FUNDAMENTAL GROUPOIDS FOR GRAPHS

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Abstract. In this paper, we define a $\times$-homotopy fundamental groupoid for graphs, and prove that it is a functorial $\times$-homotopy invariant for finite graphs. We also introduce tools to compute this fundamental groupoid, including a van Kampen theorem. We conclude with a comparison with previous definitions along these lines, including those built on polyhedral complexes of graph morphisms.

Keywords—graph, homotopy, groupoid, fundamental group

1. Introduction

There are several different definitions of homotopy for graphs in the literature. Two of particular prominence are $\times$-homotopy [2, 10, 11, 17–20] and A-homotopy [1, 3, 14, 22, 23]. In this paper, we focus on $\times$-homotopy. Taking our cue from topology, we define an algebraic invariant that captures information about the $\times$-homotopy type of a graph. This fundamental groupoid builds on a related groupoid defined by Kwak and Nedela in [21].

In this paper, we define the fundamental groupoid, and its related fundamental group. We prove that our fundamental groupoid is both functorial and a homotopy invariant, thus providing us with an algebraic tool for studying the $\times$-homotopy category of graphs. We also develop tools for computing our fundamental groupoid, including a van Kampen theorem.

The idea of defining a fundamental groupoid for graphs has been tackled from several angles in the literature. In addition to the definition in [21] which serves as the starting point for our definition, a fundamental group which is $\times$-homtopy invariant has been defined specifically for exponential graphs in [11]. Our definition applies more generally to any graph, and differs somewhat from the definition of [11] in the way it treats repeated vertices. A more precise comparison is given in the last section. Additionally, the idea of a fundamental group for A-homotopy theory has been explored in [1, 3, 13]. Since A-homotopy is built on a different category of graphs where moprhisms are allowed to collapse connected edges, these groups measure different properties of graphs. In particular, these groups treat 3-cycles as contractible, which our groupoid does not.

Our definition of the fundamental groupoid offers a computable tool for studying the $\times$-homotopy of graphs. In future work we plan to use it to study lifting properties of graphs and homotopy covers of graphs which allow liftings of $\times$-homotopies. This will be explored in [9], currently in progress.

Our paper is structured as follows. Section 2 contains background results. In Section 3, we recall a groupoid of walks in G from [21], and show that it is a functor from Gph to $\mathsf{Groupoids}$. In Section 4, we define our fundamental groupoid as a quotient of the walk groupoid, where morphisms are homotopy classes of walks. We show that this defines a functor from the homotopy category of graphs to groupoids, giving a homotopy invariant. In Section 5, we develop further tools for computing this fundamental groupoid, showing how it behaves with respect to product graphs and proving a modified Van Kampen Theorem [7, 15]. We end with Section 6 describing a variant of our groupoid for graphs where all vertices are looped. We show that this generalizes the fundamental group of the exponential graph defined by Dochtermann [11], and prove there is an equivalence of categories between the looped groupoid of an exponential graph H^G and the fundamental groupoid of the polyhedral hom space associated with H^G studied in [2, 11, 17–20].

2. Background

In this section, we summarize background material. A more complete exposition can be found in [8]. We work in the category Gph of undirected graphs, without multiple edges. Moreover, throughout this paper, we will assume that graphs are both finite, and contain no isolated vertices. Graph theory terminology and notation follows [5] and category theory terminology and notation follows [25].

Definition 2.1. [16] The category of finite graphs Gph is defined by:

- An object is a graph G, consisting of a finite set of vertices $V(G) = \{v_\lambda\}$ and a set $E(G)$ of edges connecting them. Each edge is given by an unordered pair of vertices. Any pair of vertices has at most one edge connecting them, and loops are allowed but isolated vertices are not: each vertex must be connected to at least one other (possibly itself). A connecting edge will be notated by $v_1 \sim v_2$.
- An morphism in the category Gph is a graph homomorphism $f : G \to H$, given by a set map $f : V(G) \to V(H)$ such that for $v_1, v_2 in V(G)$, if $v_1 \sim v_2 \in E(G)$ then $f(v_1) \sim f(v_2) \in E(H)$.

Throughout this paper, we will assume that 'graph' always refers to an object in Gph.

In defining our fundamental groupoids, we will make use of the path graphs, both looped and unlooped.

Definition 2.2. [5, 10] Let P_n be the path graph with $n+1$ vertices $\{0, 1, \ldots, n\}$ such that $i \sim i+1$ for $i = 0, \ldots, n-1$. Let I_n^ℓ be the looped path graph with $n+1$ vertices $\{0, 1, \ldots, n\}$ such that $i \sim i$ and $i \sim i+1$ for $i = 0, \ldots, n-1$.

Definition 2.3. [5] A **walk** in G of length n is a morphism $\alpha : P_n \to G$ from $\alpha(0)$ to $\alpha(n)$. A **looped walk** in G of length n is a morphism $\alpha : I_n^\ell \to G$. Note that we allow length 0 walks, defined by a single vertex.

We will usually describe a walk by a list of image vertices $(v_0 v_1 v_2 \ldots v_n)$ such that $v_i \sim v_{i+1}$. In a looped walk, all vertices along the walk are looped.

Definition 2.4. ([8], Definition 2.1) Given a walk $\alpha : P_n \to G$ from x to y, and a walk $\beta : P_m \to G$ from y to z, the **concatenation of walks** $\alpha * \beta : P_{m+n} \to G$ by

$$(\alpha * \beta)(i) = \begin{cases} \alpha(i) & \text{if } i \leq n \\ \beta(i-n) & \text{if } n < i \leq n+m \end{cases}$$

Thus the concatenation $(x v_1 v_2 \ldots v_{n-1} y) * (y w_1 w_2 \ldots w_{m-1} z) = (x v_1 v_2 \ldots v_{n-1} y w_1 \ldots w_{m-1} z)$. Contatenation of looped walks is defined in the same way.

Homotopies are defined using the product graph $G \times I_n^\ell$.

Definition 2.5. [16] For graphs G and H, the (categorical) **product graph** $G \times H$ is defined by:

- A vertex is a pair (v, w) where $v \in V(G)$ and $w \in V(H)$.
- An edge is defined by $(v_1, w_1) \sim (v_2, w_2) \in E(G \times H)$ whenever $v_1 \sim v_2 \in E(G)$ and $w_1 \sim w_2 \in E(H)$.

Definition 2.6. [10] Given $f, g : G \to H$, we say that f is $\times$**-homotopic** to g, written $f \simeq g$, if there is a map $\Gamma : G \times I_n^\ell \to H$ such that $\Gamma|_{G\times\{0\}} = f$ and $\Gamma|_{G\times\{n\}} = g$. We will say Γ is a length n homotopy.

Other authors have considered alternate definitions of homotopies of graphs, and this is sometimes referred to as $\times$-homotopy to distinguish it. Since this is the primary version of homotopy that we will consider in this paper, we will also refer to it simply as 'homotopy'.

WE can use the notion of honmotopy to define a 2-category Gph.

Theorem 2.7 ([8], Theorem 3.18)**.** *We can define a 2-category of graphs as follows:*

- *Objects [0-cells] are given by objects of* Gph*, the finite undirected graphs.*
- *Morphisms [1-cells] are given by the morphisms of* Gph*, the graph homomorphisms*
- *2-cells are defined as homotopies between moprhisms.*

It is also shown in [8] that we can create the homotopy category for Gph as a quotient category of this 2-category, identifying morphisms which are connected by a 2-cell.

Homotopy can also be defined by a looped walk in the exponential graph G^H. A priori, even a length 1 homotopy (given by a single edge of G^H) can connect morphisms whose images differ on many vertices. However, in [8], we analyzed the structure of homotopies to show that we can shift one vertex at a time.

Definition 2.8 ([8], Definition 4.1)**.** Let $f, g : G \to H$ be graph morphisms. We say that f and g are a **spider pair** if there is a single vertex x of G such that $f(y) = g(y)$ for all $y \neq x$. If x is unlooped there are no additional conditions, but if $x \sim x \in E(G)$, then we require that $f(x) \sim g(x) \in E(H)$. When we replace f with g we refer to it as a **spider move**.

Proposition 2.9 ([8], Proposition 4.4: Spider Lemma)**.** *If $f, g : G \to H$ are graph morphisms then $f \simeq g$ if and only if is a finite sequence of spider moves connecting f and g.*

This result relies on the domain graph G being finite. Since all graphs considered in this paper are finite, we can apply this proposition here without restriction.

We can also use the framework of spider moves to analyze homotopy equivalences. In the literature, homotopy equivalence has been linked to the idea of a **fold** [11,16]. This can be thought of as a special case of our spider moves.

Definition 2.10 ([6,10,12,16])**.** If G is a graph, we say that a morphism $f : G \to G$ is a **fold** if f and the identity map are a spider pair.

Proposition 2.11 ([8], Lemma 6.8)**.** *If f is a fold, then $f : G \to Im(f)$ is a homotopy equivalence.*

In the literature, graphs that cannot be folded are called **stiff** graphs [4,6]. Since each homotopy class contains exactly one stiff graph, this gives us a unique choice of representative for homotopy equivalent graphs. In fact, the subcategory of stiff graphs defines a skeletal category for hFGph, meaning that in addition to giving a unique choice of representative for each homotopy class, the inclusion of the subcategory induces an equivalence of categories.

Theorem 2.12 ([8], Theorem 6.5)**.** *The stiff graphs are a skeletal subcategory of the homotopy category of finite graphs* hFGph *defined in* [8]*, Definition 5.1.*

Thus every graph is homotopy equivalent to a unique stiff graph, and the homotopy classes of morphisms between graphs can be determined by the homotopy classes of morphisms between their stiff representatives.

3. The Walk Groupoid

In this section we describe the walk groupoid of a graph G. This was first defined by Kwak and Nedela in [21], where they refer to it as the "fundamental groupoid". They give a definition and show it is a groupoid, but do not develop any further properties. Here, we define this groupoid using the language consistent with how we will later present our homotopy invariant fundamental groupoid. We also show that it defines a functor.

Definition 3.1. Let $\alpha = (v_0v_1v_2 \dots v_n)$ be a walk in G. We say that α is **prunable** if $v_i = v_{i+2}$ for some i. We define a **prune** of α to be given by a walk α' obtained by deleting the vertices v_i and v_{i+1} from the walk when $v_i = v_{i+2}$: if

$$\alpha = (v_0v_1v_2 \dots v_{i-1}v_iv_{i+1}v_iv_{i+3} \dots v_n)$$

then the prune of α is

$$\alpha' = (v_0v_1v_2 \dots v_{i-1}v_iv_{i+3} \dots v_n)$$

We define an equivalence relation on walks in G generated by the prunes. Concretely, $\alpha \simeq \beta$ if there is a finite sequence of prunings between them: $\alpha = \gamma_0 \simeq \gamma_1 \simeq \gamma_2 \simeq \cdots \simeq \gamma_{k-1} \simeq \gamma_k = \beta$ where either γ_i is a prune of γ_{i+1} or γ_{i+1} is a prune of γ_i.

Observation 3.2. Since any prune always removes two edges, the parity of a prune equivalence class is well-defined and each prune class of walks consists of all even length or all odd length walks.

Each prune equivalence class has a unique non-prunable representative, as shown by the next two results. This is described as the *reduction* of a walk in [21].

Proposition 3.3. *Repeated pruning of a walk results in a unique non-prunable walk.*

Proof. We proceed via induction. If α is length 0 or 1, then there are no prunings possible and hence α is itself the unique non-prunable walk. Now consider a walk $\alpha : P_n \to G$. If there exists a unique i such that $v_i = v_{i+2}$, then pruning α results in a unique α' of length $n-2$.

Now suppose that there are two values i, j such that $v_i = v_{i+2}$ and $v_j = v_{j+2}$, and hence two possible prunings of α. We will show that either order of pruning will lead to the same result. Without loss of generality, assume $i < j$. If $i + 1 < j$, then the repeated vertices are separated in the walk and it is easily checked that pruning at i and then j results in the same walk as pruning at j and then i. If $j + i = 1$, then α is of the form

$$\alpha = (v_0v_1 \dots, v_{i-1}v_iv_{i+1}v_iv_{i+1}v_{i+4} \dots v_n)$$

Pruning at i removes the first v_iv_{i+1} pair, while pruning at $j = i+1$ removes the $v_{i+1}v_i$ pair. Both pruning orders result in

$$\alpha' = (v_0v_1 \dots, v_{i-1}v_iv_{i+1}v_{i+4} \dots v_n)$$

Thus by induction any choice of successive prunings on α will eventually result in the same non-prunable walk.

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Corollary 3.4. *Each prune class of walks has a unique non-prunable representative.*

Proof. If we have two non-prunable walks α, β such that $[\alpha] = [\beta]$ then then there is a sequence of forward and backward prune moves connecting them: $\alpha \longleftarrow \gamma_1 \longrightarrow \gamma_2 \longleftarrow \gamma_3 \longrightarrow \dots \gamma_k \longrightarrow \beta$ where each morphism represents a sequence of prunes in the indicated direction. We induct on k: if $k = 1$ then we have $\alpha \longleftarrow \gamma_1 \longrightarrow \beta$, and Proposition 3.3 ensures that $\alpha = \beta$ since they both result from prunings of the same path γ_1. If $k > 1$, then consider the left portion of the sequence of prune moves $\alpha \longleftarrow \gamma_1 \longrightarrow \gamma_2$: letting γ' be the walk that results from completely pruning γ_2, we have that $\alpha = \gamma'$ by Proposition 3.3 again. But then we have a sequence of prune moves of length $k-2$ connecting γ' to β, and by our inductive hypothesis we can say that $\gamma' = \beta$. □

The walk groupoid consists of prune classes of walks under concatenation, as defined in Definition 2.4. To define this, we need:

Lemma 3.5. *Concatenation is well-defined on prune classes.*

Proof. The endpoints of any representatives of a prune class are always the same, and so the start and end vertices are well-defined on prune classes. If α prunes to α' and β prunes to β' then $\alpha * \beta$ prunes to $\alpha' * \beta'$, and by Proposition 3.3 that the order in which the pruning is done will not matter. □

The following is equivalent to the 'fundamental groupoid' of Section 1.2 in [21].

Definition 3.6. For a graph G, we define the **walk groupoid** of G, $\mathfrak{W}G$, as follows:

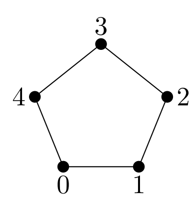
- objects of $\mathfrak{W}G$ are vertices of the graph G
- a morphism from v_0 to v_n in $\mathfrak{W}G$ is given by a prune class of walks from v_0 to v_n
- composition of morphisms is defined using concatenation of walks.

To see that $\mathfrak{W}G$ is a groupoid, observe that concatenation of walks is associative ([8], Lemma 2.17) and the length 0 walk at a vertex v gives an identity morphism from v to v ([8], Observation 2.16.) Lastly, given any walk $\alpha = (v_0 v_1 v_2 \dots v_{n-1} v_n)$ we define $\alpha^{-1} = (v_n v_{n-1} \dots v_2 v_1 v_0)$; it is easy to see that $\alpha * \alpha^{-1}$ prunes down to a length 0 identity walk.

Since every prune class has a unique non-prunable representative, we can also think of this groupoid as having morphisms given by non-prunable walks, where the composition operation is given by concatenation followed by pruning.

To get a group from this groupoid, we can fix a vertex v and consider the isotropy group consisting of all morphisms from v to v. Any choice of vertices in the same connected component of G will result in isomorphic groups.

Example 3.7. Consider the graph C_5



The groupoid $\mathfrak{W}C_5$ has objects given by the vertex set $\{0, 1, 2, 3, 4\}$. We consider the isotropy group at 0, given by prune classes of walks from 0 to 0. If any walk reverses orientation and goes from clockwise to counterclockwise or vice versa, there will be a subwalk which can be pruned, and so all non trivial walks from 0 to 0 may be represented by strictly clockwise or counterclockwise walks, generated by (043210) and (012340) respectively. Since the concatenation of these walks prune to the identity walk (0), these are inverse morphisms. Either of them is a free generators for the isotropy group, which is isomorphic to $\mathbb{Z}$. Since C_5 is connected, the groupoid $\mathfrak{W}C_5$ has isotropy $\mathbb{Z}$ for any object.

Theorem 3.8. *$\mathfrak{W}G$ defines a functor from* Gph *to* Groupoids*, the category of groupoids.*

Proof. If we have a graph homomorphism $\phi : G \to H$, we can define a functor $\phi_* : \mathfrak{W}G \to \mathfrak{W}H$ by $\phi_*(v) = \phi(v)$ on objects, and $\phi_*(\alpha) = \phi_*(v_0v_1v_2 \dots v_n) = (\phi(v_0)\phi(v_1)\phi(v_2)\dots\phi(v_n))$; the fact that ϕ is a graph homomorphism ensures that this a walk in H. If α prunes to α', then $\phi_*(\alpha)$ also prunes to $\phi_*(\alpha')$, and concatenation is respected, and so ϕ_* defines a moprhism of groupoids $\mathfrak{W}G \to \mathfrak{W}H$.

To verify functoriality, observe that if $id : G \to G$ is the identity, then id_* is the identity map on groupoids; and if $\phi : G \to H$ and $\psi : H \to K$, then $(\psi\phi)_*$ is the same as $\psi_*\phi_*$ since they are both defined by $(\psi\phi(v_0)\psi\phi(v_1)\psi\phi(v_2)\dots\psi\phi(v_n))$. □

4. The Fundamental Groupoid

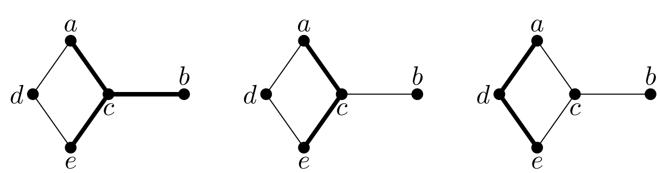
This section defines our primary homotopy invariant, the fundamental groupoid. This groupoid is a variant of the walk groupoid of the previous section, defined using homotopy classes of walks.

Definition 4.1. Suppose that α, β are walks in G from x to y. We say α and β are **homotopic rel endpoints** if α and β are homotopic via a homotopy where all intermediate walks $\Gamma|_{G\times\{i\}}$ are also walks from x to y, so the endpoints of the walk remain fixed throughout the homotopy. A similar definition holds for looped walks.

Definition 4.2. For a graph G, we define the **fundamental groupoid** of G, $\Pi(G)$, as follows:

- objects of $\Pi(G)$ are vertices of the graph G
- an morphism from v_0 to v_n in $\Pi(G)$ is given by a prune class of walks from v_0 to v_n, up to homotopy rel endpoints
- composition of morphisms is defined using concatenation of walks.

Example 4.3. Let $\alpha = (acbce)$ and $\beta = (ade)$ be walks in the graph below. Then $[\alpha] = [\beta] \in \Pi(G)$ since we have a prune of α to $\alpha' = (ace)$ and then a spider move to $\beta = (ade)$.



In order to verify that this definition gives us a well-defined groupoid, we check the following.

Proposition 4.4. *Concatenation is well-defined on elements of* $\Pi(G)$.

Proof. We have already shown that concatenation is well-defined with respect to pruning in Proposition 3.5, so we need to check that it respects homotopy. Suppose that we have walks that are homotopic rel endpoints: $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$. Then there is a sequence of spider moves connecting

α to α', and β to β'. So we can produce a sequence of spider moves connecting $\alpha * \beta$ to $\alpha' * \beta'$ by holding β fixed and moving $\alpha * \beta$ to $\alpha' * \beta$, and then holding α' fixed and moving $\alpha' * \beta$ to $\alpha' * \beta'$. □

Theorem 4.5. $\Pi(G)$ *defines a groupoid.*

Proof. The concatenation operation is associative as shown in [8], Lemma 2.17, and given a vertex $v \in G$ we have the length 0 walk (v) acting as an identity element by [8], Observation 2.16. If $\alpha = (v_0 v_1 \dots v_{n-1} v_n)$ then we can define an inverse $\alpha^{-1} = (v_n v_{n-1} \dots v_1 v_0)$. Then

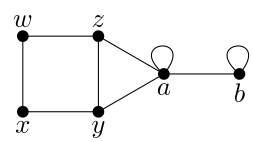
$$\alpha * \alpha^{-1} = (v_0 v_1 v_2 \dots v_{n-2} v_{n-1} v_n v_{n-1} v_{n-2} \dots v_2 v_1 v_0).$$

Successive pruning operations will reduce this to the identity walk (v_0). □

Any length 4 closed walk is contractible in $\Pi(G)$ since $(x v_1 v_2 v_3 x) \simeq (x v_1 x v_3 x) = (x)$. Thus we consider these walks to be special, and give them a name. Any walk of length 4 which can be pruned would necessarily prune to the trivial walk, so we do not include these in our definition but focus on the non-prunable but contractible walks.

Definition 4.6. A **diamond** is a prune-free length 4 closed walk.

Example 4.7. Consider the graph G depicted below.



The length 4 closed walks $(wxyzw)$, $(aayza)$ and $(aabba)$ are prune-free and thus are diamonds. However the walk $(wxwzw)$ is pruneable, and thus not a diamond.

We can describe the morphisms of $\Pi(G)$ more concretely with the following result.

Lemma 4.8. *Let α be prunable at i, so $v_i = v_{i+2}$ and α has the form*

$$(v_0 v_1 \dots, v_{i-1} v_i v_{i+1} v_i v_{i+3} \dots v_n)$$

Then α is homotopic rel endpoints to the walk

$$(v_0 v_1 \dots, v_{i-1} v_i v_{i+2} \dots v_n v_{n-1} v_n).$$

Proof. As maps from $P_n \to G$, we apply successive spider moves to α to move the repeated vertex down the walk:

$$\begin{aligned} \alpha &= (v_0 v_1 \dots, v_{i-1} v_i v_{i+1} v_i v_{i+3} v_{i+4} v_{i+5} \dots v_n) \\ &\simeq (v_0 v_1 \dots, v_{i-1} v_i v_{i+3} v_i v_{i+3} v_{i+4} v_5 \dots v_n) \\ &\simeq (v_0 v_1 \dots, v_{i-1} v_i v_{i+3} v_{i+4} v_{i+3} v_{i+4} v_5 \dots v_n) \end{aligned}$$

Repeatedly applying spider moves will shift the repeat down to the end of the walk. □

Thus for morphisms of $\Pi(G)$, we can consider only prunes of the last two edges. This allows us to identify morphisms with homotopy classes of walks of infinite length, which eventually stabilize and end with a string of alternating vertices $v_n v_{n-1} v_n v_{n-1} v_n v_{n-1} v_n \dots$. Two such walks α, β will be equivalent if there is some extension of each which become homotopic rel endpoints: if $\alpha = (v_0 v_1 \dots v_n)$ and $\beta = (w_0 w_1 \dots w_m)$ then there exists extensions $(v_0 v_1 \dots v_n, v_{n-1} v_n v_{n-1} \dots v_n v_{n-1} v_n)$ and $(w_0 w_1 \dots w_m w_{m-1} w_m w_{m-1} \dots w_m w_{m-1} w_m)$ which are homotopic rel endpoints.

As with our walk gropuoid, our fundamental groupoid defines a functor from Gph to groupoids.

Theorem 4.9. Π *defines a functor from* Gph *to groupoids.*

Proof. Suppose that $f : G \to H$ is a graph homomorphism, and define $f_* : \Pi(G) \to \Pi(H)$ by applying f to each vertex as in Theorem 3.8. We have shown this is a functor from $\mathfrak{W}G$, and so respects prune classes. If α and β are homotopic rel endpoints, there is a sequence of spider moves connecting them, shifting one vertex $(v_0 v_1 \dots v_{i-1} v_i v_{i+1} \dots v_n)$ to $(v_0 v_1 \dots v_{i-1} \hat{v}_i v_{i+1} \dots v_n)$, and applying f will give a sequence of walks where each pair similarly differs by a single vertex, and hence is a sequence of spider moves. So $f_*(\alpha)$ will be homotopic rel endpoints to $f_*(\beta)$.

Functoriality also follows from the argument from Theorem 3.8.

□

We wish to show that Π is actually a homotopy invariant. The homotopy category hFGph is defined in [8] as a quotient of the 2-category Gph: the morphisms of hFGph are equivalence classes of morphisms of Gph, where two morphisms are equivalent if there is a 2-cell between them, ie they are homotopic. Thus in order to prove Theorem 4.10, we will show that we can extend Π to a 2-functor from Gph to groupoids which takes 2-cells to natural isomorphisms.

We consider groupoids to be a 2-category by considering them to be a subcategory of the 2-category of categories: a groupoid morphism is a functor, and a 2-cell is a natural transformation between functors.

Theorem 4.10. Π *defines a functor from the homotopy category of graphs* hFGph *to the category of groupoids and functors up to natural isomorphism.*

Proof. We begin by extending the functor Π of Theorem 4.9 to a strict 2-functor from Gph to groupoids. We have defined Π on objects and morphisms. To define it on 2-cells, we need to assign a natural transformation of functors to each homtopy of graph morphisms. We know by Proposition 2.9 that any homotopic maps are connected by a sequence of spider moves, so we may assume that we have morphisms $f, g : G \to K$ which are a spider pair, differing only on a single vertex v of G. We define a natural transformation $\gamma : f_* \Rightarrow g_*$. This means that for each vertex w in G, we need an arrow $\gamma_w : f(w) \to g(w)$ in $\Pi(K)$.

We define γ_w to be the length 0 walk from $f(w) = g(w)$ if $w \neq v$. For $w = v$, choose a vertex v' such that $v \sim v'$ and define γ_v to be the walk $(f(v) f(v') g(v))$. This walk is independent of choice of v', since all choices result in walks that are homotopic rel endpoints, connected by a spider move shifting the middle vertex of the walk.

To verify the naturality square, consider a walk $\alpha = (w_0 w_1 w_2 \dots w_n)$ in G. We need to compare the walks $\gamma_{w_0} * g(\alpha)$ and $f(\alpha) * \gamma_{w_n}$ and show that they define the same morphism in $\Pi(K)$. If the walk α does not include the vertex v moved by the spider move, or if $v = w_i$ for some $i \neq 0, n$ then γ_{w_0} and γ_{w_n} are empty and the walks are either identical or connected by a spider move shifting

$f(w_i)$ to $g(w_i)$ (or multiple spider moves, if that vertex shows up multiple times). If $v = w_0$ then

$$\begin{aligned}\gamma_{w_0} * g(\alpha) &= (f(v)f(v')g(v)g(w_1)g(w_2)\dots g(w_n))\\ &= (f(v)f(v')g(v)f(w_1)f(w_2)\dots f(w_n))\\ &= (f(v)f(w_1)g(v)f(w_1)f(w_2)\dots f(w_n))\end{aligned}$$

which prunes to $f(\alpha) = f(\alpha) * \gamma_{w_n}$. A similar argument gives the equality if $v = w_n$.

The 2-functor Π lands in groupoids, where all morphisms are invertible. So the natural transformations γ are automatically natural isomorphisms. Thus Π passes to the quotient category and we obtain a functor hFGph to the category of groupoids and functors up to natural isomorphism. □

Corollary 4.11. *The category $\Pi(G)$ is a homotopy invariant, defined up to equivalence of categories.*

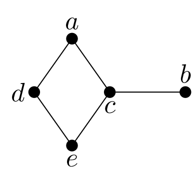
Proof. If $f : G \to H$ is a homotopy equivalence, then there is $g : H \to G$ such that $fg \simeq id$ and $fg \simeq id$. Then there is a natural isomorphism from $\Pi(G)$ to $g_* f_* \Pi(G)$, and from $\Pi(H)$ to $f_* g_* \Pi(H)$ and so f_* and g_* are equivalences of categories between $\Pi(G)$ and $\Pi(H)$.

□

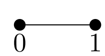
Corollary 4.12. *Let G be a graph, and G' the (unique) stiff graph which is homotopy equivalent to G. Then the fundamental groupoid $\Pi(G)$ is equivalent to the fundamental groupoid of $\Pi(G')$.*

Observation 4.13. We have chosen to work with the fundamental groupoid here. It is easy to recover a more familiar fundamental group by choosing a basepoint vertex v in G, and looking at the group $\Pi_1(G, v)$ of all morphisms in $\Pi(G)$ which start and end at v; this is the isotropy subgroup of v in the groupoid. Because $\Pi(G)$ is a groupoid, we have an isomorphism between the isotropy groups of any two choices of vertex in the same component of G.

Example 4.14. Let G be the graph from Example 4.3:



By Corollary 4.12 we have $\Pi(G) \cong \Pi(K_2)$, since K_2 is the stiff homotopy equivalent representative of G.



The objects of $\Pi(K_2)$ are the vertices $0, 1$ and the morphisms are identity morphisms given by length 0 walks at 0 and 1, and the length 1 walks between them. Any other walk would consist of alternating 0's and 1's, and may thus be pruned to a length 1 walk. Choosing a basepoint, we get a trivial fundamental group.

We can describe the fundamental group $\Pi_1(G, v)$ as a quotient of the isotropy subgroup of the walk groupoid $\mathfrak{W}G$.

Theorem 4.15. *The fundamental group $\Pi_1(G, v)$ can be defined by I_v/D where I_v denotes the isotropy group of v in $\mathfrak{W}G$, and D is the normal subgroup generated by all diamonds as defined in*

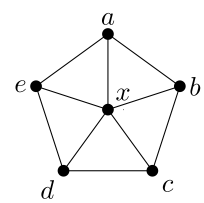
Definition 4.6. Explicitly, the subgroup D consists of products of walks of the form $\gamma * (v_1v_2v_3v_4v_1) * \gamma^{-1}$ for $(v_1v_2v_3v_4v_1)$ a diamond and γ a walk from v to v_1.

Proof. We observed that all diamonds are contractible and thus any element of D is trivial in $\Pi(G)$. Conversely, if two walks from v to v are homotopic, then there is a sequence of spider moves connecting them. Each spider move will shift one vertex, so consider

$$\alpha = (vw_1w_2 \dots w_{i-1}w_iw_{i+1} \dots w_{n-1}v)$$
$$\beta = (vw_1w_2 \dots w_{i-1}\hat{w}_iw_{i+1} \dots w_{n-1}v).$$

Define $\gamma = (vw_1w_2 \dots w_{i-1})$ and the diamond $d = (w_{i-1}\hat{w}_iw_{i+1}w_iw_{i-1})$ Then $\gamma d\gamma^{-1} * \alpha$ prunes to β and so α and β are equivalent in $\mathfrak{W}G/D$. $\square$

Example 4.16. Let G be the graph depicted below:



By Theorem 4.15, $\Pi_1(G, x)$ can be calculated by I_x/D where I_x is the isotropy of x in the walk groupoid, given by the free group with generators $e_1 = (xabx), e_2 = (xbcx), e_3 = (xcdx), e_4 = (xdex), e_5 = (xeax)$. The diamonds are given by $(xabcx), (xbcdx), (xcdex), (xdeax), (xeabx)$ which are equal to $e_1e_2, e_2e_3, e_3e_4, e_4e_5, e_5e_1$. This means that in $\Pi_1(G, x)$, $e_2 = e_1^{-1}$, and $e_2 = e_3^{-1}$, etc. Thus we find that $e_1 = e_3 = e_5$ and $e_2 = e_4 = e_1^{-1}$ and the group is generated by a single generator e_1 under the relationship $e_1^2 = 1$. Thus $\Pi_1(G, x) \cong \mathbb{Z}/2$. This also shows that our fundamental group can contain torsion, even if G does not contain loops.

5. Fundamental Groupoid of Product and Union Graphs

In this section we further examine the structure of our fundamental groupoid $\Pi(G)$, looking deeper into the parity structure of even and odd length walks and analyzing the fundamental groupoid of product and union graphs. We observed earlier that the parity of the walk is independent of the choice of representative, and so even and odd length walks are well-defined in $\Pi(G)$. Now we examine this phenomenon a little deeper.

Let T be the terminal object of Gph which has one vertex and one loop edge τ as shown in [16, 24]. Then $\Pi(G)$ is a groupoid with one object, hence a group, and it has two morphisms: the identity morphism given by the length 0 walk, and the length 1 walk (τ). The walk $(\tau\tau)$ can be pruned to the identity empty walk, so as a group we have $\tau^2 = id$ and so $\Pi(T)$ is isomorphic to $\mathbb{Z}/2$.

The parity structure of the fundamental groupoid is connected to the fact that the terminal object of Gph has a groupoid which is not the terminal identity groupoid. Every graph G has a unique canonical morphism to T, and so we have a groupoid morphism $\Pi(G) \to \Pi(T)$ and our fundamental groupoids live in the category of groupoids over $\mathbb{Z}/2$, with all morphisms of groupoids induced by graph maps respecting this structure. Explicitly, we have that the canonical map $\Pi(G) \to \mathbb{Z}/2$ takes even length walks to id and odd length walks to τ, and every map from $\Pi(G) \to \Pi(H)$ that comes from a graph map $G \to H$ will commute with the map to $\Pi(T)$ and hence preserve parity. We can define the even subgroupoid $ev(\Pi(G)) = p^{-1}(id)$.

The product $G \times H$ is the pullback over the terminal object

$$\begin{tikzcd} G \times H \arrow[r] \arrow[d] & H \arrow[d] \\ G \arrow[r] & T \end{tikzcd}$$

Functoriality says that the projections $p_1 : G \times H \to G$ and $p_2 : G \times H \to H$ give maps $\Pi(G \times H) \to \Pi(G)$ and $\Pi(G \times H) \to \Pi(H)$ and so we will have the following diagram:

$$\begin{tikzcd} \Pi(G \times H) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & \Pi(G) \times_{\mathbb{Z}/2} \Pi(H) \arrow[r] \arrow[d] & \Pi(H) \arrow[d] \\ & \Pi(G) \arrow[r] & \Pi(T) = \mathbb{Z}/2 \end{tikzcd}$$

where $\Pi(G) \times_{\mathbb{Z}/2} \Pi(H)$ denotes the pullback groupoid. Explicitly, the pullback is defined as follows: the objects are the product of the objects of $\Pi(G)$ and $\Pi(H)$, and morphisms are given by $\{(\alpha, \beta) | p_1(\alpha) = p_2(\beta)\}$, meaning (α, β) such that the parity of the walks are the same.

Theorem 5.1. *The induced map* $\Phi : \Pi(G \times H) \to \Pi(G) \times_{\mathbb{Z}/2} \Pi(H)$ *is an isomorphism of groupoids.*

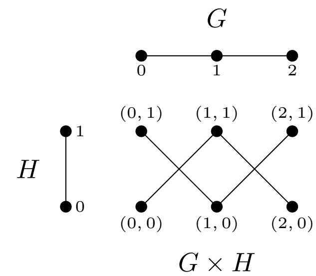
Proof. Objects of $\Pi(G \times H)$ are given by vertices of $G \times H$ which is the set $V(G) \times V(H)$, the objects of $\Pi(G) \times_{\mathbb{Z}/2} \Pi(H)$, so this is an isomorphism on objects.

On morphisms, the map is defined by $\Phi(\omega) = (\alpha, \beta)$ where $p_1(\omega) = \alpha$ in G and $p_2(\omega) = \beta$ in H. We need to show that this is both full and faithful (injective and surjective). To show that Φ is surjective on morphisms, suppose we have $(\alpha, \beta) \in \Pi(G) \times_{\mathbb{Z}/2} \Pi(H)$ given by $\alpha \in \Pi(G)$ and $\beta \in \Pi(H)$ with the same parity. Take the shorter one and repeat the last two vertices to extend so that both representative walks have the same length. This will create a walk ω in $G \times H$ such that $\Phi(\omega) = (\alpha, \beta)$.

If $\Phi(\omega) = \Phi(\omega')$ then $\alpha = \alpha'$ in $\Pi(G)$ and $\beta = \beta'$ in $\Pi(H)$. This means there are extensions of α, α' which are homotopic rel endpoints in G (via Γ), and extensions of β, β' which are homotopic rel enpoints in H (via Γ'). We are assuming that these have the same parity, and so by extending further, we may assume all are the same length. Then we can combine the homotopies $\Gamma \times \Gamma'$ to get a homotopy in $G \times H$. This shows that the functor is also injective on morphisms.

$\square$

Example 5.2. Let $G = P_2, H = K_2$ and consider $G \times H$:



There is an odd length walk from $(0,0)$ to $(1,1)$ since there is an odd length walk from 0 to 1 in both G and H. Similarly, there is an even length walk from $(0,0)$ to $(2,0)$. However, there is no walk from $(0,0)$ to $(1,0)$, since the walks from 0 to 1 in G and 0 to 0 in H have different parity.

If we consider reflexive graphs (where all vertices have loops) then the parity plays less of a role and our fundamental groupoid winds up with odd and even portions isomorphic to each other. To make this precise, we look at a product groupoid $X \times \mathbb{Z}/2$. The objects of this product are the same as the objects of X and the morphisms from x to x' are defined by (α, id) and (α, τ) for $\alpha : x \to x'$, with composition definied by X in the first coordinate and multiplication in $\mathbb{Z}/2$ in the second.

Proposition 5.3. *Suppose that G is a reflexive graph, and $\Pi(G) = \Pi$ is its fundamental groupoid, and $E = ev(\Pi(G))$ its even subgroupoid. Then $\Pi \simeq E \times \mathbb{Z}/2$.*

Proof. Define $\Psi : \Pi \to E \times \mathbb{Z}/2$ by: $\alpha \to (\alpha, id)$ if $\alpha \in E$ and $\alpha \to (\alpha v_n, \tau)$ if α is odd, where v_n is the last vertex of the walk α. Thus if α is odd, we repeat the last vertex (which we can do since all vertices are looped) to create an even walk.

The map Ψ is an isomorphism on objects, since the objects of E are the same as the objects of Π. We check that it is a functor. If α is even then it is easy to see that $\Psi(\alpha\beta) = \Psi(\alpha)\Psi(\beta)$. If α is odd and β is even we need to compare $\alpha v_n \beta$ with $\alpha\beta w_n$. But these are homotopic rel enpoints, since all vertices are looped and so we have a sequence of spider moves that move the repeated vertex down through β to the end. Similarly, if α and β are both odd we are comparing $\alpha v_n \beta w_n$ to $\alpha\beta$; again we have a sequence of spider moves that take the repeated vertex to the end to get $\alpha\beta w_n w_n$ which prunes to $\alpha\beta$.

We define an inverse map $\Lambda(\alpha, id) = \alpha$ and $\Lambda(\alpha, \tau) = \alpha v_n$. Then $\Lambda\Psi$ and $\Psi\Lambda$ are identities since on evens they are identities and on odds they send α to $\alpha v_n v_n$ which prunes to α, showing that Ψ is an isomorphism.

□

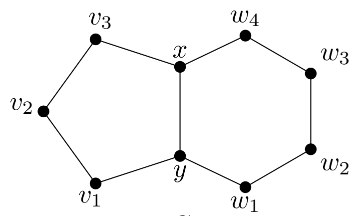
Now we look at a graph created from a union and prove a modified van Kampen theorem [7, 15]. Recall that the classic van Kampen theorem allows us to calulate the fundamental groupoid of a pushout of two spaces using the free product of the fundamental groupoid of the component spaces, amalgamated over the fundamental groupoid of the intersection. We obtain the analogous result here for the union of graphs provided a technical condition on diamonds is met.

Theorem 5.4. *If $G = G_1 \cup G_2$ and all diamonds (as in Definition 4.6) of G are fully contained in either G_1 or G_2 then $\Pi(G) = \Pi(G_1) *_{\Pi(G_1 \cap G_2)} \Pi(G_2)$, the free product of $\Pi(G_1)$ and $\Pi(G_2)$ amalgamated over the common subgroupoid $\Pi(G_1 \cap G_2)$.*

Proof. We verify that $\Pi(G)$ has the universal property for a pushout diagram: suppose we have two groupoid maps $\varphi_1, \varphi_2 : \Pi(G_i) \to R$ for some groupoid R. Then we can define a map $\varphi : \Pi(G) \to R$

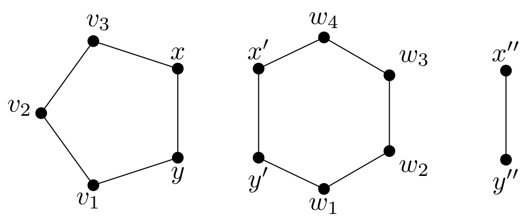
as follows. For any morphisms $\alpha = (vw_1w_2 \dots w)$, we can break it up into pieces $\alpha = \alpha_1 * \alpha_2\alpha_3 \dots$ where each piece is contained in either G_1 or G_2. Then we define $\varphi(\alpha) = \varphi_i(\alpha_k)$ where we apply the map φ_1 to pieces in G_1 and φ_2 to pieces in G_2. This is well-defined, since if any piece is in both G_1 and G_2 then $\varphi_1 = \varphi_2$, and any spider move will take place in either G_1 or G_2 by our diamond condition. It is unique since the functor φ needs to agree with φ_1 and φ_2 and respect the concatenation operation. Thus $\Pi(G)$ is the groupoid pushout, given by the free product with amalgamation. □

Example 5.5. Let G be the graph depicted below:



G

The graph G is the union of subgraphs $G_1 \cong C_5, G_2 \cong C_6$ who have an intersection $K \cong K_2$. Since the original graph G does not contain diamonds, all diamonds of G are vacuously contained in G_1 or G_2.



G_1 G_2 K

The objects of $\Pi(G_1)$ are $\{x, y, v_1, v_2, v_3\}$, and given any of these objects, the isotropy group is isomorphic to $\mathbb{Z}$. Thus $\Pi(G_1)$ has a $\mathbb{Z}$ worth of morphisms between any objects. Similarly the objects of $\Pi(G_2)$ are $\{x', y', w_1, w_2, w_3, w_4\}$ and the isotropy groups are $\mathbb{Z}$, with a $\mathbb{Z}$ worth of morphisms between objects. The objects of $\Pi(K)$ are $\{x'', y''\}$, and since the isotropy is trivial here, the only morphisms are $x''y''$ and $y''x''$.

So when we take $\Pi_1(G_1) *_K \Pi(G_2)$, we identify the objects x, x', x'' to be a single object, and similarly identify y, y', y''. This gives us a groupoid whose objects are $\{x, y, v_1, v_2, v_3, w_1, w_2, w_3, w_4\}$. Since the isotropy of K is trivial, we have that the isotropy of any element is the free product $\mathbb{Z} * \mathbb{Z}$. Thus, the morphisms between elements of $\Pi(G)$ are in 1-1 correspondence with $\mathbb{Z} * \mathbb{Z}$.

6. Comparison with Other Fundamental Groups for Graphs

Fundamental groups and groupoids based on A-homotopy have been studied by [1,3,13]. These are all based on a different category of graphs, in which the morphisms allow two connected vertices to be collapsed. Because the underlying category of graphs for this theory is different, we did not make a direct comparison functor between our fundamental group and these constructions. However, we observe that the definitions are similar, but the fundamental group of A-homotopy theory treats both 3- and 4-cycles as contractible, while our fundamental group contracts 4-cycles but not 3-cycles. The parity related patterns that we see with our definition thus do not appear in the A-homotopy setting.

There is another fundamental group which has been defined based on $\times$-homotopy by [11]. This relates to a looped version of our fundamental groupoid which we sketch here. Our fundamental groupoid $\Pi(G)$ is based on homotopy classes walks defined by $P_n \to G$. It is also possible to define a looped fundamental groupoid based on homotopy classes of walks $I_n^\ell \to G$, so that all the vertices in the objects and in any walk need to be looped.

Definition 6.1. Let $\alpha = (v_0 v_1 v_2 \dots v_n)$ be a looped walk in G. We say that α is **ℓ-prunable** if it is prunable or if $v_i = v_{i+1}$ for some i. We define a **ℓ-prune** of α either to be a prune or to be given by a walk α' obtained by deleting one of the repeated vertices v_i from the walk when $v_i = v_{i+1}$: if

$$\alpha = (v_0 v_1 v_2 \dots v_{i-1} v_i v_i v_{i+2} v_{i+3} \dots v_n)$$

then the ℓ-prune of α is

$$\alpha' = (v_0 v_1 v_2 \dots v_{i-1} v_i v_{i+2} \dots v_n)$$

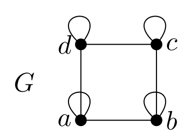
Then we can make the following definition.

Definition 6.2. For a graph G, we define the **looped fundamental groupoid** of G, $\Pi^\ell(G)$, as follows:

- objects of $\Pi^\ell(G)$ are looped vertices of the graph G
- a morphism from v_0 to v_n in $\Pi^\ell(G)$ is given by a ℓ-prune class of walks from v_0 to v_n defined up to homotopy rel endpoints
- composition of morphisms is defined using concatenation of walks

This definition applies to any graph, but will only depend on the induced subgraph of looped vertices. Verifying that this is a well-defined groupoid and that Π^ℓ defines a homotopy invariant for finite graphs is a straightforward adaptation of the arguments given in Section 4 for the unlooped version. However, the looped groupoid is NOT equivalent to the unlooped even if all vertices are looped, since the requirement for a homotopy of I_n^ℓ is stricter than that for P_n and any spider move must swap images between **connected** vertices. This is illustrated in the example below.

Example 6.3. Consider G depicted below:



Consider the walk (abc) from a to c. In $\Pi(G)$, this walk is homotopic to (adc) via a spider-move from b to d. However in $\Pi^\ell(G)$ $(abc) \neq (adc)$, since there is no homotopy from I_3^ℓ taking b to d: since the vertices of I_3^ℓ are looped, such a spider move would require an edge from b and d.

We can think of looped walks as infinite length walks which stabilize at some point, so for some n, then for all $m \geq n$ the walk is the same vertex v_n. This gives us a presentation that is very similar to the definition of the fundamental group given by [11] for exponential objects H^G, developed in the context of pointed graphs. In fact, results below will show that the isotropy of our groupoid at a chosen base vertex coincides with Dochtermann's based group. Our definition does not require a choise of basepoint and applies to any graph, not just an exponential one.

To show that we recover the base group of Dochtermann, we make use of [11], Corollary 4.8 giving a connection to the polyhedral hom complex. We will prove a version of this result that applies to our groupoids, using the same approximation techniques.

Definition 6.4. [2] The polyhedral complex $\Delta = \mathrm{Hom}(G, H)$ has cells indexed by functions $\eta : V(G) \to 2^{V(H)} \backslash \{\emptyset\}$, such that if $x \sim y \in E(G)$, then $\eta(x) \times \eta(y) \subseteq E(H)$. The boundary attachments of the cells are defined by inclusions $\eta \subseteq \eta'$.

The 2-skeleton of this complex is described explicitly by:

- 0-cells are indexed by graph homomorphism $G \to H$.
- 1-cells will have a single vertex f such that $|\eta(f)| = 2$. Then η defines a 1-cell connecting the two 0-cells indexed by the morphisms defined by the two choices of image of v, and these two are connected by a spider move.
- 2-cells are of two types: A single vertex v with $|\eta(v)| = 3$, giving a 2-cell filling in a triangle of shape (A), or two vertices v, w with $|\eta(v)| = |\eta(w)| = 2$, giving a 2-cell filling in square of shape (B):

(A)
$$\begin{tikzcd}[arrows=-] g_1 f \arrow[r] \arrow[d] & g_2 f \arrow[dl] \\ g_3 f & \end{tikzcd}$$

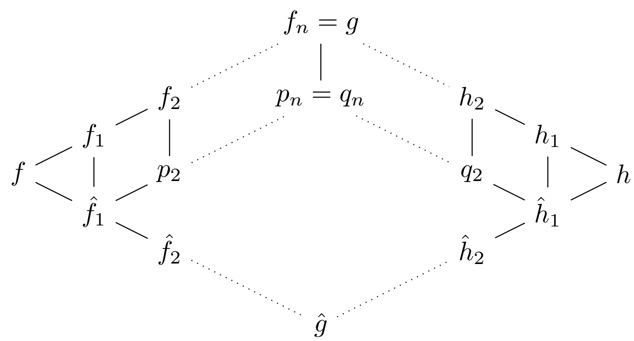
(B)
$$\begin{tikzcd}[arrows=-] g_1 h_1 f \arrow[r] \arrow[d] & g_2 h_1 f \arrow[d] \\ g_1 h_2 f \arrow[r] & g_2 h_2 f \end{tikzcd}$$

We will use this to show that the fundamental groupoid of $\mathrm{Hom}(G, H)$ is equivalent to the looped groupoid of the exponential graph H^G. To do this, we need the following lemma.

Lemma 6.5. *Given any four morphisms $f, g, \hat{g}, h : G \to H$ such that $(fgh) = (f\hat{g}h)$ in $\Pi^{\ell}(H^G)$ we can fill the interior of the 4-cycle $f, g, h, \hat{g}$ in with triangles and squares of the form (A) and (B).*

Proof. We induct on the total number of vertices which have different images under one or more pair of the morphisms $f, g, \hat{g}, f$. If $k = 1$ then all of these morphisms agree on everything but a single vertex, and we can fill in with triangles of form (A).

Now suppose the images of n vertices differ. We will choose an ordering for these vertices $a_1, \dots, a_n$, and assume that all morphisms to be discussed will agree on any other vertex of G. By Proposition 2.9, we have a sequence of n spider moves from f to g, consisting of morphisms $\{f_1, \dots, f_k\}$ where each f_j agrees with f on a_k for $k \geq j$, and agrees with f on a_k for $k < j$. Thus as we work through the d_j, we move the images of a_k in increasing order. Similarly, we have spider moves from h to g, consisting of h_j changing the images of the vertices in order from h to x; and $\hat{f}_j$ from f to $\hat{g}$, and lastly $\hat{g}_j$ from h to $\hat{g}$. We then fill out the 4-cycle with morphisms p_k, q_k as follows:



where p_j is defined to agree with f_j on all vertices except for a_1, and take the same value as $\hat{g}$ on a_1; and similarly q_j agrees with h_j on all vertices except for a_1, and with $\hat{g}$ on a_1. Then the bars in the diagram above are spider pairs because they only differ on a single vertex (a_1 for the vertical bars, and successive a_k for the diagonals), and the squares and triangles between the top and second lines are of the form (A) and (B).

Thus what remains is paths $(\hat{f}_1 p_n \hat{h}_1)$ and $(\hat{f}_1 \hat{g} \hat{h}_1)$ who all agree on a_1, and thus disagree on $n-1$ vertices. Our inductive hypothesis fills in the interior of this interior 4-cycle.

□

Theorem 6.6. *Let $K = H^G$ be the exponential graph, and let $\Delta = \mathrm{Hom}(G, H)$ of Definition 6.4. There is an equivalence of categories $\Pi^\ell(K) \simeq \Pi(\Delta)$ where $\Pi^\ell(K)$ is the looped groupoid from Definnition 6.2, and $\Pi(\Delta)$ is the topological fundamental groupoid of the space Δ.*

Proof. Define $\Phi : \Pi^\ell(K) \to \Pi(\Delta)$ as follows: if v is an object of $\Pi^\ell(K)$ then v is a looped vertex of $K = H^G$ which defines a morphism $G \to H$ which corresponds to a 0-cell. Send the object v to the object represented by this 0-cell in $\Pi(\Delta)$. If $\alpha = (v_0 v_1 v_2 \dots v_n)$ represent an morphism of $\Pi^\ell(K)$, then $v_i \sim v_{i+1}$ in H^G, and so we have a sequence of spider moves $v_i f_1 f_2 \dots f_m v_{i+1}$ connecting the morphisms v_i and v_{i+1}, each connecting morphisms which differ in the image of a single vertex v, and thus corresponding to a 1-cell of Δ. Send α to the path along the 1-cells. This is independent of choice of spider move, since a different choice would correspond to a different order of moving the vertices one at a time, and we can fill in two such choices with a square of type (B) from the 2-skeleton. Thus two choices of spider realizations are homotopic in Δ.

Now if $[\alpha] = [\beta]$ in $\Pi^\ell(K)$, then they are homotopic rel endpoints up to ℓ-pruning. A prune comes from a repeated vertex, which would be mapped under Φ to a path in Δ which was constant at that vertex, homotopic to the walk without the pause. And any homotopy rel endpoints could be realized by a sequence of spider moves which could be filled in by Lemma 6.5.

To show that Φ is essentially surjective on objects, let $x \in \Pi(\Delta)$ be an object of the fundamental groupoid and hence a point in Δ. Choose any corner y of its simplex and a path γ from x to y. Then y is in the image of Φ and γ represents an morphism from x to y.

To show that Φ is full on morphisms, suppose that there is a path in Δ from 0-cell v to w. Then γ is homotopic to γ' that lies in the 1-skeleton of Δ by cellular approximation [15], and γ' is in the image of Φ. to show that Φ is faithful on morphisms, suppose that $\alpha, \beta : v \to w$ in $\Pi(K)$ given by paths $\alpha = (v v_1 v_2 \dots w)$ and $\beta = (v w_1 w_2 \dots w)$, such that $\Phi(\alpha) = \Phi(\beta)$ in $\Pi(\Delta)$. This means that there is a homotopy from α to β in Δ which we may assume lives in the 2-skeleton, so lives on triangles of type (A) and squares of type (B). Each of these corresponds to spider moves showing that $[\alpha] = [\beta]$ in $\Pi^\ell(K)$. □

Corollary 6.7. *The isotropy group of a vertex v in $\Pi^\ell(H^G)$ is isomorphic to the based group $[1_*, \Omega(H^G)]_\times$ defined in* [11].

Proof. It is shown in [11] that the based group $[1_*, \Omega(H^G)]_\times$ is isomoprhic to the fundamental group of the simplicial complex $\pi_1(\mathrm{Hom}(G, H))$. □

7. Future Directions

This work is part of a broader effort to understand and develop a theory of ×-homotopy for graphs. This work introduced our fundamental groupoid, a computable ×-homotopy invariant. Future directions for expanding on this include using the fundamental groupoids defined here to develop a theory of covers and deck transformations of graphs which lift ×-homotopy, currently a work in progress [9]. The definition of the fundamental groupoid also opens up the natural question of whether it is possible to develop higher homotopy groups for ×-homotopy, analogous to those which have been defined for A-homotopy [1, 3, 22].

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