $f_P$  is an isomorphism of groups.

*Proof.* Note that  $f_R : R^* \otimes_R N \rightarrow \mathrm{Hom}_R(R, N)$  is an isomorphism with inverse given by  $(g : R \rightarrow N) \mapsto \mathrm{id}_R \otimes_R g(1_R)$ . The result follows from the fact that  $f_{(-)} : (-)^* \otimes_R N \rightarrow \mathrm{Hom}_R(-, N)$  preserves finite direct sums.  $\square$

The Hattori-Stallings trace is a generalization of the classical trace, which is the map taking a matrix to the sum of its diagonal entries.

## CHAPTER 2

### BACKGROUND ON GROUPOIDS AND RINGOIDS

In order to construct a base point free Lefschetz-Nielsen theory, we replace the fundamental group with the fundamental groupoid. In this chapter, we generalize to the “oid” setting the basic algebraic definitions and results which we will need for our constructions. This treatment is based on [21, § 9], though we have developed additional material as needed. In particular, in Sections 2.2 and 3.1, we generalize the Hattori-Stallings trace.

We use the following notation. If  $C$  is a category, we denote the collection of objects in  $C$  by  $\text{Ob}(C)$ . If  $x$  and  $y$  are objects in  $C$ , we denote the collection of maps from  $x$  to  $y$  in  $C$  by  $C(x, y)$ . The category of sets will be denoted  $\mathbf{Sets}$ , the category of abelian groups will be denoted  $\mathbf{Ab}$ , and the category of left  $R$ -modules will be denoted  $R\text{-mod}$ . For two functors  $F$  and  $F'$ ,  $\text{Nat}(F, F')$  is the collection of natural transformations  $F \rightarrow F'$ .

Throughout, “ring” will mean an associative ring with unit.

#### 2.1 General Definitions and Results

##### 2.1.1 Groupoids

Let  $G$  be a group. We may view  $G$  as a category, denoted  $\mathbf{G}$ , in which there is one object,  $*$ , and for which all of the maps are isomorphisms. Each map corresponds to an element of  $G$  with composition of maps corresponding to the multiplication in the group.

This idea generalizes to define a groupoid.

**Definition 2.1.1.** *A groupoid  $G$  is a small category (the objects form a set) such that all maps are isomorphisms.*

Notice that if  $G$  is a groupoid, then for each object  $x$ ,  $G(x, x)$  is a group. The group structure comes from composition in the category  $G$ .

### 2.1.2 Ringoids

The same game can be played with rings. Let  $R$  be a ring. We may view  $R$  as a category, denoted  $\mathbf{R}$ , in which there is one object,  $*$ ,  $\mathbf{R}(*, *)$  is an abelian group, and composition is bilinear. Each map corresponds to an element of  $R$ , composition of maps corresponds to the multiplicative operation in  $R$ , and the group operation in  $\mathbf{R}(*, *)$  corresponds to the additive operation in  $R$ .

We generalize these notions to define a ringoid, also known as a linear category or a small category enriched in the category of abelian groups.

**Definition 2.1.2.** *A ringoid  $\mathcal{R}$  is a small category such that for each pair of objects  $x$  and  $y$ ,  $\mathcal{R}(x, y)$  is an abelian group and the composition function  $\mathcal{R}(y, z) \times \mathcal{R}(x, y) \rightarrow \mathcal{R}(x, z)$  is bilinear.*

Notice that if  $\mathcal{R}$  is a ringoid, then for each object  $x$ ,  $\mathcal{R}(x, x)$  is a ring.

**Example 2.1.3.** Recall that if  $H$  is a group, then the group ring  $\mathbb{Z}H$  is the free abelian group generated by  $H$ . Since  $\mathbb{Z}G$  is a ring, we can think of it as a category  $\mathbb{Z}\mathbf{H}$ , as above. This group ring construction can be generalized to a “groupoid ringoid” (though we will call it the group ring): Let  $G$  be a groupoid. The group ring of  $G$  with respect to  $\mathbb{Z}$ , denoted  $\mathbb{Z}G$ , is the category with the same objects as  $G$ , but with maps given by  $\mathbb{Z}G(x, y) = \{\sum n_i f_i \mid n_i \in \mathbb{Z}, f_i \in G(x, y)\}$  where only a finite number of the  $n_i$  are nonzero. More generally, let  $R$  be a commutative ring and  $G$  a groupoid. Define  $RG$  to be the ringoid with the same objects as  $G$ , but

with  $RG(x, y) = R(G(x, y))$ , the additive abelian group of formal sums of elements of  $G(x, y)$  with coefficients in  $R$ . This will be an important example.

### 2.1.3 Modules

Let  $R$  be a ring and  $\mathbf{R}$  the associated category. Let  $M$  be a left  $R$ -module. We can view  $M$  categorically as the (covariant) functor  $\mathbf{M} : \mathbf{R} \rightarrow \mathbf{Ab}$  taking the object  $*$  of  $\mathbf{R}$  to the abelian group  $M$ , and the map  $r : * \rightarrow *$  to the map on  $M$  given by left multiplication by  $r$  (recall that  $R = \mathbf{R}(*, *)$ .) That is,  $\mathbf{M}(r) : M \rightarrow M$  is given by  $a \mapsto ra$  for  $a \in M$ . Notice that  $\mathbf{M}$  gives a linear function  $\mathbf{R}(*, *) \rightarrow \mathbf{Ab}(M, M)$  and this linearity corresponds to the distributive properties in the module.

If  $N$  is a right  $R$ -module, then we view it categorically as a (covariant) functor  $\mathbf{N} : \mathbf{R}^{\text{op}} \rightarrow \mathbf{Ab}$ . This works the same as for a left  $R$ -module, except that on maps,  $\mathbf{N}(r) : N \rightarrow N$  is given by  $a \mapsto ar$  for  $a \in N$ , i.e., right multiplication by  $r$ .

**Definition 2.1.4.** *Let  $\mathcal{R}$  be a ringoid. A left  $\mathcal{R}$ -module is a (covariant) functor  $M : \mathcal{R} \rightarrow \mathbf{Ab}$  such that on maps,  $M : \mathcal{R}(x, y) \rightarrow \mathbf{Ab}(M(x), M(y))$  is a linear function. For right  $\mathcal{R}$ -modules, we take (covariant) functors  $\mathcal{R}^{\text{op}} \rightarrow \mathbf{Ab}$ .*

**Example 2.1.5.** In the case that  $R$  is a commutative ring and  $G$  is a groupoid, we reinterpret the definition of a left  $RG$ -module as a functor  $M : G \rightarrow R\text{-mod}$ . Similarly, a right  $RG$ -module is a functor  $M : G^{\text{op}} \rightarrow R\text{-mod}$ .

For the remainder of this section, unless otherwise noted, let  $G$  be a groupoid,  $R$  a commutative ring, and  $\mathcal{R}$  a ringoid. A homomorphism of (left)  $R$ -modules corresponds, in the category setting, to a natural transformation of functors.

**Definition 2.1.6.** *Let  $M$  and  $N$  be  $\mathcal{R}$ -modules. An  $\mathcal{R}$ -module homomorphism from  $M$  to  $N$  is a natural transformation from  $M$  to  $N$ . The set of all  $\mathcal{R}$ -module homomorphisms from  $M$  to  $N$  is denoted  $\text{Hom}_{\mathcal{R}}(M, N)$ .*

Let  $\mathcal{R}\text{-mod}$  denote the category of left  $\mathcal{R}$ -modules, and let  $\text{mod-}\mathcal{R}$  denote the category of right  $\mathcal{R}$ -modules.

**Definition 2.1.7.** *Let  $M$  and  $N$  be  $\mathcal{R}$ -modules. The direct sum  $M \oplus N$  of  $M$  and  $N$  is the left  $\mathcal{R}$ -module defined on an object  $x$  by  $(M \oplus N)(x) = M(x) \oplus N(x)$  and on a map  $g : x \rightarrow y$  by  $(M \oplus N)(g) = M(g) \oplus N(g)$ .*

**Definition 2.1.8.** *Let  $N$  be a left  $\mathcal{R}$ -module and  $M$  a right  $\mathcal{R}$ -module. Define the tensor product over  $\mathcal{R}$  of  $M$  and  $N$  to be the abelian group*

$$M \otimes_{\mathcal{R}} N = P/Q$$

where  $P$  is the abelian group

$$P = \bigoplus_{x \in \text{Ob}(\mathcal{R})} M(x) \otimes_{\mathbb{Z}} N(x)$$

and  $Q$  is the subgroup of  $P$  generated by

$$\{M(f)(m) \otimes n - m \otimes N(f)(n) \mid m \in M(y), n \in N(x), f \in \mathcal{R}(x, y)\}.$$

Next, we define free  $RG$ -modules, where  $R$  is a commutative ring and  $G$  is a groupoid. First, we need the following notions.

Given a category  $C$ , we can view  $\text{Ob}(C)$  as the subcategory of  $C$  whose objects are the same as the objects of  $C$ , but whose maps are only the identity maps. A covariant (contravariant) functor  $\text{Ob}(C) \rightarrow \text{Sets}$  will be called a left (right)  $\text{Ob}(C)$ -set. A map of  $\text{Ob}(C)$ -sets is a natural transformation. Let  $\text{Ob}(C)\text{-Sets}$  denote the category of left  $\text{Ob}(C)$ -sets, and let  $\text{Sets-Ob}(C)$  denote the category of right  $\text{Ob}(C)$ -sets.

Given either a left or right  $\text{Ob}(C)$ -set  $B$ , let

$$\mathcal{B} = \bigsqcup_{x \in \text{Ob}(C)} B(x),$$

where  $\sqcup$  denotes disjoint union, and let

$$\beta : \mathcal{B} \rightarrow \text{Ob}(C)$$

send  $b$  to  $x$  if  $b \in B(x)$ . Given  $\text{Ob}(C)$ -sets  $B$  and  $B'$ , we say  $B$  is an  $\text{Ob}(C)$ -subset of  $B'$  if for every  $x \in \text{Ob}(C)$ ,  $B(x) \subset B'(x)$ .

Suppose  $C$  is a small category and  $D$  is a category equipped with a “forgetful functor”  $D \rightarrow \text{Sets}$ . For a functor  $F : C \rightarrow D$ , let  $|F| : \text{Ob}(C) \rightarrow \text{Sets}$  be the composition  $\text{Ob}(C) \hookrightarrow C \rightarrow D \rightarrow \text{Sets}$ , where the functor  $D \rightarrow \text{Sets}$  is the forgetful functor. In particular,  $|-| : \text{RG-mod} \rightarrow \text{Ob}(C)\text{-Sets}$  and  $|-| : \text{mod-RG} \rightarrow \text{Sets-Ob}(G)$ .

**Definition 2.1.9.** Let  $RG(x, y)$  denote the free  $R$ -module generated by the set  $G(x, y)$ . Then for each  $x \in \text{Ob}(G)$ , we can define a left  $RG$ -module  $\overline{\text{RG}}_x = RG(x, -)$  by  $\overline{\text{RG}}_x(y) = RG(x, y)$ . For a map  $g : y \rightarrow z$  in  $G$ , let  $\overline{\text{RG}}_x(g) = g \circ (-)$ . Similarly, we define a right  $RG$ -module  $\overline{\text{RG}}^x = RG(-, x)$  by  $\overline{\text{RG}}^x(y) = RG(y, x)$ . For a map  $h : y \rightarrow z$  in  $G^{op}$ , let  $\overline{\text{RG}}^x(h) = (-) \circ h$ .

**Definition 2.1.10.** Define a functor

$$\overline{\text{RG}}_{(-)} : \text{Ob}(G)\text{-Sets} \rightarrow \text{RG-mod}$$

by

$$\overline{\text{RG}}_B = \bigoplus_{b \in \mathcal{B}} \overline{\text{RG}}_{\beta(b)} = \bigoplus_{b \in \mathcal{B}} RG(\beta(b), -).$$

Similarly, define  $\overline{\text{RG}} = \overline{\text{RG}}^{(-)} : \text{Sets-Ob}(G) \rightarrow \text{mod-RG}$  by

$$\overline{\text{RG}}^B = \bigoplus_{b \in \mathcal{B}} \overline{\text{RG}}^{\beta(b)} = \bigoplus_{b \in \mathcal{B}} RG(-, \beta(b)).$$

**Proposition 2.1.11.** The functor  $\overline{\text{RG}}_{(-)}$  is a left adjoint to  $|-| : \text{RG-mod} \rightarrow \text{Ob}(G)\text{-Sets}$ . The functor  $\overline{\text{RG}}^{(-)}$  is a left adjoint to  $|-| : \text{mod-RG} \rightarrow \text{Sets-Ob}(G)$ .

*Proof.* We prove this for left  $RG$ -modules. The proof for right  $RG$ -modules is similar.

For an  $\text{Ob}(G)$ -set  $B$  and an  $RG$ -module  $M$ , define a set map

$$\psi = \psi_{B,M} : \text{RG-mod}(\overline{\text{RG}}_B, M) \rightarrow \text{Ob}(G)\text{-Sets}(B, |M|)$$

by  $\psi(\eta)_y(b) = \eta_y(\text{id}_y) \in |M(y)|$ , where  $\eta : \overline{\text{RG}}_B \rightarrow M$  is a natural transformation and  $b \in B(y)$ . Then  $\psi$  is a bijection with inverse  $\psi^{-1}$  given by  $\psi^{-1}(\tau)_y(h) = M(h)(\tau_x(b))$ , where  $\tau : B \rightarrow |M|$  is a natural transformation,  $h : \beta(b) \rightarrow y$  is a map in  $G$  and  $\beta(b) = x$ .  $\square$

Notice that for each  $\text{Ob}(G)$ -set  $B$ , we get a natural transformation

$$\eta_B = \psi(\text{id}_{\overline{\text{RG}}_B}) : B \rightarrow |\overline{\text{RG}}_B|$$

which is universal. To be precise, for each  $RG$ -module  $N$  and each natural transformation  $f : B \rightarrow |N|$ , there exists a unique natural transformation  $F : \overline{\text{RG}}_B \rightarrow N$  such that the following diagram commutes.

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & |\overline{\text{RG}}_B| \\ & \searrow f & \downarrow |F| \\ & & |N| \end{array}$$

This leads to the following definition of a free  $RG$ -module with base  $B$ .

**Definition 2.1.12.** *An  $RG$ -module  $M$  is free with base an  $\text{Ob}(G)$ -set  $B \subset |M|$  if for each  $RG$ -module  $N$  and natural transformation  $f : B \rightarrow |N|$  there is a unique natural transformation  $F : M \rightarrow N$  with  $|F| \circ i = f$ , where  $i$  is the inclusion  $B \rightarrow |M|$ .*

**Example 2.1.13.** The  $RG$ -module  $\overline{\text{RG}}_x$  is a free left  $RG$ -module with base  $B_x : \text{Ob}(G) \rightarrow \text{Sets}$  given by

$$B_x(y) = \begin{cases} \{x\}, & \text{if } y = x; \\ \emptyset, & \text{if } y \neq x. \end{cases}$$

If  $B$  is any  $\text{Ob}(G)$ -set,

$$\overline{RG}_B = \bigoplus_{b \in B} \overline{RG}_{\beta(b)} = \bigoplus_{b \in B} RG(\beta(b), -)$$

is a free  $RG$ -module with base  $B$ . We call this the canonical free left  $RG$ -module with base  $B$ .

Similarly,  $\overline{RG}^x$  and  $\overline{RG}^B$  are free right  $RG$ -modules with bases  $B_x$  and  $B$ , respectively.

Let  $\mathcal{R}$  be a ringoid and  $M$  an  $\mathcal{R}$ -module. Let  $S$  be an  $\text{Ob}(\mathcal{R})$ -subset of  $|M|$  and let  $\text{Span}(S)$  be the smallest  $\mathcal{R}$ -submodule of  $M$  containing  $S$ ,

$$\text{Span}(S) = \bigcap \{N \mid N \text{ is an } \mathcal{R}\text{-submodule of } M, S \subset N\}.$$

**Definition 2.1.14.** *We say  $M$  is generated by  $S$  if  $M = \text{Span}(S)$ , and  $M$  is finitely generated if  $S$  is finite.*

**Proposition 2.1.15.** *If  $R$  is a commutative ring,  $G$  is a groupoid,  $M$  is a left  $RG$ -module, and  $B$  is an  $\text{Ob}(G)$ -subset of  $|M|$ , then  $\text{Span}(B)$  is the image of the unique natural transformation  $\tau : \overline{RG}_B \rightarrow M$  extending  $\text{id} : B \rightarrow B \subset |M|$ . Furthermore,  $M$  is generated by  $B$  if  $\tau$  is surjective.*

**Proposition 2.1.16.** *Let  $R$  be a commutative ring and  $G$  a groupoid. Let  $B$  be an  $\text{Ob}(G)$ -set. If  $M$  is a free left  $RG$ -module with base  $B$ , then  $M$  is generated by  $B$ . In particular, there is a natural equivalence  $\tau : \overline{RG}_B \rightarrow M$ .*

*Proof.* Define  $\tau : \overline{RG}_B \rightarrow M$ . For  $x \in \text{Ob}(G)$ , let

$$\tau_x : \overline{RG}_B(x) = \bigoplus_{b \in B} RG(\beta(b), x) \rightarrow M(x)$$

be given by  $(g : \beta(b) \rightarrow x) \mapsto M(g)(b)$ . One can show that  $\tau$  is a natural transformation. To construct an inverse natural transformation, define  $\eta : B \rightarrow |\overline{RG}_B|$  by setting  $\eta_x(b) = \text{id}_x$ . Since  $M$  is free with base  $B$ ,  $\eta$  extends to a unique natural transformation  $M \rightarrow \overline{RG}_B$ . One can show that this is an inverse to  $\tau$ .  $\square$

**Definition 2.1.17.** *An  $RG$ -module  $P$  is projective if it is the direct summand of a free  $RG$ -module.*

#### 2.1.4 $RG$ -bimodules

At this point, we restrict our attention to  $RG$ -modules, where  $R$  is a commutative ring and  $G$  is a groupoid. In this setting, we define an  $RG$ -bimodule. We describe a way to think of the ringoid  $RG$  as a “free rank one”  $RG$ -bimodule. Such notions as Hom, tensor product, and the dual of a module will be defined. As needed, the usual results about such objects will be proven.

See Example 2.1.5 for the definition of an  $RG$ -module. Recall that the category of left  $RG$ -modules is denoted  $\text{RG-mod}$  and that the category of right  $RG$ -modules is denoted  $\text{mod-RG}$ .

**Definition 2.1.18.** *An  $RG$ -bimodule is a (covariant) functor*

$$M : G \times G^{\text{op}} \rightarrow \text{R-mod}.$$

*Denote the category of  $RG$ -bimodules by  $\text{RG-bimod}$ .*

**Example 2.1.19.** We put an  $RG$ -bimodule structure on  $RG$ , and denote it by  $\overline{\text{RG}}$ . For  $(x, y) \in G \times G^{\text{op}}$ , set  $\overline{\text{RG}}(x, y) = RG(y, x)$  (the free  $R$ -module on the set  $G(y, x)$ ). Notice the change in the order of  $x$  and  $y$ . For maps  $g : x \rightarrow x'$  in  $G$  and  $h : y \rightarrow y'$  in  $G^{\text{op}}$ , set  $\overline{\text{RG}}(g, h) = g \circ (-) \circ h : RG(y, x) \rightarrow RG(y', x')$ .

Notice that if  $G$  is a group, then  $\overline{\text{RG}}$  is just  $RG$  itself with left and right  $RG$ -module structure given by the multiplication in  $RG$ .

We would like to be able to view an  $RG$ -bimodule  $N$  as either a right or a left  $RG$ -module. However, there is no canonical way to do this; each choice of object in  $G$  produces a left and a right  $RG$ -module structure on  $N$ . Instead, we define two functors:  $(-)\text{ad}$  and  $\text{ad}(-)$ . In essence,  $N\text{ad}$  encapsulates all of the right  $RG$ -

module structures on  $N$  induced by objects of  $G$ . Similarly,  $\text{ad } N$  encapsulates all of the left  $RG$ -module structure on  $N$ .

**Definition 2.1.20.** *Define a covariant functor*

$$(-) \text{ ad} : \text{RG-bimod} \rightarrow (\text{mod-RG})^G$$

as follows. Let  $N$  be an  $RG$ -bimodule. For  $x \in \text{Ob}(G)$  let

$$N \text{ ad}(x) = N(x, -).$$

For  $g$  a map in  $G$  let

$$N \text{ ad}(g) = N(g, -).$$

Explicitly,

$$N \text{ ad}(x) : G^{\text{op}} \rightarrow \text{R-mod}$$

is given by

$$N \text{ ad}(x)(y) = N(x, y)$$

for  $y \in \text{Ob}(G^{\text{op}})$  and

$$N \text{ ad}(x)(h) = N(\text{id}_x, h)$$

for  $h : y \rightarrow z$  a map in  $G^{\text{op}}$ .

**Definition 2.1.21.** *Define a covariant functor*

$$\text{ad}(-) : \text{RG-bimod} \rightarrow (\text{RG-mod})^{G^{\text{op}}}$$

as follows: let  $N$  be an  $RG$ -bimodule. For  $x \in \text{Ob}(G^{\text{op}})$  let

$$\text{ad } N(x) = N(-, x).$$

For  $g$  a map in  $G^{\text{op}}$  let

$$\text{ad } N(g) = N(-, g).$$

Explicitly,

$$\text{ad } N(x) : G \rightarrow \text{R-mod}$$

is given by

$$\text{ad } N(x)(y) = N(y, x)$$

for  $y \in \text{Ob}(G)$  and

$$\text{ad } N(x)(h) = N(h, \text{id}_x)$$

for  $h : y \rightarrow z$  a map in  $G$ .

**Example 2.1.22.** Apply the ad functors to the  $RG$ -bimodule  $\overline{RG}$ . For  $x \in \text{Ob}(G)$ ,  $\overline{RG} \text{ad}(x) = RG(-, x) = \overline{RG}^x$  (the canonical free right  $RG$ -module with base consisting of one element.) Hence,  $\overline{RG} \text{ad}(x) : G^{\text{op}} \rightarrow \text{R-mod}$ , with  $\overline{RG} \text{ad}(x)(y) = RG(y, x)$  and  $\overline{RG} \text{ad}(x)(h) = (-) \circ h$  for  $h : y \rightarrow z$  a map in  $G^{\text{op}}$ . Also, for  $g : x \rightarrow x'$  a map in  $G$ ,  $\overline{RG} \text{ad}(g) = \overline{RG}(g, -) : RG(-, x) \rightarrow RG(-, x')$  is the natural transformation of right  $RG$ -modules given by  $\overline{RG} \text{ad}(g)_y = g \circ (-) : RG(y, x) \rightarrow RG(y, x')$ . Similarly, one can see what  $\text{ad } \overline{RG}$  looks like.

Next, if  $N$  is an  $RG$ -bimodule and  $M$  is an  $RG$ -module, one expects to define  $\text{Hom}_{RG}(N, M)$  and  $\text{Hom}_{RG}(M, N)$  in such a way that they are also  $RG$ -modules. Let  $M_l$  (respectively,  $M_r$ ) denote a left (respectively, right)  $RG$ -module. Note that  $\text{Hom}_{RG}(M_l, -)$  is a covariant functor  $\text{RG-mod} \rightarrow \text{R-mod}$  and  $\text{Hom}_{RG}(-, M_l)$  is a contravariant functor  $\text{RG-mod} \rightarrow \text{R-mod}$ . Similarly  $\text{Hom}_{RG}(M_r, -)$  is a covariant functor  $\text{mod-RG} \rightarrow \text{R-mod}$  and  $\text{Hom}_{RG}(-, M_r)$  is a contravariant functor  $\text{mod-RG} \rightarrow \text{R-mod}$ .

**Definition 2.1.23.** Let  $N$  be an  $RG$ -bimodule.  $\text{Hom}_{RG}(M_l, N)$  is defined to be the right  $RG$ -module given by the composition

$$G^{\text{op}} \xrightarrow{\text{ad } N} \text{RG-mod} \xrightarrow{\text{Hom}_{RG}(M_l, -)} \text{R-mod}.$$

$\text{Hom}_{RG}(N, M_l)$  is defined to be the left  $RG$ -module given by the composition

$$G^{\text{op}} \xrightarrow{\text{ad } N} \text{RG-mod} \xrightarrow{\text{Hom}_{RG}(-, M_l)} \text{R-mod}.$$

$\text{Hom}_{RG}(M_r, N)$  is defined to be the left  $RG$ -module given by the composition

$$G \xrightarrow{N \text{ ad}} \text{mod-}RG \xrightarrow{\text{Hom}_{RG}(M_r, -)} R\text{-mod}.$$

$\text{Hom}_{RG}(N, M_r)$  is defined to be the right  $RG$ -module given by the composition

$$G \xrightarrow{N \text{ ad}} \text{mod-}RG \xrightarrow{\text{Hom}_{RG}(-, M_r)} R\text{-mod}.$$

Similarly, we define the tensor products  $N \otimes_{RG} M_l$  and  $M_r \otimes_{RG} N$  where  $N$  is an  $RG$ -bimodule. As above,  $M_l$  (respectively,  $M_r$ ) is a left (respectively, right)  $RG$ -module. Notice that both functors  $(-) \otimes_{RG} M_l : \text{mod-}RG \rightarrow R\text{-mod}$  and  $M_r \otimes_{RG} (-) : RG\text{-mod} \rightarrow R\text{-mod}$  are covariant.

**Definition 2.1.24.** Let  $N$  be an  $RG$ -bimodule. Define  $N \otimes_{RG} M_l$  to be the left  $RG$ -module given by the composition

$$G \xrightarrow{N \text{ ad}} \text{mod-}RG \xrightarrow{(-) \otimes_{RG} M_l} R\text{-mod}.$$

Define  $M_r \otimes_{RG} N$  to be the right  $RG$ -module given by the composition

$$G^{\text{op}} \xrightarrow{\text{ad } N} RG\text{-mod} \xrightarrow{M_r \otimes_{RG} (-)} R\text{-mod}.$$

Applying the above definitions to the  $RG$ -bimodule  $\overline{RG}$ , we get the results for Hom and tensor product which we would expect from algebra. These next three propositions justify viewing  $\overline{RG}$  as “the free rank one”  $RG$ -module. Notice that it is not, however, a free  $RG$ -module.

**Proposition 2.1.25.** Given an  $RG$ -module  $M$ ,

$$\text{Hom}_{RG}(\overline{RG}, M) \cong M$$

as  $RG$ -modules.

*Proof.* Let  $M$  be a right  $RG$ -module. For an object  $x$  of  $G$ ,

$$\text{Hom}_{RG}(\overline{RG}, M)(x) = \text{Nat}(RG(-, x), M) \cong M(x)$$

by Yoneda's lemma. Note that the isomorphism is given by  $\tau \mapsto \tau_x(id_x)$ .

For a map  $g : x \rightarrow y$  in  $G$ , we must show that we get a commutative diagram

$$\begin{array}{ccc} \text{Nat}(RG(-, x), M) & \xrightarrow{\cong} & M(x) \\ \phi_g \uparrow & & \uparrow M(g) \\ \text{Nat}(RG(-, y), M) & \xrightarrow{\cong} & M(y) \end{array}$$

where  $\phi_g$  is defined by  $\phi_g(\tau)_z(\sigma) = \tau_z(g \circ \sigma)$  for  $z \in \text{Ob}(G)$  and  $\sigma : z \rightarrow x$  a map in  $G$ . To check commutativity, take  $\tau \in \text{Nat}(RG(-, y), M)$ . Moving clockwise around the diagram,

$$\tau \mapsto \phi_g(\tau) \mapsto \phi_g(\tau)_x(id_x) = \tau_x(g).$$

Moving counterclockwise,

$$\tau \mapsto \tau_y(id_y) \mapsto M(g)(\tau_y(id_y)).$$

But by naturality of  $\tau$ ,  $M(g)(\tau_y(id_y)) = \tau_x(g)$  and hence the diagram commutes.

The proof if  $M$  is a left  $RG$ -module is similar. □

**Proposition 2.1.26.** *Given a left (right)  $RG$ -module  $M$ ,*

$$\overline{RG} \otimes_{RG} M \cong M$$

*as left (right)  $RG$ -modules.*

*Proof.* For  $x \in \text{Ob}(G)$ , define

$$\phi : RG(-, x) \otimes_{RG} M \rightarrow M(x)$$

by

$$(\sigma : y \rightarrow x) \otimes m \mapsto M(\sigma)(m).$$

Define

$$\psi : M(x) \rightarrow RG(-, x) \otimes_{RG} M$$

by

$$m \mapsto id_x \otimes m.$$

Clearly  $\phi \circ \psi = id$ . On the other hand,  $\psi \circ \phi(\sigma \otimes m) = id_x \otimes M(\sigma)(m)$ . By definition of tensor product,  $\sigma \otimes m - id_x \otimes M(\sigma)(m) = 0$ . Hence  $\psi \circ \phi = id$  and  $(\overline{RG} \otimes_{RG} M)(x) \cong M(x)$ .

For  $g : x \rightarrow y$  a map in  $G$ , we need to show that we get a commutative diagram

$$\begin{array}{ccc} RG(-, x) \otimes_{RG} M & \xrightarrow{\phi_x} & M(x) \\ (g \circ (-)) \otimes id \downarrow & & \downarrow M(g) \\ RG(-, y) \otimes_{RG} M & \xrightarrow{\phi_y} & M(y) \end{array}$$

Let  $(\sigma : z \rightarrow x) \otimes m \in RG(z, x) \otimes_R M(z) \subset RG(-, x) \otimes_{RG} M$ . Moving counter-clockwise,

$$\sigma \otimes m \mapsto g \circ \sigma \otimes m \mapsto M(g \circ \sigma)(m),$$

and moving clockwise,

$$\sigma \otimes m \mapsto M(\sigma)(m) \mapsto M(g)M(\sigma)(m) = M(g \circ \sigma)(m).$$

□

**Proposition 2.1.27.** *Given right  $RG$ -module  $M$ ,*

$$M \otimes_{RG} \overline{RG} \cong M$$

*as right  $RG$ -modules.*

The proof is similar to the previous proof.

In particular, we can now define the dual of an  $RG$ -module.

**Definition 2.1.28.** *Let  $M$  be a left (right)  $RG$ -module. The dual of  $M$  is the right (left)  $RG$ -module  $M^* = \text{Hom}_{RG}(M, \overline{RG})$ .*

### 2.1.5 More About Modules

We will need a few more unsurprising results and definitions involving  $RG$ -modules, Hom and tensor product.

**Proposition 2.1.29.** *Let  $M, N$  and  $P$  be  $RG$ -modules. Then*

$$\mathrm{Hom}_{RG}(M \oplus N, P) \cong \mathrm{Hom}_{RG}(M, P) \oplus \mathrm{Hom}_{RG}(N, P).$$

*Proof.* Define

$$\phi : \mathrm{Hom}_{RG}(M \oplus N, P) \rightarrow \mathrm{Hom}_{RG}(M, P) \oplus \mathrm{Hom}_{RG}(N, P)$$

by

$$\tau \mapsto \tau_M \oplus \tau_N$$

for  $\tau : M \oplus N \rightarrow P$ . Here  $\tau_M : M \rightarrow P$  is the natural transformation given by  $(\tau_M)_x(m) = \tau_x(m, 0_{N(x)})$  for  $m \in M(x)$  and  $0_{N(x)}$  denoting the additive identity in the  $R$ -module  $N(x)$ . Note that naturality follows from naturality of  $\tau$ . Define  $\tau_N$  analogously.

Define

$$\psi : \mathrm{Hom}_{RG}(M, P) \oplus \mathrm{Hom}_{RG}(N, P) \rightarrow \mathrm{Hom}_{RG}(M \oplus N, P)$$

as follows: let  $\eta : M \rightarrow P$  and  $\nu : N \rightarrow P$ . For  $x \in \mathrm{Ob}(G)$ , let

$$\psi(\eta \oplus \nu)_x : M(x) \oplus N(x) \rightarrow P(x)$$

be given by

$$(m, n) \mapsto \eta_x(m) + \nu_x(n)$$

for  $m \in M(x)$  and  $n \in N(x)$ .

We show that  $\phi$  and  $\psi$  are inverse homomorphisms.

Step 1:  $\phi \circ \psi = id$ . Notice that  $\phi \circ \psi(\eta \oplus \nu) = \psi(\eta \oplus \nu)_M \oplus \psi(\eta \oplus \nu)_N$ . Let  $x \in \mathrm{Ob}(G)$  and  $m \in M(x)$ . Then  $(\psi(\eta \oplus \nu)_M)_x(m) = \eta_x(m) + \nu_x(0) = \eta_x(m)$  and it follows that  $\psi(\eta \oplus \nu)_M = \eta$ . Similarly,  $\psi(\eta \oplus \nu)_N = \nu$ .

Step 2:  $\psi \circ \phi = id$ . Let  $x \in \mathrm{Ob}G$ ,  $m \in M(x)$  and  $n \in N(x)$ . Then  $\psi \circ \phi(\tau)_x(m, n) = (\tau_M)_x(m) + (\tau_N)_x(n) = \tau_x(m, 0) + \tau_x(0, n) = \tau_x(m, n)$ .  $\square$

**Proposition 2.1.30.** *Let  $M, N$  and  $P$  be  $RG$ -modules. Then*

$$(M \oplus N) \otimes_{RG} P \cong (M \otimes_{RG} P) \oplus (N \otimes_{RG} P).$$

*Proof.* Define

$$\phi : (M \oplus N) \otimes_{RG} P \rightarrow (M \otimes_{RG} P) \oplus (N \otimes_{RG} P)$$

by

$$(m, n) \otimes p \mapsto (m \otimes p, n \otimes p)$$

where  $x \in \text{Ob}(G)$ ,  $m \in M(x)$ ,  $n \in N(x)$  and  $p \in P(x)$ .

Define

$$\phi : (M \otimes_{RG} P) \oplus (N \otimes_{RG} P) \rightarrow (M \oplus N) \otimes_{RG} P$$

by

$$(m \otimes p_1, n \otimes p_2) \mapsto (m, 0) \otimes p_1 + (0, n) \otimes p_2$$

where  $x, y \in \text{Ob}(G)$ ,  $m \in M(x)$ ,  $n \in N(y)$ ,  $p_1 \in P(x)$  and  $p_2 \in P(y)$ .

It is straightforward to show that  $\phi$  and  $\psi$  are inverse homomorphisms.  $\square$

**Proposition 2.1.31.** *Let  $M$  and  $N$  be  $RG$ -modules. Then there is a natural equivalence*

$$(M \oplus N)^* \cong M^* \oplus N^*$$

*Proof.* Recall that  $\overline{RG}_x$  is the free left  $RG$ -module with base  $B_x \subset \text{Ob}(G)$  (i.e.,  $\overline{RG}_x(y) = RG(x, y)$  for  $y \in \text{Ob}(G)$ .)

Define a natural transformation

$$\phi : (M \oplus N)^* \rightarrow M^* \oplus N^*$$

as follows: for  $x \in \text{Ob}(G)$  and  $\tau : M \oplus N \rightarrow \overline{RG}_x$  a natural transformation, let  $\phi_x(\tau) = \tau_M \oplus \tau_N$ . The natural transformations  $\tau_M$  and  $\tau_N$  are as defined above in the proof of Proposition 2.1.29.

Define a natural transformation

$$\psi : M^* \oplus N^* \rightarrow (M \oplus N)^*$$

as follows: for  $x \in \text{Ob}(G)$  and natural transformations  $\eta : M \rightarrow \overline{RG}_x$  and  $\nu : N \rightarrow \overline{RG}_x$ , let  $\psi_x(\eta \oplus \nu) = \eta + \nu$ . Here,  $(\eta + \nu)_z(m, n) = \eta_z(m) + \nu_z(n) \in RG(x, z)$  for  $z \in \text{Ob}(G)$ ,  $m \in M(z)$ , and  $n \in N(z)$ .

It is straightforward to show that  $\phi$  and  $\psi$  are inverse natural transformations. □

### 2.1.6 Chain Complexes

In this section,  $R$  is a commutative ring and  $G$  is a groupoid. Let  $\text{Ch}(R)$  be the category of chain complexes over the ring  $R$ .

**Definition 2.1.32.** *An  $RG$ -chain complex is a (covariant) functor  $C_\bullet : G \rightarrow \text{Ch}(R)$ .*

**Lemma 2.1.33.** *The following are equivalent:*

- (i)  $C_\bullet$  is an  $RG$ -chain complex;
- (ii) there exists a family  $\{C_n\}$  of  $RG$ -modules together with a family of natural transformations  $\{d_n : C_n \rightarrow C_{n-1}\}$ , called differentials, such that  $d_{n-1} \circ d_n = 0$ .

Using the second characterization of  $RG$ -chain complexes, we can now define finitely generated projective chain complexes, chain maps and chain homotopies in the usual manner.

**Definition 2.1.34.** *An  $RG$ -chain complex  $P_\bullet$  is said to be finitely generated projective if each  $P_n$  is a finitely generated projective  $RG$ -module and  $P_\bullet$  is bounded (i.e.,  $P_n = 0$  for all but a finite number of  $n$ .)*

**Definition 2.1.35.** *An  $RG$ -chain map  $f : C_\bullet \rightarrow D_\bullet$  is a family  $\{f_n : C_n \rightarrow D_n\}$  of natural transformations such that  $d'_n \circ f_n = f_{n-1} \circ d_n$  for all  $n$ , where the  $d_n$  are the differentials of  $C_\bullet$  and the  $d'_n$  are the differentials of  $D_\bullet$ .*

**Definition 2.1.36.** *Two  $RG$ -chain maps  $f : C_\bullet \rightarrow D_\bullet$  and  $g : C_\bullet \rightarrow D_\bullet$  are  $RG$ -chain homotopic, denoted  $f \sim_{ch} g$ , if there exists a family  $\{s_n : C_n \rightarrow D_{n-1}\}$  of natural transformations such that*

$$f_n - g_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n.$$

**Definition 2.1.37.** *Two  $RG$ -chain complexes  $C_\bullet$  and  $D_\bullet$  are chain homotopy equivalent if there exists  $RG$ -chain maps  $f : C_\bullet \rightarrow D_\bullet$  and  $g : D_\bullet \rightarrow C_\bullet$  such that  $f \circ g \sim_{ch} \text{id}_{D_\bullet}$  and  $g \circ f \sim_{ch} \text{id}_{C_\bullet}$ . In this case,  $f$  is said to be a chain homotopy equivalence.*

Let  $\text{Ch}(RG)$  denote the category of  $RG$ -chain complexes and chain maps, and let  $\mathcal{P}(RG)$  denote the subcategory of finitely generated projective  $RG$ -chain complexes.

## 2.2 Generalized Hattori-Stallings Trace

In this section we define Hattori-Stallings trace for  $RG$ -modules. Our definition is a direct generalization of the classical definition given in Section 1.6. When  $\alpha : G \rightarrow G$  is a functor, we consider the Hattori-Stallings trace for “ $\alpha$ -twisted” endomorphisms, and prove that the trace is commutative. We also extend the trace to  $RG$ -chain complexes. Throughout this section,  $G$  is a groupoid and  $R$  is a commutative ring.

### 2.2.1 The Definition

Given left  $RG$ -modules  $N$  and  $P$ , define an  $R$ -module homomorphism

$$\phi_P = \phi_{P,N} : P^* \otimes_{RG} N \rightarrow \text{Hom}_{RG}(P, N) \tag{2.2.1}$$

by letting:  $\phi_P(\tau \otimes m) : P \rightarrow N$  be the natural transformation given by

$$\phi_P(\tau \otimes m)_z(p) = N(\tau_z(p))(m)$$

where  $\tau \in P^*(y)$ ,  $m \in N(y)$ ,  $p \in P(z)$ , and  $z \in \text{Ob}(G)$ .

**Proposition 2.2.1.** *If  $P$  is a finitely generated projective  $RG$ -module, then  $\phi_P$  is an isomorphism.*

The proof will use the following three lemmas.

**Lemma 2.2.2.** *Given  $x \in \text{Ob}(G)$ ,  $\phi_{\overline{RG}_x}$  is an isomorphism.*

*Proof.* In the proof, write  $\phi$  for  $\phi_{\overline{RG}_x}$ . We define a map  $\psi$  and show that it is an inverse of  $\phi$ .

Define

$$\psi : \text{Hom}_{RG}(\overline{RG}_x, N) \rightarrow \overline{RG}_x^* \otimes_{RG} N$$

by

$$\eta \mapsto \text{id}_{\overline{RG}_x} \otimes \eta_x(\text{id}_x)$$

where  $\eta : \overline{RG}_x \rightarrow N$  is a natural transformation.

Step 1: Show that  $\psi \circ \phi = \text{id}$ . Let  $\tau \in \overline{RG}_x^*(y)$  and  $m \in N(y)$ . Note that  $\tau$  is a natural transformation  $\tau : RG(x, -) \rightarrow RG(y, -)$ . Now,

$$\psi \circ \phi(\tau \otimes m) = \text{id}_{\overline{RG}_x} \otimes N(\tau_x(\text{id}_x))(m)$$

and, recalling the definition of tensor product of  $RG$ -modules, we need only show that

$$\tau \otimes m - \text{id}_{\overline{RG}_x} \otimes N(\tau_x(\text{id}_x))(m) \tag{2.2.2}$$

is zero in  $\overline{RG}_x^* \otimes_{RG} N$ . Given the map  $\tau_x(\text{id}_x) : y \rightarrow x$  in  $G$  and the elements  $\text{id}_{\overline{RG}_x} \in \overline{RG}_x^*$  and  $m \in N(y)$ , we get that

$$\overline{RG}_x^*(\tau_x(\text{id}_x))(\text{id}_{\overline{RG}_x}) \otimes m - \text{id}_{\overline{RG}_x} \otimes N(\tau_x(\text{id}_x))(m) = 0 \tag{2.2.3}$$

in  $\overline{RG}_x^* \otimes_{RG} N$ . To compare  $\overline{RG}_x^*(\tau_x(\text{id}_x))(\text{id}_{\overline{RG}_x})$  to  $\tau$ , let  $z \in \text{Ob}(G)$  and  $h : x \rightarrow z$  in  $G$ . Then  $\overline{RG}_x^*(\tau_x(\text{id}_x))(\text{id}_{\overline{RG}_x})_z(h) = h \circ \tau_x(\text{id}_x) = \tau_z(h \circ \text{id}_x) = \tau_z(h)$  by naturality of  $\tau$ . Hence, 2.2.2 and 2.2.3 are equal and we have completed Step 1.

Step 2: Show that  $\phi \circ \psi = \text{id}$ . For a natural transformation  $\eta : \overline{RG}_x \rightarrow N$ , an object  $z$  and a map  $h : x \rightarrow z$  in  $G$ ,  $\phi \circ \psi(\eta)$  is given by

$$(\phi \circ \psi)(\eta)_z(h) = N((\text{id}_{\overline{RG}_x})_z(h))(\tau_x(\text{id}_x)) = N(h)(\tau_x(\text{id}_x)).$$

But, naturality of  $\eta$  implies  $N(h)(\eta_x(\text{id}_x)) = \eta_z(h \circ \text{id}_x) = \eta_z(h)$ . Thus,  $(\phi \circ \psi)(\eta) = \eta$ .  $\square$

**Lemma 2.2.3.** *If  $P$  and  $Q$  are left  $RG$ -modules, then  $\phi_{P \oplus Q} = \phi_P \oplus \phi_Q$ .*

*Proof.* Consider the following diagram:

$$\begin{array}{ccc} (P \oplus Q)^* \otimes_{RG} N & \xrightarrow{\phi_{P \oplus Q}} & \text{Hom}_{RG}(P \oplus Q, N) \\ \cong \downarrow & & \downarrow \cong \\ (P^* \oplus Q^*) \otimes_{RG} N & & \\ \cong \downarrow & & \\ (P^* \otimes_{RG} N) \oplus (Q^* \otimes_{RG} N) & \xrightarrow{\phi_P \oplus \phi_Q} & \text{Hom}_{RG}(P, N) \oplus \text{Hom}_{RG}(Q, N) \end{array}$$

The vertical isomorphisms are described in Propositions 2.1.29, 2.1.30 and 2.1.31. Using those descriptions, one can see that the diagram commutes.  $\square$

**Lemma 2.2.4.** *Let  $P$  and  $Q$  be left  $RG$ -modules and let  $N = P \oplus Q$ . If  $\phi_N$  is an isomorphism, then  $\phi_P$  is an isomorphism also.*

*Proof.* By the previous Lemma,  $\phi_N = \phi_P \oplus \phi_Q$ . The result follows immediately.  $\square$

*Proof of Proposition 2.2.1.* The proof is in two steps.

Step 1: Suppose that  $P$  is a finitely generated free  $RG$ -module. Then  $P$  is naturally equivalent to  $\overline{RG}_B = \bigoplus_{b \in B} \overline{RG}_{\beta(b)}$  for some  $\text{Ob}(G)$ -set  $B$ . By Lemma 2.2.3,  $\phi_P = \bigoplus_{b \in B} \phi_{\overline{RG}_{\beta(b)}}$ , and by Lemma 2.2.2, it is an isomorphism.

Step 2: Suppose that  $P$  is a finitely generated projective  $RG$ -modules and so  $P$  is a direct summand of a finitely generated free  $RG$ -module. Combining Step 1 and Lemma 2.2.4 we see that  $\phi_P$  is an isomorphism.  $\square$

**Definition 2.2.5.** Let  $R$  be a ring. Given an  $RG$ -bimodule  $M$ , define  $M/[RG, M]$  to be the  $R$ -module

$$\left( \bigoplus_{x \in \text{Ob}(G)} M(x, x) \right) / \{m - M(g, g^{-1})(m) \mid g : x \rightarrow y, m \in M(x, x)\}.$$

We call this the zero dimensional Hochschild homology of  $RG$  with coefficients in  $M$ ,

$$HH_0(RG, M).$$

Let  $P$  be a finitely generated projective  $RG$  module, and let  $M$  be an  $RG$ -bimodule. As in the classical situation, we want to define an Hattori-Stallings trace map as the composition of two maps. The first map is the inverse of the isomorphism

$$P^* \otimes_{RG} M \otimes_{RG} P \rightarrow \text{Hom}_{RG}(P, M \otimes_{RG} P)$$

given by applying Proposition 2.2.1 to  $N = M \otimes_{RG} P$ . The second map is given by the next lemma.

**Lemma 2.2.6.** *There is an  $R$ -module homomorphism*

$$P^* \otimes_{RG} M \otimes_{RG} P \rightarrow M/[RG, M]$$

defined by

$$(\tau, m, p) \mapsto M(\tau_z(p), \text{id}_z)(m),$$

where  $\tau \in P^*(x)$ ,  $m \in M(x, z)$  and  $p \in P(z)$ .

*Proof.* We need to check that this map is well defined. Considering the definition of  $P^* \otimes_{RG} M \otimes_{RG} P$ , we need to show that an element of the form  $\eta f \otimes m \otimes p - \eta \otimes M(f, \text{id}_z)(m) \otimes p$  maps to zero. But, this is clear.  $\square$

**Definition 2.2.7.** *The Hattori-Stallings trace, denoted  $\text{tr}$ , is the composition*

$$\begin{array}{ccc} \text{Hom}_{RG}(P, M \otimes_{RG} P) & \xleftarrow{\cong} & P^* \otimes_{RG} M \otimes_{RG} P & \longrightarrow & M/[RG, M] \\ & & \searrow \text{tr} & & \parallel \\ & & & & HH_0(RG; M) \end{array}$$

### 2.2.2 The $\alpha$ -twisted Case

In this section we consider a special case of the Hattori-Stallings trace. Fix a functor  $\alpha : G \rightarrow G$ . We show that in the case when  $M$  is the  $RG$ -bimodule  ${}_{\alpha}RG$ , defined below, then the Hattori-Stallings trace is commutative. We also consider the trace when  $G$  is a connected groupoid.

**Definition 2.2.8.** Define an  $RG$ -bimodule  ${}_{\alpha}RG : G \times G^{\text{op}} \rightarrow \text{R-mod}$  by

$${}_{\alpha}RG(x, y) = RG(y, \alpha(x))$$

for  $x, y \in \text{Ob}(G)$ , and

$${}_{\alpha}RG(g, h) = \alpha(g) \circ (-) \circ h$$

for  $g$  a map in  $G$  and  $h$  a map in  $G^{\text{op}}$ . This is the  $RG$ -bimodule  $\overline{RG}$ , but with the left modules structure “twisted by  $\alpha$ .”

**Definition 2.2.9.** Let  $M$  and  $N$  be  $RG$ -modules. An  $\alpha$ -linear homomorphism  $M \rightarrow N$  is defined to be a natural transformation  $\eta : M \rightarrow N \circ \alpha$ .

**Definition 2.2.10.** Let  $M$  be an  $RG$ -module. The  $\alpha$ -dual of  $M$  is

$$M^{\alpha} = \text{Hom}_{RG}(M, {}_{\alpha}RG).$$

**Lemma 2.2.11.** Given left  $RG$ -modules  $P$  and  $Q$ , there is an isomorphism

$$\text{Hom}_{RG}(P, Q \circ \alpha) \cong \text{Hom}_{RG}(P, {}_{\alpha}RG \otimes_{RG} Q).$$

*Proof.* Define  $A : \text{Hom}_{RG}(P, Q \circ \alpha) \rightarrow \text{Hom}_{RG}(P, {}_{\alpha}RG \otimes_{RG} Q)$  by  $A(f) = \bar{f}$  where  $\bar{f} : P \rightarrow {}_{\alpha}RG \otimes_{RG} Q$  is the natural transformation given by  $\bar{f}_x(p) = \text{id}_{\alpha(x)} \otimes f_x(p)$ , where  $f \in \text{Hom}_{RG}(P, Q \circ \alpha)$ ,  $x \in \text{Ob}(G)$ , and  $p \in P(x)$ . We show that  $A$  is one-to-one and onto.

Suppose  $A(f) = A(g)$ . Then for all  $x \in \text{Ob}(G)$  and for all  $p \in P(x)$ , we have  $\text{id}_{\alpha(x)} \otimes f_x(p) = \text{id}_{\alpha(x)} \otimes g_x(p)$  in  $RG(-, \alpha(x)) \otimes_{RG} Q$ . But, this can happen only if  $f_x(p) = g_x(p)$  for all  $x$  and  $p$ . Hence,  $f = g$ .

Suppose  $g \in \text{Hom}_{RG}(P, {}_\alpha RG \otimes_{RG} Q)$ . Given  $x \in \text{Ob}(G)$  and  $p \in P(x)$ , then  $g_x(p) = \sum_i \sigma_i \otimes q_i$  for some collection of  $x_i \in \text{Ob}(G)$ ,  $\sigma_i : x_i \rightarrow \alpha(x)$ , and  $q_i \in Q(x_i)$ . This can be rewritten as  $g_x(p) = \text{id}_{\alpha(x)} \otimes \sum_i Q(\sigma_i)(q_i)$ . Note that naturality of  $f$  follows from naturality of  $g$ . Define  $f \in \text{Hom}_{RG}(P, Q \circ \alpha)$  by  $f_x(p) = \sum_i Q(\sigma_i)(q_i)$ . Then  $A(f) = g$ .  $\square$

Now, letting  $M = {}_\alpha RG$  and using Lemma 2.2.11, the Hattori-Stallings trace can be given by the composition

$$\begin{array}{ccc} \text{Hom}_{RG}(P, P \circ \alpha) & \xleftarrow{\cong} & P^\alpha \otimes_{RG} P \longrightarrow {}_\alpha RG / [RG, {}_\alpha RG] \\ & \searrow \text{tr} & \parallel \\ & & HH_0(RG; {}_\alpha RG) \end{array}$$

where  $P$  is a finitely generated projective left  $RG$ -module, and the isomorphism is given by the map

$$\phi_{P^\alpha, P} : P^\alpha \otimes_{RG} P \rightarrow \text{Hom}_{RG}(P, P \circ \alpha)$$

defined in (2.2.1).

**Proposition 2.2.12 (Commutativity).** *Let  $P$  and  $Q$  be finitely generated projective left  $RG$ -modules. If  $f \in \text{Hom}_{RG}(P, Q \circ \alpha)$  and  $g \in \text{Hom}_{RG}(Q, P)$ , then*

$$\text{tr}(f \circ g) = \text{tr}(g \circ f).$$

*Proof.* We will need the following maps.

Define

$$B : (P^\alpha \otimes_{RG} Q) \times (Q^* \otimes_{RG} P) \rightarrow P^\alpha \otimes_{RG} P$$

by  $B((\tau \otimes q) \times (\eta \otimes p)) = \tau \otimes P(\eta_x(q))(p)$ , where  $\tau \in P^\alpha(x)$ ,  $q \in Q(x)$ ,  $\eta \in Q^*(y)$ , and  $p \in P(y)$ .

Define

$$B' : (Q^* \otimes_{RG} P) \times (P^\alpha \otimes_{RG} Q) \rightarrow Q^\alpha \otimes Q$$

by  $B'((\eta \otimes p) \times (\tau \otimes q)) = (\alpha \circ \eta) \otimes Q(\tau_y(p))(q)$ , where  $\tau \in P^\alpha(x)$ ,  $q \in Q(x)$ ,  $\eta \in Q^*(y)$ , and  $p \in P(y)$ .

Define

$$T_P : P^\alpha \otimes P \rightarrow HH_0(RG, {}_\alpha RG)$$

by  $T_P(\tau \otimes p) = \tau_x(p)$ , where  $\tau \in P^\alpha(x)$  and  $p \in P(x)$ .

Define

$$T' : (P^\alpha \otimes_{RG} Q) \times (Q^* \otimes_{RG} P) \rightarrow HH_0(RG, {}_\alpha RG)$$

by  $T'_{P,Q}((\tau \otimes q) \times (\eta \otimes p)) = \alpha(\eta_x(q)) \circ \tau_y(p)$ , where  $\tau \in P^\alpha(x)$ ,  $q \in Q(x)$ ,  $\eta \in Q^*(y)$ , and  $p \in P(y)$

We now consider three diagrams. To see that each of them commutes is an exercise in applying the definitions.

The first diagram we will consider is

$$\begin{array}{ccc} \text{Hom}_{RG}(P, Q \circ \alpha) \times \text{Hom}_{RG}(Q, P) & \longleftarrow & (P^\alpha \otimes_{RG} Q) \times (Q^* \otimes_{RG} P) \\ \downarrow & & \downarrow B \\ \text{Hom}_{RG}(P, P \circ \alpha) & \xleftarrow{\phi_{P^\alpha, P}} & P^\alpha \otimes_{RG} P \end{array} \quad (2.2.4)$$

where the unlabeled vertical map is given by  $(f, g) \mapsto g \circ f$  and the unlabeled horizontal map is  $\phi_{P^\alpha, Q} \times \phi_{Q, P}$ .

Transposing the products in the top line of diagram (2.2.4) gives us the second diagram.

$$\begin{array}{ccc} \text{Hom}_{RG}(Q, P) \times \text{Hom}_{RG}(P, Q \circ \alpha) & \longleftarrow & (Q^* \otimes_{RG} P) \times (P^\alpha \otimes_{RG} Q) \\ \downarrow & & \downarrow B' \\ \text{Hom}_{RG}(Q, Q \circ \alpha) & \xleftarrow{\phi_{Q^\alpha, Q}} & Q^\alpha \otimes_{RG} Q \end{array}$$

where the unlabeled vertical map is  $(g, f) \mapsto f \circ g$ .

The third diagram is

$$\begin{array}{ccc}
& Q^\alpha \otimes_{RG} Q & \\
& \uparrow B' & \\
(Q^* \otimes_{RG} P) \times (P^\alpha \otimes_{RG} Q) & \xrightarrow{T_Q} & \\
& \uparrow & \\
(P^\alpha \otimes_{RG} Q) \times (Q^* \otimes_{RG} P) & \xrightarrow{T'} & HH_0(RG, {}_\alpha RG) \\
& \downarrow B & \\
& P^\alpha \otimes_{RG} P & \\
& \nearrow T_P & 
\end{array}$$

where the unlabeled arrow is transposition.

Commutativity of the three diagrams together proves commutativity of the Hattori-Stallings trace.  $\square$

Consider the following setup. Let  $G$  be a connected groupoid, i.e., one for which there exists a map between any two objects. Let  $\alpha : G \rightarrow G$  be a functor and let  $P$  be a finitely generated projective left  $RG$ -module. Choose an object  $*$  of  $G$  and choose a map  $\tau : * \rightarrow \alpha(*)$  in  $G$ . Also, let  $\beta \in \text{Hom}_{RG}(P, P \circ \alpha)$ .

Let  $RG(*)$  be the subcategory of  $RG$  with a single object,  $*$ , and with maps given by the maps in  $RG$  from  $*$  to  $*$ . Then the inclusion  $RG(*) \rightarrow RG$  is an equivalence of categories. The proof amounts to choosing a map  $\mu_x : * \rightarrow x$  for each  $x \in \text{Ob}(G)$ . For each  $x$ , we fix a choice of  $\mu_x$ .

The functor  $\alpha$  induces a functor  $\alpha_\tau : RG(*) \rightarrow RG(*)$  which maps the object  $*$  to itself. If  $g : * \rightarrow *$ , let  $\alpha_\tau(g) = \tau^{-1} \circ \alpha(g) \circ \tau$ . In the obvious way, the  $RG$ -module  $P$  induces a finitely generated projective left  $RG(*)$ -module, denoted  $P(*)$ . The natural transformation  $\beta$  induces a natural transformation  $\beta_\tau = P(\tau^{-1}) \circ \beta_* \in \text{Hom}_{RG(*)}(P(*), P(*) \circ \alpha_\tau)$ .

We also have the  $RG$ -bimodules  $\overline{RG}$  and  ${}_\alpha RG$ , and the  $RG(*)$ -bimodule  ${}_{\alpha_\tau} RG(*)$ , as defined in Example 2.1.19 and Definition 2.2.8.

**Lemma 2.2.13.** *There is an isomorphism of groups*

$$A : HH_0(RG(*), \alpha_\tau RG(*)) \rightarrow HH_0(RG, \alpha RG)$$

given by  $A(m) = \tau \circ m$  for  $m \in HH_0(RG(*), \alpha_\tau RG(*))$ .

*Proof.*  $A$  is well defined. For, if  $g : * \rightarrow *$ , then  $m = \alpha_\tau(g) \circ m \circ g^{-1}$  and  $A(\alpha_\tau(g) \circ m \circ g^{-1}) = \tau \circ \tau^{-1} \circ \alpha(g) \circ m \circ g^{-1} = \tau \circ m$  in  $HH_0(RG, \alpha RG)$ .

$A$  is a monomorphism. If  $m$  and  $m'$  are elements of  $HH_0(RG(*), \alpha_\tau RG(*))$  such that  $A(m) = A(m')$ , then there exists a map  $g : * \rightarrow *$  such that  $\tau \circ m' = \alpha(g) \circ \tau \circ m \circ g^{-1}$ . Composing with  $\tau^{-1}$  on both sides gives

$$m' = \alpha_\tau(g) \circ m \circ g^{-1} = m$$

in  $HH_0(RG(*), \alpha_\tau RG(*))$ .

$A$  is an epimorphism. Suppose  $m : x \rightarrow \alpha(x) \in HH_0(RG, \alpha RG)$  for some  $x \in \text{Ob}(G)$ . Then  $A(\tau^{-1} \circ \alpha(\mu_x^{-1}) \circ m \circ \mu_x) = \alpha(\mu_x^{-1}) \circ m \circ \mu_x = m$  in  $HH_0(RG, \alpha RG)$ . □

**Proposition 2.2.14.** *The Hattori-Stallings trace of  $\beta_\tau$  and  $\beta$  are equivalent, i.e.,*

$$A(\text{tr}(\beta_\tau)) = \text{tr}(\beta).$$

*Proof.* Given  $\eta \in P^\alpha(x)$  for some  $x \in \text{Ob}(G)$ , define  $\bar{\eta} : P(*) \rightarrow RG(*, *) \in P(*)^{\alpha_\tau}$  by  $\bar{\eta}(p) = \tau^{-1} \circ \eta_*(p) \circ \mu_x$ , where  $p \in P(*)$ . Naturality of  $\bar{\eta}$  follows from naturality of  $\eta$ . This gives us a map  $P^\alpha \rightarrow P(*)^{\alpha_\tau}$ .

Define a map  $B : P^\alpha \otimes_{RG} P \rightarrow P(*)^{\alpha_\tau} \otimes RG(*)P(*)$  by  $\eta \otimes p \mapsto \bar{\eta} \otimes P(\mu_x^{-1})(p)$ , where  $\eta \in P^\alpha(x)$  and  $p \in P(x)$  for some  $x \in \text{Ob}(G)$ . It is an exercise to show that  $B$  is well-defined. Also, define a map  $C : \text{Hom}_{RG}(P, P \circ \alpha) \rightarrow \text{Hom}_{RG(*)}(P(*), P(*) \circ \alpha_\tau)$  by  $\gamma \mapsto \gamma_\tau = P(\tau^{-1}) \circ \gamma_*$  for  $\gamma \in \text{Hom}_{RG}(P, P \circ \alpha)$ .

Commutativity of the following two diagrams is an application of the definitions

and implies that  $A(\text{tr}(\beta_\tau)) = \text{tr}(\beta)$ .

$$\begin{array}{ccc}
\text{Hom}_{RG(*)}(P(*), P(*) \circ \alpha_\tau) & \xleftarrow{\phi_{P(*), P(*)}^{\alpha_\tau}} & P(*)^{\alpha_\tau} \otimes_{RG(*)} P(*) \\
\uparrow C & & \uparrow B \\
\text{Hom}_{RG}(P, P \circ \alpha) & \xleftarrow{\phi_{P, P}^\alpha} & P^\alpha \otimes_{RG} P \\
\\ 
P(*)^{\alpha_\tau} \otimes_{RG(*)} P(*) & \xrightarrow{T} & HH_0(RG(*), \alpha_\tau RG(*)) \\
\uparrow B & & \downarrow A \\
P^\alpha \otimes_{RG} P & \xrightarrow{T} & HH_0(RG, \alpha RG)
\end{array}$$

□

Notice that  $A(\text{tr}(\beta_\tau))$  is independent of the choices of maps  $\mu_x$ .

### 2.2.3 The Trace for Chain Complexes

We begin with the general case.

**Definition 2.2.15.** *Let  $P_\bullet$  be a finitely generated projective  $RG$ -chain complex. Define the Hattori-Stallings trace*

$$\text{Tr} : \text{Hom}_{\mathcal{P}(RG)}(P_\bullet, P_\bullet) \rightarrow HH_0(RG, RG)$$

by

$$f \mapsto \sum_i (-1)^i \text{tr}(f_i),$$

where  $f : P_\bullet \rightarrow P_\bullet$  is given by the family  $\{f_i \in \text{Hom}_{RG}(P_i, P_i)\}$ .

Commutativity follows from commutativity of the Hattori-Stallings trace for  $RG$ -modules, taking  $\alpha = \text{id}_G$ .

**Proposition 2.2.16 (Commutativity).** *Let  $P_\bullet$  and  $Q_\bullet$  be finitely generated projective  $RG$ -chain complexes, and let  $f : P_\bullet \rightarrow Q_\bullet$  and  $g : Q_\bullet \rightarrow P_\bullet$  be chain maps.*

*Then*

$$\text{Tr}(f \circ g) = \text{Tr}(g \circ f).$$

The Hattori-Stallings trace is also invariant up to chain homotopy.

**Proposition 2.2.17.** *Let  $P_\bullet$  be a finitely generated projective  $RG$ -chain complex. If  $f : P_\bullet \rightarrow P_\bullet$  and  $g : P_\bullet \rightarrow P_\bullet$  are chain homotopic, then  $\text{Tr}(f) = \text{Tr}(g)$ .*

*Proof.* Let  $\{s_n : P_n \rightarrow P_{n+1}\}$  be a chain homotopy from  $f$  to  $g$ . Then

$$\begin{aligned} \text{Tr}(f) - \text{Tr}(g) &= \sum_i (-1)^i \text{tr}(f_i) - \sum_i (-1)^i \text{tr}(g_i) \\ &= \sum_i (-1)^i \text{tr}(f_i - g_i) \\ &= \sum_i (-1)^i \text{tr}(d_{i+1} \circ s_i + s_{i-1} \circ d_i) \\ &= \sum_i (-1)^i [\text{tr}(s_i \circ d_{i+1}) + \text{tr}(s_{i-1} \circ d_i)]. \end{aligned}$$

The last equality comes from applying commutativity. Rearranging the terms in the last sum gives  $\text{Tr}(f) - \text{Tr}(g) = 0$ .  $\square$

Now suppose that  $C_\bullet$  is an  $RG$ -chain complex which is chain homotopy equivalent to a finitely generated projective  $RG$ -chain complex. Suppose further that  $\phi : C_\bullet \rightarrow C_\bullet$  is a chain map. Choose a finitely generated projective  $RG$ -chain complex  $P_\bullet$ , choose a chain homotopy equivalence  $f : C_\bullet \rightarrow P_\bullet$ , and choose a lift  $\psi : P_\bullet \rightarrow P_\bullet$  of  $\phi$ . We get the diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\psi} & P_\bullet \\ f \uparrow & & \uparrow f \\ C_\bullet & \xrightarrow{\phi} & C_\bullet \end{array}$$

which commutes up to chain homotopy.

**Definition 2.2.18.** *The Hattori-Stallings trace of  $\phi : C_\bullet \rightarrow C_\bullet$  is defined to be the trace of  $\psi : P_\bullet \rightarrow P_\bullet$ .*

$$\text{Tr}(\phi) = \text{Tr}(\psi).$$

To see that  $\text{Tr}$  is well-defined, we must show that it is independent of the choices we made. First, suppose that  $\phi'$  is another lift of  $\phi$ . Then  $\psi \sim_{ch} f \circ \phi \circ f^{-1} \sim_{ch} \psi'$  and by Proposition 2.2.17,  $\text{Tr}(\psi) = \text{Tr}(\psi')$ . Second, suppose that  $Q_\bullet$  is another finitely generated projective  $RG$ -chain complex and  $g : C_\bullet \rightarrow Q_\bullet$  is a chain homotopy equivalence. Then

$$\begin{aligned} \text{Tr}(g \circ \phi \circ g^{-1}) &= \text{Tr}(g \circ f \circ f^{-1} \circ \phi \circ f^{-1} \circ f \circ g^{-1}) \\ &= \text{Tr}(f \circ g^{-1} \circ g \circ f^{-1} \circ f \circ \phi \circ f^{-1}) \\ &= \text{Tr}(f \circ \phi \circ f^{-1}). \end{aligned}$$

We end this section by considering the special case of  $\alpha$ -linear chain maps, where  $\alpha : G \rightarrow G$  is a functor. We state the definitions and results without proof as the proofs are analogous to those above.

**Definition 2.2.19.** *A chain map  $f : C_\bullet \rightarrow D_\bullet$  of  $RG$ -chain complexes is called  $\alpha$ -linear if for each  $n$ ,  $f_n$  is  $\alpha$ -linear. The collection of  $\alpha$ -linear chain maps from  $C_\bullet$  to  $D_\bullet$  is denoted  $\text{Hom}_{\text{Ch}(RG)}(C_\bullet, D_\bullet \circ \alpha)$ .*

**Definition 2.2.20.** *Let  $P_\bullet$  be a finitely generated projective  $RG$ -module. The Hattori-Stallings trace of a chain map  $f : P_\bullet \rightarrow P_\bullet$  is*

$$\text{Tr}(f) = \sum_i (-1)^i \text{tr}(f_i) \in HH_0(RG, {}_\alpha RG).$$

**Proposition 2.2.21 (Commutativity).** *Let  $P_\bullet$  and  $Q_\bullet$  be finitely generated projective chain complexes. If  $f \in \text{Hom}_{\mathcal{P}(RG)}(P_\bullet, Q_\bullet \circ \alpha)$  and  $g \in \text{Hom}_{\mathcal{P}(RG)}(Q_\bullet, P_\bullet)$  then*

$$\text{Tr}(f \circ g) = \text{Tr}(g \circ f).$$

**Proposition 2.2.22.** *Let  $P_\bullet$  be a finitely generated projective  $RG$ -chain complex. Suppose  $f : P_\bullet \rightarrow P_\bullet$  and  $g : P_\bullet \rightarrow P_\bullet$  are chain homotopic  $\alpha$ -linear chain maps. Then*

$$\text{Tr}(f) = \text{Tr}(g).$$

Suppose  $C_\bullet$  is an  $RG$ -chain complex which is chain homotopy equivalent to a finitely generated projective  $RG$ -chain complex, and suppose  $\phi : C_\bullet \rightarrow C_\bullet$  is an  $\alpha$ -linear chain map. Choose a finitely generated projective  $RG$ -chain complex  $P_\bullet$ , a chain homotopy equivalence  $f : C_\bullet \rightarrow P_\bullet$ , and a lift  $\psi$  of  $\phi$ . Then we can define the Hattori-Stallings trace of  $\phi$  by

$$\mathrm{Tr}(\phi) = \mathrm{Tr}(\psi).$$

## CHAPTER 3

### BASE POINT FREE LEFSCHETZ-NIELSEN INVARIANTS

In this chapter, we present our base point free refinements of the classical geometric and algebraic Lefschetz-Nielsen invariants. We begin by defining the fundamental groupoid, and describing the way in which we think of the universal cover.

#### 3.1 Fundamental Groupoid

An important example of a groupoid is the fundamental groupoid. Let  $X$  be a topological space.

**Definition 3.1.1.** *The fundamental groupoid,  $\Pi X$ , is the category whose objects are the points in  $X$ , whose maps are the homotopy classes rel endpoints of paths in  $X$ . Composition is given by concatenation of paths. To be precise, if  $f$  and  $g$  are paths in  $X$  such that  $f(1) = g(0)$ , then*

$$[g] \circ [f] = [f \cdot g].$$

*For each morphism, an inverse is given by traversing a representative path backwards.*

This groupoid deserves to be called the fundamental groupoid since for a given point  $x \in X$ , the subcategory of  $\Pi X$  generated by  $x$  is  $\pi_1(X, x)$ . The subcategory generated by  $x$  is the category with one object,  $x$ , and whose morphism set is  $\Pi X(x, x)$ . In a sense, then, the fundamental groupoid is a way of encoding in one object the fundamental groups with all possible choices of base point.

Let  $f : X \rightarrow X$  be a continuous map. Then  $f$  induces a functor  $\Pi f : \Pi X \rightarrow \Pi X$  given by  $\Pi f(x) = f(x)$  and  $\Pi f(g) = f \circ g$  where  $x \in X$  and  $g$  is a path in  $X$ .

### 3.2 Universal Cover

Let  $X$  be a path connected, locally path connected, semilocally simply-connected space. For each  $x \in X$ , one can describe the universal cover [16, page 64] of  $X$  as the space

$$\tilde{X}_x = (X, x)^{(I,0)} / \sim$$

where  $I$  is the closed unit interval and  $\sim$  is the equivalence relation given by homotopy rel endpoints. The set  $(X, x)^{(I,0)}$  is given the compact-open topology, and  $\tilde{X}_x$  is given the quotient topology. The projection map  $p : \tilde{X}_x \rightarrow X$  is given by  $p(\sigma) = \sigma(1)$ .

Recall  $\Pi X$ , the fundamental groupoid of  $X$ . Let  $\text{Top}$  be the category of topological spaces.

**Definition 3.2.1.** *The universal cover functor*

$$U : \Pi X \rightarrow \text{Top}$$

is defined by  $U(x) = \tilde{X}_x$  for  $x \in \text{Ob}(\Pi X)$ . For  $g : x \rightarrow y$  a map in  $\Pi X$ , define  $U(g) : \tilde{X}_x \rightarrow \tilde{X}_y$  by  $U(g)[\sigma] = [g^{-1} \cdot \sigma]$ , where  $[\sigma] \in \tilde{X}_x$ .

### 3.3 The Geometric Invariant

Fix a compact, path connected  $n$ -dimensional manifold  $X$  and a continuous endomorphism  $f : X \rightarrow X$  such that  $\text{Fix}(f)$  is finite.

Let  $\Pi$  be the fundamental groupoid of  $X$ . The map  $f$  induces a functor  $\varphi = \Pi f : \Pi \rightarrow \Pi$  defined by  $\varphi(x) = f(x)$ , where  $x \in \text{Ob}(\Pi)$ . For  $g : x \rightarrow y$  a map in  $\Pi$  let  $\varphi(g) = f \circ g$ .

Let  $\text{Fix}(\varphi)$  be the subcategory of  $\Pi$  whose set of objects is  $\text{Fix}(f)$ , and whose maps are the maps  $g : x \rightarrow y$  in  $\Pi$  ( $x, y \in \text{Fix}(f)$ ) such that  $f \circ g = g$ . The category  $\text{Fix}(\varphi)$  decomposes into a finite number of connected components; denote them by  $\mathbb{F}_1, \dots, \mathbb{F}_r$ .

Define an  $\mathbb{Z}\Pi$ -bimodule  ${}_{\varphi}\mathbb{Z}\Pi : \Pi \times \Pi^{\text{op}} \rightarrow \mathbf{Ab}$  given by  $(x, y) \mapsto \mathbb{Z}\Pi(y, \varphi(x))$ , where  $x, y \in \text{Ob}(\Pi)$ . For  $g : x \rightarrow x'$  a map in  $\Pi$  and  $h : y \rightarrow y'$  a map in  $\Pi^{\text{op}}$ , let  ${}_{\varphi}\mathbb{Z}\Pi(g, h) = \varphi(g) \circ (-) \circ h$ . By definition,

$$\begin{aligned} HH_0(\mathbb{Z}\Pi, {}_{\varphi}\mathbb{Z}\Pi) &= {}_{\varphi}\mathbb{Z}\Pi / [{}_{\varphi}\mathbb{Z}\Pi] \\ &= \bigoplus_{x \in \text{Ob}(\Pi)} \mathbb{Z}\Pi(x, \varphi(x)) / Q \end{aligned}$$

where  $Q$  is generated by elements of the form  $\sigma - \varphi(g) \circ \sigma \circ g^{-1}$  for maps  $\sigma : x \rightarrow \varphi(x)$  and  $g : x \rightarrow y$  in  $\Pi$ .

Define

$$\Phi : \{\mathbb{F}_k\}_{k=1}^r \rightarrow HH_0(\mathbb{Z}\Pi, {}_{\varphi}\mathbb{Z}\Pi)$$

by choosing an object  $x$  in  $\mathbb{F}_k$  and mapping  $\mathbb{F}_k$  to  $\text{id}_x : x \rightarrow x = \varphi(x)$ . One can check that this is a well-defined injection.

Also, let

$$i(f, \mathbb{F}_k) = \sum_{x \in \text{Ob}(\mathbb{F}_k)} i(f, x) \in \mathbb{Z},$$

where  $i(f, x)$  is the fixed point index.

**Definition 3.3.1.** *The geometric Lefschetz invariant of  $f : X \rightarrow X$  is*

$$L^{geo}(f) = \sum_k i(f, \mathbb{F}_k) \Phi(\mathbb{F}_k) \in HH_0(\mathbb{Z}\Pi, {}_{\varphi}\mathbb{Z}\Pi).$$

**Proposition 3.3.2.** *The classical geometric Lefschetz invariant and the base point free geometric Lefschetz invariant correspond under an isomorphism*

$$A : \mathbb{Z}\pi_{\phi} \rightarrow HH_0(\mathbb{Z}\Pi, {}_{\varphi}\mathbb{Z}\Pi).$$

The isomorphism  $A$  is not canonical; it depends on choosing a path from  $*$  to  $f(*)$ . On the other hand,  $HH_0(\mathbb{Z}\Pi, {}_\varphi\mathbb{Z}\Pi)$  is canonical.

*Proof.* Recall that in the classical definition, we have chosen a base point  $*$  and a base path  $\tau$ . The fundamental group  $\pi_1(X, *)$  is denoted by  $\pi$ , the map on  $\pi$  induced by  $f : X \rightarrow X$  and the base path  $\tau$  is denoted by  $\phi$ , and the injection  $\{F_i\}_{i=1}^s \rightarrow \pi_\phi$  is denoted by  $\Phi$ .

Step 1: After appropriate reordering of the fixed point classes  $F_1, \dots, F_s$ ,  $s = r$  and  $F_i = \text{Ob}(\mathbb{F}_i)$ . This can be seen as follows. If  $x$  and  $y$  are equivalent in  $\text{Fix}(f)$ , then there exists a path  $\nu$  from  $x$  to  $y$  in  $X$  such that  $\nu \cdot (f \circ \nu)^{-1} \simeq *$ . But this is equivalent to saying that  $\nu$  is a map in  $\text{Fix}(\varphi)$  from  $x$  to  $y$ , and hence that  $x$  and  $y$  are in the same connected component of  $\text{Fix}(\varphi)$ .

Step 2: Define an isomorphism of abelian groups

$$A : \mathbb{Z}\pi_\phi \rightarrow HH_0(\mathbb{Z}G, {}_\varphi\mathbb{Z}G)$$

by  $A([\omega]) = \omega \cdot \tau = \tau \circ \omega$ , where  $[\omega] \in \pi$ .

To see that  $A$  is well defined, suppose that  $[\omega]$  and  $[\omega_1]$  are equivalent in  $\mathbb{Z}\pi_\phi$ . By definition, there exists  $g \in \pi$  such that  $\omega_1 = g \cdot \omega \cdot \tau \cdot (f \circ g)^{-1} \cdot \tau^{-1}$ . Hence,  $\tau \circ \omega_1 = \varphi(g^{-1}) \circ \tau \circ \omega \circ g = \tau \circ g$  in  $HH_0(\mathbb{Z}G, {}_\varphi\mathbb{Z}G)$ , and  $A$  is well-defined.

To see that  $A$  is an epimorphism, suppose that  $\sigma : x \rightarrow \varphi(x) \in HH_0(\mathbb{Z}G, {}_\varphi\mathbb{Z}G)$ . Choose a path  $\mu$  in  $X$  from  $*$  to  $x$ , i.e., a map  $\mu : * \rightarrow x$  in  $G$ . Then  $\sigma = \varphi(\mu^{-1}) \circ \sigma \circ \mu$  in  $HH_0(\mathbb{Z}G, {}_\varphi\mathbb{Z}G)$ , and  $\mu \cdot \sigma \cdot (f \circ \mu)^{-1} \cdot \tau^{-1}$  gives an element in  $\pi$  which is mapped to  $\sigma$  by  $A$ .

The last thing to check is that  $A$  is a monomorphism. Suppose  $[\omega]$  and  $[\omega_1]$  are elements of  $\pi$  such that  $\tau \circ \omega = \tau \circ \omega_1$ . Then there exists  $g \in \text{Ob}(G)$  such that  $\tau \circ \omega_1 = \varphi(g^{-1}) \circ \tau \circ \omega \circ g$ . It follows that  $\omega_1 = g \cdot \omega \cdot \tau \cdot (f \circ g)^{-1} \cdot \tau^{-1}$  and hence that  $[\omega_1]$  is equivalent to  $[\omega]$  in  $\mathbb{Z}\pi_\phi$ .

Step 3: Let  $F$  be a fixed point class, and  $\mathbb{F}$  the corresponding connected component of  $\text{Fix}(\varphi)$ . For any choice of  $x \in F$  and path  $\mu$  from  $*$  to  $x$ , we have that

$$A(\Phi(F)) = A(\mu \cdot (f \circ \mu)^{-1} \cdot \tau^{-1}) = \varphi(\mu^{-1}) \circ \mu = \text{id}_x \text{ in } HH_0(\mathbb{Z}G, \varphi\mathbb{Z}G).$$

Therefore,

$$L^{geo}(f, *, \tau) = \sum_{k=1}^s i(f, F_k) \Phi(F_k) \in \mathbb{Z}\pi_\phi$$

maps equivalent to

$$L^{geo}(f) = \sum_{k=1}^r i(f, \mathbb{F}_k) \Phi(\mathbb{F}_k) \in HH_0(\mathbb{Z}G, \varphi\mathbb{Z}G).$$

□

### 3.4 The Algebraic Invariant

Let  $X$  be a finite CW complex and  $f : X \rightarrow X$  a continuous map. Let  $\Pi = \Pi X$  be the fundamental groupoid of  $X$  and let  $\varphi : \Pi \rightarrow \Pi$  be the functor induced by  $f$ , as above.

Recall the universal cover functor  $U : \Pi \rightarrow \text{Top}$ . The map  $f$  induces a natural transformation  $\tilde{f} : U \rightarrow U \circ \varphi$ . Given an object  $x$  in  $\Pi$ ,  $\tilde{f}_x : \tilde{X}_x \rightarrow \tilde{X}_{f(x)}$  is defined by  $[\alpha] \mapsto [f \circ \alpha]$ , where  $[\alpha] \in \tilde{X}_x$ . One can check naturality.

There is a functor  $S : \text{Top} \rightarrow \text{Ch}(\mathbb{Z})$  given by taking the singular chain complex of a space. If  $f : X \rightarrow Y$  is a continuous map, then  $S(f) : S(X) \rightarrow S(Y)$  is given by  $\sigma \mapsto f \circ \sigma$ , where  $\sigma : \Delta^n \rightarrow X$ . Here,  $\Delta^n$  is the standard  $n$ -simplex.

Let  $C_\bullet$  be the  $\mathbb{Z}\Pi$ -chain complex given by the composition

$$\Pi \xrightarrow{U} \text{Top} \xrightarrow{S} \text{Ch}$$

The map  $f$  induces a natural transformation  $\tilde{f}_* : SU \rightarrow SU\varphi$ . Given an object  $x$  in  $\Pi$ , let  $\tilde{f}_*(x) : S(\tilde{X}_x) \rightarrow S(\tilde{X}_{f(x)})$  be given by  $\sigma \mapsto \tilde{f}_x \circ \sigma$ , where  $\sigma : \Delta^n \rightarrow \tilde{X}_x$ . Naturality of  $\tilde{f}_*$  follows from naturality of  $\tilde{f}$ . Hence,  $\tilde{f}_*$  is a  $\varphi$ -linear chain map  $C_\bullet \rightarrow C_\bullet$ . As usual,  $\tilde{f}_*$  is given by a family of  $\varphi$ -linear natural transformations  $\tilde{f}_n : C_n \rightarrow C_n$ .

The singular chain complex of a finite CW complex is chain homotopy equivalent to a finitely generated projective  $\mathbb{Z}\Pi$  chain complex. Hence, the Hattori-Stallings trace of  $\tilde{f}_*$  is defined, and we can define the algebraic Lefschetz invariant as follows.

**Definition 3.4.1.** *The algebraic Lefschetz invariant of  $f : X \rightarrow X$  is*

$$L^{alg}(f) = \text{Tr}(\tilde{f}_*) = \sum_{k \geq 0} (-1)^k \text{tr}(\tilde{f}_k) \in HH_0(\mathbb{Z}\Pi, {}_{\varphi}\mathbb{Z}\Pi).$$

As an immediate corollary of Proposition 2.2.14 we get the following.

**Proposition 3.4.2.** *The classical algebraic Lefschetz invariant and the base point free algebraic Lefschetz invariant correspond under the isomorphism*

$$A : \mathbb{Z}\pi_{\phi} \rightarrow HH_0(\mathbb{Z}\Pi, {}_{\varphi}\mathbb{Z}\Pi).$$

## CHAPTER 4

### THE FAMILY VERSION

In this chapter we discuss the family (parametrized) version of Lefschetz-Nielsen fixed point theory. This is a “work in progress.” We begin with the bordism theoretic definition of the invariant and the Hopf-Lefschetz theorem, which are complete. We go on to describe our long term goals. More specifically, we describe the properties of the Lefschetz-Nielsen invariant for a single manifold and a single endomorphism which should have analogues in the family version. We finish by outlining our program for constructing an algebraic definition of  $L^{bord}$  in the parametrized situation.

#### 4.1 The Bordism Invariant

Let  $p : E^{n+k} \rightarrow B^n$  be a smooth fiber bundle with compact fibers and  $n > 2$ . Assume that  $B$  is a closed manifold. Let  $f : E \rightarrow E$  be a map such that  $p \circ f = p$  (i.e., a map of fiber bundles over  $B$ ).

The fibered product  $E \times_B E$  is a fiber bundle over  $B$  with the fiber over  $b \in B$  given by  $F_b \times F_b$ , where  $F_b$  is the fiber of  $p$  over  $b$ . The dimension of  $E \times_B E$  is  $n + 2k$ . Let  $\Delta : E \rightarrow E \times_B E$  be the diagonal map given by  $\Delta(x) = (x, x)$ ; note that  $\Delta$  is a map of fiber bundles over  $B$ . Let  $\Delta_f : E \rightarrow E \times_B E$  be the map of fiber bundles given by  $\Delta_f(x) = (x, f(x))$ .

Take the homotopy pullback

$$\begin{array}{ccc} \mathcal{L}_f^B E & \longrightarrow & E \\ \downarrow & & \downarrow \Delta \\ E & \xrightarrow{\Delta_f} & E \times_B E \end{array}$$

By Lemma 1.4.6,  $\Delta_f$  is fiber homotopic to a map  $\Gamma : E \rightarrow E \times_B E$  such that  $\Gamma$  is transversal to  $\Delta$ . Take the transversal pullback

$$\begin{array}{ccc} \Gamma \overline{\cap} \Delta & \longrightarrow & E \\ \downarrow & & \downarrow \Delta \\ E & \xrightarrow{\Gamma} & E \times_B E \end{array}$$

By universality of the homotopy pullback, there is a map  $\Gamma \overline{\cap} \Delta \rightarrow \mathcal{L}_f^B E$ . By transversality,  $\nu(\Gamma \overline{\cap} \Delta)$  is stably isomorphic to  $\nu(B)$ . Thus, we get that

$$[\Gamma \overline{\cap} \Delta] \in \Omega_n^{fr}(\mathcal{L}_f^B E; \nu(B)).$$

**Definition 4.1.1.** *The Lefschetz invariant is*

$$L^{bord}(f) = [\Gamma \overline{\cap} \Delta] \in \Omega_n^{fr}(\mathcal{L}_f^B E; \nu(B)).$$

## 4.2 The Hopf-Lefschetz theorem

**Theorem 4.2.1 (Hopf-Lefschetz Theorem with Converse).** *If  $f$  is fiber homotopic to a map with no fixed points, then  $L^{bord}(f) = 0$ . If  $k > n+2$  and  $L^{bord}(f) = 0$ , then  $f$  is fiber homotopic to a map with no fixed points.*

The proof relies heavily on the following results: a theorem of Hatcher and Quinn [17, Theorem 4.2] and a lemma. In the theorem,  $E_M$ ,  $E_Q$  and  $E_P$  are smooth fiber bundles over a compact manifold  $R$  with fibers  $M$ ,  $Q$  and  $P$  respectively. Recall that  $E(i_P, i_Q)$  is the homotopy pullback.

**Theorem 4.2.2.** Let  $P^p, Q^q$  be closed manifolds,  $M^m$  a manifold, and  $R^k$  a compact manifold. Assume that there is a diagram

$$\begin{array}{ccccc}
 Q & \longrightarrow & M & \longleftarrow & P \\
 \downarrow & & \downarrow & & \downarrow \\
 E_Q & \xrightarrow{i_Q} & E_M & \xleftarrow{i_P} & E_P \\
 & \searrow & \downarrow & \swarrow & \\
 & & R & & 
 \end{array}$$

such that  $i_P$  and  $i_Q$  are bundle maps which are immersions (embeddings) in each fiber. Suppose  $(N, h, \bar{h})$  and  $(i_P \bar{\cap} i_Q, i, \bar{i})$  both represent elements of

$$\Omega_{p+q+k-m}^{fr}(E(i_P, i_Q); \nu(E_P, E_M) \oplus \nu(E_Q, E_M) \oplus \nu(E_M)).$$

Assume further that  $m > p + (q + k)/2 + 1$ ,  $m > q + (p + k)/2 + 1$  and  $p, q > 0$ . Then  $i_Q$  is fiber regularly homotopic (isotopic) to a fiber immersion (embedding)  $i'_Q$  with  $i_P \bar{\cap} i'_Q = N$  if and only if

$$[i_P \bar{\cap} i_Q] = [N] \in \Omega_{p+q+k-m}^{fr}(E(i_P, i_Q); \nu(E_P, E_M) \oplus \nu(E_Q, E_M) \oplus \nu(E_M)).$$

**Lemma 4.2.3.** If  $q : A \rightarrow R$  is a fibration and  $X$  is a space, then

$$\text{Map}(X, A) \rightarrow \text{Map}(X, R)$$

is a fibration and the fiber over  $f : X \rightarrow R$  is  $\text{Map}_R(X, A)$ , where  $X$  is a space over  $R$  via  $f$  and  $A$  is a space over  $R$  via  $q$ .

*Proof of Theorem 4.2.1.* Assume  $f$  is homotopic to  $f'$  such that  $\text{Fix}(f') = \emptyset$ . Then  $\Delta_{f'} \cap \Delta$  is zero bordant, and so  $\Delta_{f'} \bar{\cap} \Delta$ . By definition, then,

$$L^{\text{bord}}(f) = [\emptyset] = 0 \in \Omega_n^{fr}(\mathcal{L}_f^B E, \nu(B)).$$

For the converse, assume  $L^{\text{bord}}(f) = 0$ . Note that  $k > n + 2$  gives the dimension

requirements, so that we can apply Theorem 4.2.2 to the following diagram:

$$\begin{array}{ccccc}
F_b & \longrightarrow & F_b \times F_b & \longleftarrow & F_b \\
\downarrow & & \downarrow & & \downarrow \\
E & \xrightarrow{\Gamma} & E \times_B E & \xleftarrow{\Delta} & E \\
& & \downarrow & & \swarrow \\
& & B & & 
\end{array}$$

The theorem gives a fiber homotopy (over  $B$ ) of  $\Gamma$  to  $\Gamma' : E \rightarrow E \times_B E$  such that  $\text{im}(\Gamma') \cap \text{im}(\Delta) = \emptyset$ .

It remains to show that  $\Gamma'$  is fiber homotopic to a map of the form  $\Delta_{f'}$  such that  $\text{im}(\Delta_{f'}) \cap \text{im}(\Delta) = \emptyset$ . This is done by considering the following diagram:

$$\begin{array}{ccc}
\text{Map}_E(E, E \times_B E - \Delta) & \hookrightarrow & \text{Map}_E(E, E \times_B E) \\
\downarrow & & \downarrow \\
\text{Map}_B(E, E \times_B E - \Delta) & \hookrightarrow & \text{Map}_B(E, E \times_B E) \\
\downarrow & & \downarrow \\
\text{Map}_B(E, E) & \xlongequal{\quad} & \text{Map}_B(E, E)
\end{array}$$

By a mild generalization of Lemma 4.2.3 applied to the projection  $pr_1 : E \times_B E \rightarrow E$ , the columns are fiber bundle sequences. This implies that the upper square is a homotopy pullback. Thus, since we have  $\Delta_f \in \text{Map}_E(E, E \times_B E)$  and  $\Gamma' \in \text{Map}_B(E, E \times_B E - \Delta)$  such that  $\Delta_f \simeq \Gamma'$  in  $\text{Map}_B(E, E \times_B E)$ , it follows that there is a map

$$\Gamma'' \in \text{Map}_E(E, E \times_B E - \Delta)$$

such that  $\Gamma' \simeq \Gamma''$  in  $\text{Map}_B(E, E \times_B E - \Delta)$ . Further,  $\Gamma'' \in \text{Map}_E(E, E \times_B E)$  implies that  $\Gamma''$  is of the form  $\Delta_{f'}$  for some  $f' : E \rightarrow E$  such that  $f'$  is fiber homotopic to  $f$  and  $f'$  has no fixed points.

□

### 4.3 Long Term Goals

Lefschetz-Nielsen theory in the case of a single space and a single endomorphism has been studied extensively. Here, we list some of the properties and results from this theory for which we expect to have analogues in the family version.

1. Our first goal is to give an algebraic description of the invariant  $L^{bord}(f)$  which is described in Section 4.1. This will be based on our base point free description of  $L^{alg}$  given in Section 3.4. In the next section, we describe our program for producing the definition.
2. Suppose  $E$  is a fiberwise ENR over a compact ENR  $B$ , and suppose  $f : U \rightarrow E$  is a compactly fixed map, where  $U$  is an open subset of  $E$ . Then Crabb and James [6, Section 6] have defined an invariant  $N(f, U)$ , which they call the Nielsen-Reidemeister index.

**Conjecture 4.3.1.** *If  $E \rightarrow B$  is a smooth fiber bundle and  $U = E$ , then  $L^{bord}(f)$  can be identified with  $N(f, U)$ .*

3. The Nielsen-Reidemeister index is an enhancement of the Lefschetz-Hopf fixed point index  $\tilde{L}(f, U)$  defined by Dold, and Becker and Gottlieb (an account can be found in [6, Section 6].) The index  $\tilde{L}(f, U)$  satisfies various properties: Naturality, Localization, Additivity, Multiplicativity, Homotopy invariance, and Commutativity.

**Question 4.3.2.** *Does the invariant  $L^{bord}$  satisfy analogous properties?*

4. In this thesis, we have discussed a converse to the Hopf-Lefschetz Theorem for endomorphisms of manifolds and for endomorphisms of smooth fiber bundles. However, Weber (see [5, Chapter VIII]) proved a converse for  $f : X \rightarrow X$  where  $X$  is a simplicial complex such that the dimension of  $X$  is at least three

and for each vertex  $v$ , the boundary of the star of  $v$  is connected. Recall that the star of  $v$  is the collection of all simplices of  $X$  containing  $v$ .

**Question 4.3.3.** *For what class of fibrations is there a family version of Weber’s result?*

5. The Nielsen number of an endomorphism  $f$  of a single compact, connected manifold  $X$  is defined to be the number of fixed point classes  $F$  of  $f$  such that  $i(f, F) \neq 0$ . Jiang [19, pages 17–23] proves that if  $\dim(X) > 2$ , then the Nielsen number of  $f$  is equal to the least number of fixed points among all self-maps having the same homotopy type as  $f$ . Now, suppose  $E \rightarrow B$  is a smooth fiber bundle with compact fibers.

**Question 4.3.4.** *Is there a family version of Jiang’s result?*

6. We would like to develop techniques for evaluating the family version of  $L^{bord}(f)$  in various special cases.

#### 4.4 The Algebraic Invariant: a Program

In this section we describe our program for accomplishing the first long term goal: to give an algebraic description of  $L^{bord}(f)$  in the family setting. We begin with discussions of the “home” of the invariant and of trace maps. Next, we describe a linearized version of this program. We end with the special case where  $p : E \rightarrow B$  is a product bundle.

##### 4.4.1 The Home

For any space  $Y$  let  $Q(Y) = \text{hocolim } \Omega^i \Sigma^i(Y_+)$ , where  $+$  is a disjoint base point. Note that  $\pi_0 Q(Y) \cong \Omega_0^{fr}(Y) \cong H_0(Y)$ . Suppose that  $X$  is a space with an endomorphism  $f : X \rightarrow X$ , then we can ask for an *algebraic* model for  $Q\mathcal{L}_f X$ . Recall that we have already used the isomorphism  $\pi_0 Q\mathcal{L}_f X \cong HH_0(\mathbb{Z}\Pi; {}_f\mathbb{Z}\Pi)$ , where  $\Pi$  is the fundamental groupoid of  $X$ .

For any ring  $R$  and  $R$ -bimodule  $M$ , we get the simplicial abelian group  $\mathbb{H}\mathbb{H}(R; M)$ . If  $X$  is path connected with base point  $*$ , then there exists an equivalence

$$\mathbb{Z}(\mathcal{L}X) \rightarrow \mathbb{H}\mathbb{H}(\mathbb{Z}\pi_1(X, *); \mathbb{Z}\pi_1(X, )),$$

where  $\mathbb{Z}(-)$  is the result of applying the free abelian group functor degree-wise to the singular complex of the input.

The following is an outline for generalizing this to the family situation.

Step 1: Extend the definition of  $\mathbb{H}\mathbb{H}(R; M)$  to ringoids and bimodules over them.

Give an equivalence

$$\mathbb{Z}(\mathcal{L}_f X) \rightarrow \mathbb{H}\mathbb{H}(\mathbb{Z}\Pi X; {}_f\mathbb{Z}\Pi X),$$

where  $\Pi$  is the fundamental groupoid of  $X$ .

Bökstedt [3, 23] extended the definition of  $\mathbb{H}\mathbb{H}(R; R)$  to when  $R$  is an FSP, such as  $\mathbb{S}\pi_1 Y$  or  $\mathbb{S}\Omega Y$  ( $Y$  is a based space.) McCarthy and Iwashita [18] have extended this construction to  $\mathbb{H}\mathbb{H}(R; M)$ , where  $R$  is an FSP and  $M$  is an  $R$ -bimodule. Furthermore, there exists a homotopy equivalence

$$Q\mathcal{L}Y \rightarrow \mathbb{H}\mathbb{H}(\mathbb{S}\Omega Y; \mathbb{S}\Omega Y).$$

Step 2: Extend the definition of  $\mathbb{H}\mathbb{H}(\mathbb{S}\pi_1(X, *); \mathbb{S}\pi_1(X, *))$  to  $\mathbb{H}\mathbb{H}(\mathbb{S}\Pi X; {}_f\mathbb{S}\Pi X)$ ,

where  $\Pi X$  is the fundamental groupoid of  $X$ . Furthermore, extend it to

$$\mathbb{H}\mathbb{H}(\mathbb{S}GX; {}_f\mathbb{S}GX),$$

where  $GX$  is the simplicial groupoid of  $X$  as defined by Dwyer and Kan [11] (see also Goerss and Jardine [14]). Then give a homotopy equivalence

$$\psi : Q\mathcal{L}_f X \rightarrow \mathbb{H}\mathbb{H}(\mathbb{S}GX; {}_f\mathbb{S}GX).$$

Step 3: Suppose that  $p : E \rightarrow B$  is a fibration with an endomorphism  $f : E \rightarrow E$ .

Recall the associated fibration  $Q^B\mathcal{L}_f^B E \rightarrow B$ , where for any  $b \in B$ , the fiber over  $b$

is  $Q\mathcal{L}_{f_b}(F_b)$ . Show that there is a fiberwise version of  $\psi$

$$\begin{array}{ccc} Q^B\mathcal{L}_f^B E & \xrightarrow{\psi^B} & \mathbb{H}H^B(\mathbb{S}G^B E; {}_f\mathbb{S}G^B E) \\ & \searrow & \downarrow \\ & & B \end{array}$$

Step 4: Suppose  $p : E \rightarrow B$  is a smooth fiber bundle and  $B$  is an  $n$ -dimensional, closed, smooth manifold. Show that (inverse) Poincaré duality (as in [10, Section 3]) yields an isomorphism

$$\Omega_n^{fr}(\mathcal{L}_f^B E; \nu(B)) \rightarrow \pi_0\Gamma(Q^B\mathcal{L}_f^B E \rightarrow B),$$

where  $\Gamma(Q^B\mathcal{L}_f^B E \rightarrow B)$  is the space of sections of  $Q^B\mathcal{L}_f^B E \rightarrow B$ .

By first applying this (inverse) Poincaré duality to  $L^{bord}(f)$  and then composing with  $\psi^B$ , we get a section of  $\mathbb{H}H^B(\mathbb{S}G^B E; {}_f\mathbb{S}G^B E) \rightarrow B$ . Denote this section by  $\bar{L}^{bord}(f)$ .

#### 4.4.2 Trace Maps

Suppose  $R$  is a ring and  $M$  is an  $R$ -bimodule. Let  $\text{End}(R, M)$  be the symmetric monoidal category with objects  $(P, \alpha)$ , where  $P$  is a finitely generated projective left  $R$ -module and  $\alpha : P \rightarrow M \otimes_R P$  is an  $R$ -linear map. A map  $f : (P, \alpha) \rightarrow (P', \alpha')$  is given by an  $R$ -linear map  $f : P \rightarrow P'$  such that  $\alpha' \circ f = (\text{id}_M \otimes f) \circ \alpha$ .

Let  $K\text{End}(R, M)$  be the  $K$ -theory of this category. For any finitely generated projective module we get a map

$$\text{Hom}_R(P, M \otimes_R P) \rightarrow \pi_0 K\text{End}(R, M).$$

The Dennis trace map is a natural transformation

$$\text{Tr} : K\text{End}(R, M) \rightarrow \mathbb{H}H(R; M)$$

such that the composition  $\text{Hom}_R(P, M \otimes_R M) \rightarrow \pi_0 K \text{End}(R, M) \rightarrow HH_0(R; M)$  is the Hattori-Stallings trace map. See Dundas and McCarthy [9], Iwashita [18], Dundas, Goodwillie and McCarthy [8], and Madsen [23].

Furthermore, if  $R$  is an FSP, such as  $\mathbb{S}\pi_1 Y$  or  $\mathbb{S}\Omega Y$ , and  $M$  is an  $R$ -bimodule, then there is also a Dennis trace map

$$\text{Tr} : K \text{End}(R, M) \rightarrow \mathbb{H}\mathbb{H}(R; M).$$

**Question 4.4.1.** *How is this related to Nicas's [24] extension of the Hattori-Stallings trace to symmetric spectra?*

Step 5: Extend the construction of the Dennis trace to

$$\text{Tr} : K \text{End}(\mathbb{S}GX, {}_f\mathbb{S}GX) \rightarrow \mathbb{H}\mathbb{H}(\mathbb{S}GX; {}_f\mathbb{S}GX).$$

If  $X$  is finitely dominated, then we should get a “universal” Lefschetz invariant

$$\langle X, f \rangle \in K \text{End}(\mathbb{S}GX; {}_f\mathbb{S}GX).$$

Compare this with Lück [22], and Dwyer, Weiss and Williams [10].

Suppose  $p : E \rightarrow B$  is a fibration with an endomorphism  $f : E \rightarrow E$ . Let

$$K \text{End}^B(\mathbb{S}G^B E, {}_f\mathbb{S}G^B E) \rightarrow B \tag{4.4.1}$$

be the result of applying  $K \text{End}(\mathbb{S}G-, {}_f\mathbb{S}G-)$  fiberwise.

Step 6: If the fibers of  $p$  are finitely dominated, then show that (4.4.1) has a preferred section  $s$  which sends  $b \in B$  to  $\langle F_b, f_b \rangle$ .

Step 7: A fiberwise version of the trace map from Step 5,

$$\text{Tr} : K \text{End}^B(\mathbb{S}G^B E, {}_f\mathbb{S}G^B E) \rightarrow \mathbb{H}\mathbb{H}^B(\mathbb{S}G^B E; {}_f\mathbb{S}G^B E),$$

should send  $s$  to

$$L^{alg}(f) \in \Gamma(\mathbb{H}\mathbb{H}^B(\mathbb{S}G^B E; {}_f\mathbb{S}G^B E) \rightarrow B).$$

Step 8: Show that  $L^{alg}(f)$  is equivalent to  $\bar{L}^{bord}(f)$ .

### 4.4.3 Linearized Version

In these last two sections we give two less ambitious programs for producing a family version of the algebraic Lefschetz invariant, the first is a linearized version and the second is for product bundles.

For a ring  $R$  with an endomorphism  $\alpha : R \rightarrow R$ , let  $K \text{End}(R, \alpha) = K \text{End}(R, {}_{\alpha}R)$ , where  ${}_{\alpha}R$  is the  $R$ -bimodule given by  $R$  with the usual right multiplication but with left multiplication twisted by  $\alpha$ .

Replacing modules with chain complexes, there is an analogous category  $\text{End}^{ch}(R, \alpha)$ . Objects are pairs  $(P_{\bullet}, \phi)$ , where  $P_{\bullet}$  is an  $R$ -chain complex which is chain homotopy equivalent to a finitely generated projective  $R$ -chain complex and  $\phi$  is an  $\alpha$ -linear chain endomorphism of  $P_{\bullet}$ .

**Theorem 4.4.2.** *There is an equivalence*

$$K \text{End}(R, \alpha) \rightarrow K \text{End}^{ch}(R, \alpha).$$

We outline our program in the following five steps.

Step 1. Extend Theorem 4.4.2 to ringoids.

Step 2. Suppose  $p : E \rightarrow B$  is a fibration with fibers  $F_b = p^{-1}(b)$  for  $b \in B$ . Suppose  $f : E \rightarrow E$  is a map of fibrations, i.e.,  $p \circ f = p$ . Let  $f_b = f|_{F_b}$  for each  $b \in B$ . Apply  $K \text{End}(\mathbb{Z}\Pi F_b, \Pi f_b)$  and  $HH(\mathbb{Z}\Pi F_b; \Pi f_b)$  fiberwise, where  $\Pi$  denotes the fundamental groupoid. One gets the following diagram.

$$\begin{array}{ccc} K \text{End}(\mathbb{Z}\Pi F_b, \Pi f_b) & \longrightarrow & HH(\mathbb{Z}\Pi F_b; \Pi f_b) \\ \downarrow & & \downarrow \\ K \text{End}^B(E, f) & \longrightarrow & HH^B(E; f) \\ & \searrow & \swarrow \\ & B & \end{array}$$

Step 3. Assume each fiber  $F_b$  is finitely dominated. As in our definition of the algebraic Lefschetz invariant (Definition 3.4.1), the endomorphism  $f_b$  of the fiber  $F_b$

induces  $\Pi f_b \in \text{End}^{ch}(\mathbb{Z}\Pi F_b, \Pi f_b)$ . Furthermore, this gives an invariant

$$L^{\text{Univ}}(F_b) \in K \text{End}(\mathbb{Z}\Pi F_b, \Pi f_b).$$

Compare this with Lück's Universal Lefschetz invariant [22].

Step 4. Applying Step 3 fiberwise gives a section

$$\begin{array}{c} K \text{End}^B(E, f) \\ \downarrow \Big) L^{\text{Univ}}(f) \\ B \end{array}$$

Step 5. Define  $L^{\text{alg}}(f)$  to be the composition

$$B \xrightarrow{L^{\text{Univ}}(f)} K \text{End}^B(E, f) \longrightarrow HH^B(E; f),$$

where the second map is from Step 2.

We would proceed by showing that Poincaré Duality yields a map from  $\Omega_n^{fr}(\mathcal{L}_f^B E; \nu(B))$  to the space of sections  $\Gamma(HH^B(E, f))$  which sends  $L^{\text{bord}}(f)$  to  $L^{\text{alg}}(f)$ .

#### 4.4.4 Product Bundles

Consider the fibration  $p : E \rightarrow B$ , where  $B$  is a space,  $E = B \times F$  for some space  $F$ , and  $p$  is the projection on the first factor. Let  $f : E \rightarrow E$  be a map of fibrations, i.e., a continuous map such that  $p \circ f = p$ . Our goal is to define a Lefschetz invariant,  $L(f)$  using an algebraic trace. The following is an outline of how we expect to accomplish this.

First, we define the trace. Let  $R$  be a ringoid. Given a functor  $\phi : R \rightarrow R$  and an  $R$ -chain complex  $C_\bullet : R \rightarrow \text{Ch}(\mathbb{Z})$ , define  $\text{End}_\phi(C_\bullet)$  as a simplicial monoid. Define a simplicial abelian group  $\mathbb{H}\mathbb{H}(R; \phi R)$ . Using Iwashita's trace [18] as a model, construct a map of simplicial monoids

$$\text{End}_\phi(C_\bullet) \rightarrow \mathbb{H}\mathbb{H}(R; \phi R)$$

with the following properties:

- (i). on  $\pi_0$  the map is our Hattori-Stallings trace, and
- (ii). if  $R$  is a ring, then the construction is the same as in [18].

This will be our trace map.

Next, we make the following additional assumption about  $f$ . Let  $\Pi F$  be the fundamental groupoid of  $F$ , and let  $B C$  denote the classifying space of a category  $C$ . Assume that there exists a map  $\alpha \in \text{End}(\Pi F)$  such that the diagram

$$\begin{array}{ccc} B \times F & \xrightarrow{f} & B \times F \\ \downarrow & & \downarrow \\ B \times B \Pi F & \xrightarrow{\text{id}_B \times B \alpha} & B \times B \Pi F \end{array}$$

commutes. Notice that since  $f$  is a map of fibrations, it is equivalent to a continuous map  $f : B \rightarrow \text{End}(F)$ . This additional assumption allows us to view  $f$  as a map  $f : B \rightarrow \text{End}_\alpha(F)$ .

Lifting to the singular chain complex of the universal cover,  $f$  induces a map  $\tilde{f}_* : B \rightarrow \text{End}_\alpha(S_\bullet)$ . Define the Lefschetz invariant of  $f$  to be the section given by the trace composed with  $\tilde{f}_*$ ,

$$B \xrightarrow{\tilde{f}_*} \text{End}_\alpha(S_\bullet) \xrightarrow{\text{Tr}} \mathbb{H}\mathbb{H}(\mathbb{Z}\Pi F; {}_\alpha\mathbb{Z}\Pi F).$$

That is,

$$L(f) = \text{Tr} \circ \tilde{f}_* \in \Gamma(\mathbb{H}\mathbb{H}(\mathbb{Z}\Pi F; {}_\alpha\mathbb{Z}\Pi F) \rightarrow B).$$

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