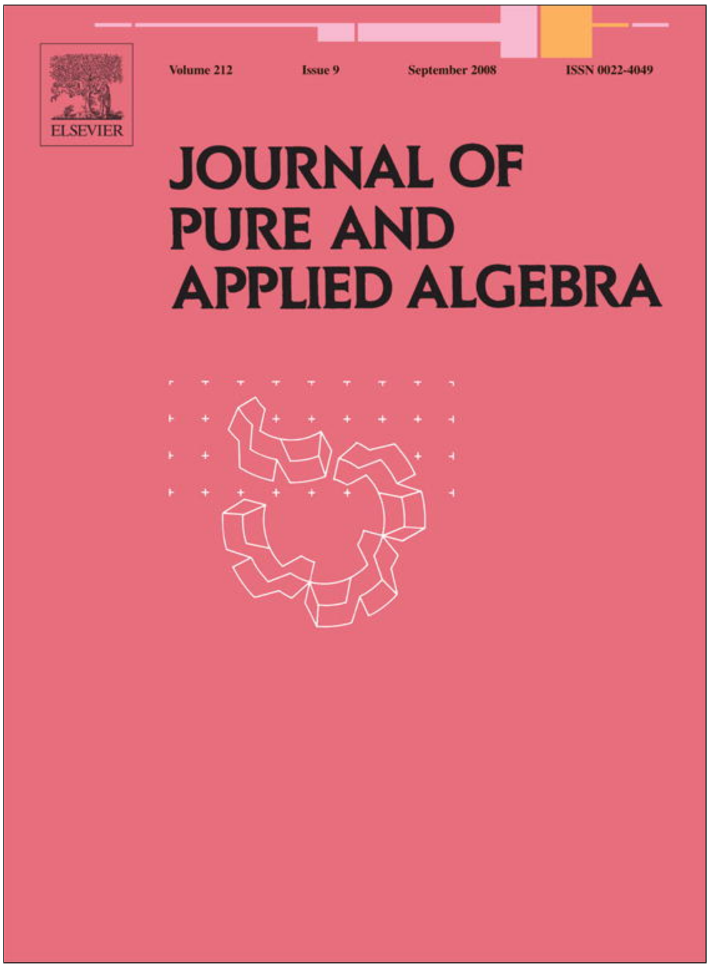

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A van Kampen theorem for equivariant fundamental groupoids

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Abstract

The equivariant fundamental groupoid of a G-space X is a category which generalizes the fundamental groupoid of a space to the equivariant setting. In this paper, we prove a van Kampen theorem for these categories: the equivariant fundamental groupoid of X can be obtained as a pushout of the categories associated to two open G-subsets covering X. This is proved by interpreting the equivariant fundamental groupoid as a Grothendieck semidirect product construction, and combining general properties of this construction with the ordinary (non-equivariant) van Kampen theorem. We then illustrate applications of this theorem by showing that the equivariant fundamental groupoid of a G-CW complex only depends on the 2-skeleton and also by using the theorem to compute an example.

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1. Introduction

The fundamental groupoid of homotopy classes of paths in a space X is a category which is closely related to the more famous fundamental group, but considers paths between different points in X. The advantage of the groupoids is that they more gracefully accommodate disconnected spaces and multiple basepoints. This is particularly relevant when studying equivariant homotopy theory, looking at spaces which have an action of a group G. If a G-space has no fixed points, then there is no good choice of a single basepoint. Moreover, the G-equivariant homotopy type of a space X depends on the structure of all of its fixed sets $X^H = \{x \in X | hx = x \; \forall h \in H\}$ for closed subgroups H of G; and a space which is itself connected may have disconnected fixed point sets. Therefore we cannot avoid the problem of disconnected spaces by looking at connected components separately.

The equivariant fundamental groupoid is a category we associate to a G-space X which generalizes the fundamental groupoid. It is designed to encompass information about the fixed sets of X as well as the space itself. This category was defined by tom Dieck [11] and has been used in a variety of equivariant constructions such as covering spaces and orientations [3,4]. Despite the name, this category is not itself a groupoid, but rather constructed out of groupoids

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via the Grothendieck semidirect product construction [6,8]; thus it has the structure of what are called variously 'catégories fibrées en groupoids' (Grothendieck [7]) or 'bundles of groupoids' in the topological setting (Costenoble, May and Waner [4]).

In this paper, we prove a van Kampen theorem which describes the equivariant fundamental groupoid of a G-space X as a pushout of the categories associated to two open G-subsets covering X. The proof follows from the ordinary van Kampen theorem [1] using general properties of the Grothendieck semidirect product construction. We then apply this theorem to show that the equivariant fundamental groupoid of a G-CW complex only depends on the 2-skeleton; and also illustrate the use of this theorem by computing an example.

The organization of this paper is as follows. The basic definitions and the statement of the main theorem are given in Section 2. Section 3 talks about the Grothendieck semidirect product construction as a weak colimit, and uses this to show that this construction commutes with pushouts (and in fact general colimits). Section 4 puts together the results of Section 3 and the ordinary van Kampen theorem, to prove the main result. Section 5 gives the application to G-CW complexes, and Section 6 contains the example calculation.

2. Equivariant fundamental groupoids

If X is a G-space, its equivariant homotopy type depends not only on the space itself but also on the homotopy type of the fixed spaces X^H for the various closed subgroups H of G. So we often think of a G-space as a collection of the spaces $\{X^H\}$, together with the inclusions and relations induced by the G-action; and any equivariant invariants defined should include information about these spaces and their relations to each other. This is the starting point for trying to define an equivariant fundamental groupoid. We start by quickly reviewing some important aspects of this theory [5,9].

One way of organizing this fixed point data is using the orbit category $\mathcal{O}_G$. This is the category whose objects are the canonical orbit types G/H for closed subgroups H of G, and whose morphisms are the equivariant maps between them. The idea is that any equivariant 'point' x of a G-space X comes with an entire orbit $\{gx | g \in G\}$, and this will be isomorphic to one of these canonical orbit types. Thus the orbit category contains building blocks for all G-spaces and the equivariant maps between them.

A little group theory reveals the form of the equivariant map between orbits $G/H \to G/K$. Any such map has the form $\alpha : gH \mapsto g\alpha K$ for some $\alpha \in G$ such that $\alpha^{-1}H\alpha \subseteq K$, and two such maps are the same if and only if $\alpha K = \alpha' K$. Thus the morphism set $\mathcal{O}_G[G/H, G/K]$ is isomorphic to $(G/K)^H$. It is also useful to observe that any morphism in $\mathcal{O}_G$ is the composite of an automorphism of conjugacy classes $G/H \to G/gHg^{-1}$ and a projection $G/H \to G/K$ for $H \subseteq K$.

To relate the orbit category to the fixed point data, we observe that a G-map $x : G/H \to X$ is equivalent to a point in X^H under the correspondence $x \leftrightarrow x(eH)$. Moreover, it is easy to check that if $x \in X^H$ then $\alpha x \in X^{\alpha H \alpha^{-1}}$; therefore if $\alpha : G/H \to G/K$ defines a map in $\mathcal{O}_G$ for $\alpha^{-1}H\alpha \subseteq K$, we can define a map $X^K \to X^H$ by $x \to \alpha x$. So we can describe the diagram of the fixed sets and their relations as a functor $\mathcal{O}_G^{op} \to \mathit{Spaces}$ defined by $G/H \to X^H$.

Thus our first attempt to define an equivariant fundamental groupoid might be to look at all of the fundamental groupoids of all the fixed sets, and define a functor $\mathcal{O}_G^{op} \to \mathsf{Cat}$ by $\underline{\Pi}_X(G/H) = \Pi(X^H)$, giving an object of $\mathsf{Cat}^{\mathcal{O}_G^{op}}$. Working with these functors can be somewhat awkward, however, and we often instead create a single category out of all of these groupoids.

To do this, we use the Grothendieck semidirect product construction, which gives a method for taking a functor $F : \mathcal{C}^{op} \to \mathsf{Cat}$ and combining the image categories $F(C)$ together into a single category. The Grothendieck semidirect product $\int_{\mathcal{C}} F$ is a category whose objects are pairs (C, X), with C an object in $\mathcal{C}$ and X an object in $F(C)$, and whose arrows $(C, X) \to (C', Y)$ are pairs (f, υ) with $f : C \to C'$ an arrow in $\mathcal{C}$, and $\upsilon : X \to F(f)(Y)$ an arrow in $F(C)$. This category has an obvious projection functor to $\mathcal{C}$, so this construction turns a functor $F \in \mathsf{Cat}^{\mathcal{C}^{op}}$ into an object in the slice category $\mathsf{Cat}/\mathcal{C}$; the projection retains information about the different pieces $F(C)$ used to create it. This category is also universal in a sense that will be discussed in Section 3.

If we apply this construction to the functor $\underline{\Pi}_X$ defined by $\Pi(X^-)$, we get a category $\Pi_G(X) = \int_{\mathcal{O}_G} \underline{\Pi}_X$, the *equivariant fundamental groupoid* of X. Explicitly, the objects of this category are pairs $(G/K, x)$ where $x \in X^K$ is a point; and the morphisms $(G/K, x) \to (G/H, y)$ are defined by pairs (α, γ) where $\alpha : G/K \to G/H$ and γ is a homotopy class of paths $\gamma : I \to X^K$ from x to αy. This is precisely the original definition given in tom Dieck [11, Definition 10.7]; we learned about the Grothendieck interpretation of this category from [10].

Note that even though all of the categories $\Pi(X^-)$ are groupoids, $\Pi_G(X)$ itself is not, since it includes morphisms (α, γ) for which the map α is not invertible. Geometrically, this is coming from the fact that the 'points' in $\Pi_G(X)$ come labeled with a subgroup H such that $x \in X^H$. However, it is entirely possible for the same point to be fixed by more than one subgroup, and therefore to appear multiple times labeled with different subgroups. In fact, since if $H \subset K$, then any point in X^K will also be fixed by H. Thus we can have a non-invertible path (α, γ) where $\alpha : G/K \to G/H$ is a projection, and γ is a path in X^K which ends at a G/H-labeled point $y \in X^K \subseteq X^H$ (note that αy is just $y \in X^H$ relabeled with (G/K).

However, we can recover the groupoids $\Pi(X^H)$ by looking at the projection to $\mathcal{O}_G$, and restricting to the subcategory over an object G/H and its identity map. In fact, $\Pi_G(X)$ has more structure coming from the fact that the component categories are *groupoids*.

Definition 2.1 (*[7]*). A *category fibred in groupoids* over a base category $\mathcal{C}$ is a small category $\mathcal{D}$ in $\mathsf{Cat}/\mathcal{C}$ such that for any choice of morphism $f : C' \to C$ of $\mathcal{C}$ and lifting D of C, there is a 'pullback'

$$\begin{tikzcd}
D' \arrow[r, dashed, "\hat{f}"] \arrow[d, dashed, "\pi"] & D \arrow[d, "\pi"] \\
C' \arrow[r, "f"] & C
\end{tikzcd}$$

which is unique up to unique isomorphism: if $\hat{f}' : D'' \to D$ is another pullback of D over f, then there is a unique isomorphism $\theta : D'' \to D'$ over $id_{C'}$ such that $\hat{f}' = \hat{f} \circ \theta$.

This definition implies that if we fix an object C of $\mathcal{C}$, the fibre subcategory over C and its identity map is a groupoid: if $f : D' \to D$ is a morphism over id_C, then D' is a pullback of D over id_C. But clearly $id : D \to D$ is also such a pullback, and so Definition 2.1 states that there is a unique isomorphism $g : D \to D'$ over id_C such that $fg = id$. Then since g is an isomorphism, g^{-1} exists and so $f = g^{-1}gf = g^{-1}$. Thus any map over id_C has an inverse map.

Conversely, it is easy to see that for any functor $F : \mathcal{C}^{op} \to \mathsf{Gpd}$, the Grothendieck construction $\int_{\mathcal{C}} F$ is a category fibred in groupoids over $\mathcal{C}$: if $f : C' \to C$ is a map in $\mathcal{C}$ and $(C, D \in F(C))$ is an object of $\int_{\mathcal{C}} F$, then a pullback (C', D') can be defined by taking $D' = F(f)D$ with the map (f, id); so $f : C' \to C$ and $id : D' \to F(f)D$ as required. Then the invertibility of all maps in the groupoid $F(C)$ can be used to show that the pullback map is unique up to unique isomorphism.

In particular, the equivariant fundamental groupoid category $\Pi_G(X)$ is fibred in groupoids over $\mathcal{O}_G$; the pullback of $(G/H, x \in X^H)$ along $\alpha : G/K \to G/H$ can be defined by $(G/K, \alpha x)$, since $\alpha x \in X^{\alpha H \alpha^{-1}}$ and $K \subseteq \alpha H \alpha^{-1}$. Notice also that this construction allows for a morphism from $(G/H, x)$ to $(G/K, y)$ to be described by a path γ which ends at αy for a point in the orbit of y. Thus we are looking at each orbit as a single piece, and paths go from one orbit to another. This also means that if x is fixed by $\alpha \in G$, we have non-trivial 'constant' paths from $(G/H, x)$ to itself over $\alpha : G/H \to G/H$, since (α, c_x) is a path from x to $x = \alpha x$. More generally, we always have constant 'relabel' maps from an orbit to itself, since there is a morphism from $(G/H, x)$ to $(G/\alpha^{-1}H\alpha, \alpha^{-1}x)$ defined by (α, c_x).

Homotopic paths in X are identified in the fundamental groupoid. For a compact Lie group G, we may also have homotopic 'constant' paths coming from the topology on the morphisms of $\mathcal{O}_G$. In certain contexts we want to identify these. Thus we consider the following construction.

We observed earlier that the morphisms $\mathcal{O}_G[G/H, G/K] \simeq (G/H)^H$, and so inherits a topology from G. Thus $\mathcal{O}_G$ is a category enriched over topological spaces, and we can form the homotopy category $h\mathcal{O}_G$ by replacing the maps $\mathcal{O}_G[G/H, G/K]$ with $\pi_0\mathcal{O}_G[G/H, G/K]$. Similarly, the Grothendieck category $\int_{\mathcal{O}_G} \underline{\Pi}_X$ is also enriched over spaces, with the topology coming from the topology on $\mathcal{O}_G$: since the morphisms are given by (α, γ) where $\alpha : G/H \to G/K$ and $\gamma : x \to \alpha y$, we can think of this as a product topology where the morphisms of $\underline{\Pi}_X$ are discrete. Therefore we can again consider the homotopy category $h \int_{\mathcal{O}_G} \underline{\Pi}_X$; this is the *discrete equivariant fundamental groupoid* $\Pi_G^d(X)$. It is a category fibred in groupoids over the homotopy orbit category $h\mathcal{O}_G$.

Explicitly, we can describe this category as follows. Suppose that $(\alpha, \gamma) : (G/H, x) \to (G/K, y)$ is one map in $\int_{\mathcal{O}_G} \underline{\Pi}_X$, and (α', γ') is another. A path between these objects is given by (σ, Λ) where $\sigma : \alpha \to \alpha'$ is a path in

$\mathcal{O}_G[G/H, G/K]$ and $\Lambda : I \times I \to X^H$ describes a homotopy of paths $\Lambda(t, -) = \gamma_t$ where $\gamma_0 = \gamma$, $\gamma_1 = \gamma'$ and γ_t is a path from x to $\sigma(t)y$. That is, we have the following diagram:

$$\begin{tikzcd}
x \arrow[r, dash, "\gamma_0"] \arrow[d, dash, "x"'] \arrow[dr, phantom, "\Lambda"] & \alpha y \arrow[d, dash, "\sigma y"] \\
x \arrow[r, dash, "\gamma_1"'] & \alpha' y
\end{tikzcd}$$

We identify (α, γ) and (α', γ') in the homotopy category $h\int_{\mathcal{O}_G} \underline{\Pi}_X = \Pi_G^d(X)$, and recover the original definition of the discrete category given by tom Dieck [11, Definition 10.9].

The definitions above use the full fundamental groupoids on the spaces X^H, with paths between all points in X^H. As in the non-equivariant case, we generally want to restrict to a smaller set of basepoints. For each $\Pi(X^H)$, we can choose a set $A_H \subseteq X^H$ and restrict to the full subcategory $\Pi(X^H, A_H)$ with objects given just by points in A_H; if A_H contains at least one point in each connected component of X^H, this restriction is an equivalence of categories.

For our semidirect product category, we need to choose enough points A_H of X_H, and also ensure that these points are compatible between the various fixed sets. One way to do this is to use a single G-subset A of X such that $X^H \cap A = A_H$ contains enough basepoints for all subgroups H.

Definition 2.2. A G-subset A of X is *thorough* in X if for each subgroup H of G, there is at least one $a \in A$ in each component of X^H.

Then we can restrict the fundamental groupoid $\Pi_G(X)$ to objects given by the basepoints A; that is, to the objects defined by all pairs $(G/H, a)$ for $a \in X^H \cap A$. These objects can also be described in the tom Dieck definition as the equivariant points $G/H \to X$ given by $eH \to a \in X^H$. (Note that the point a of X may show up as multiple objects, since it might be contained in a number of fixed sets.) Note that on each component groupoid we are restricting to $\Pi(X^H, X^H \cap A)$, which is a deformation retract of the full groupoid $\Pi(X^H)$. We denote the full subcategory of $\Pi_G(X)$ on these objects by $\Pi_G(X, A)$.

If a G-subset A is thorough in X, then there is a functor $\underline{\Pi}_{X,A} : \mathcal{O}_G^{op} \to \mathsf{Cat}$ defined by $\underline{\Pi}_{X,A}(G/H) = \Pi(X^H, A)$, and $\Pi_G(X, A) = \int_{\mathcal{O}_G} \underline{\Pi}_{X,A}$; and so $\Pi_G(X, A)$ is also a category fibred in groupoids over $\mathcal{O}_G$.

Any equivariant map between G-spaces $X \to Y$ gives functors $\Pi(X^H) \to \Pi(Y^H)$, and so naturally induces a map $\Pi_G(X) \to \Pi_G(Y)$. In particular, if U is a subset of X closed under the action of G, then the inclusion induces a map $\Pi_G(U) \to \Pi_G(X)$. If A is a set of basepoints for X, then for any subset $U \subseteq X$, we abbreviate $\Pi_G(U, U \cap A)$ by $\Pi_G(U, A)$.

The main theorem of this paper is:

Theorem 2.3. *Suppose that U_1 and U_2 are open G-subsets of X such that $X = U_1 \cup U_2$ with $U_1 \cap U_2 = U_3$; and suppose that A is a thorough subset of X such that $A \cap U_i$ is also thorough in each U_i for $i = 1, 2, 3$. Then*

$$\begin{tikzcd}
\Pi_G(U_3, A) \arrow[r] \arrow[d] & \Pi_G(U_1, A) \arrow[d] & \text{and} & \Pi_G^d(U_3, A) \arrow[r] \arrow[d] & \Pi_G^d(U_1, A) \arrow[d] \\
\Pi_G(U_2, A) \arrow[r] & \Pi_G(X, A) & & \Pi_G^d(U_2, A) \arrow[r] & \Pi_G^d(X, A)
\end{tikzcd}$$

are pushout diagrams in Cat*, and also in* $\mathsf{Fib}(\mathcal{O}_G)$*, respectively* $\mathsf{Fib}(h\mathcal{O}_G)$ *of categories fibred in groupoids over $\mathcal{O}_G$, respectively $h\mathcal{O}_G$.*

To prove this, we use the fact that the Grothendieck semidirect product category is a weak colimit to show that this construction commutes with general colimits, including pushouts. This fact, combined with the ordinary van Kampen theorem shows that we get a pushout diagram on the equivariant fundamental groupoid categories.

3. Weak Colimits and the Grothendieck semidirect product

We discuss the universal property of the Grothendieck semidirect product construction. We will use it to show that the construction commutes with pushouts, a fact that we will use in the proof of our main theorem.

Given functors $F, G : \mathcal{C}^{op} \to \mathsf{Cat}$, a weak natural transformation $\alpha : F \Rightarrow G$ consists of a family of functors $\alpha^C : F(C) \to G(C)$, one for each object C in $\mathcal{C}$, and a family of natural transformations

$$\begin{tikzcd}
F(C) \arrow[r, "\alpha^C"] & G(C) \\
F(C') \arrow[u, "F(f)"] \arrow[r, "\alpha^{C'}"'] \arrow[ur, phantom, "\Downarrow \alpha^f"] & G(C') \arrow[u, "G(f)"']
\end{tikzcd} \qquad \alpha^C F(f) \overset{\alpha^f}{\Rightarrow} G(f)\,\alpha^{C'}$$

one for each arrow $f : C \to C'$ in $\mathcal{C}$, satisfying the usual coherence law for composition. We also require that $\alpha^{id_C} = id_{\alpha^C}$. We write $\mathsf{Cat}^{\mathcal{C}^{op}_w}$ for the corresponding category of functors and weak natural transformations.

It is straightforward to extend the Grothendieck construction to weak natural transformations, so we have a functor

$$\int_{\mathcal{C}} : \mathsf{Cat}^{\mathcal{C}^{op}_w} \to \mathsf{Cat}.$$

This functor is a left adjoint to the diagonal (or constant) functor $P : \mathsf{Cat} \to \mathsf{Cat}^{\mathcal{C}^{op}_w}$. This in turn means that $\int_{\mathcal{C}}$ is a weak colimit, i.e. it has the following universal property: for any functor $F : \mathcal{C}^{op} \to \mathsf{Cat}$, there is a weak natural transformation

$$\mathbf{j} : F \Rightarrow \int_{\mathcal{C}} F,$$

given by the unit of the adjunction $\int_{\mathcal{C}} \dashv P$ (where we have identified a category with the constant functor it determines). This weak natural transformation consists of a family of functors $\mathbf{j}^C : F(C) \to \int_{\mathcal{C}} F$, one for any object $C \in \mathcal{C}$, and a family of natural transformations $\mathbf{j}^f : \mathbf{j}^{C_0} F(f) \Rightarrow \mathbf{j}^{C_1}$, one for each arrow $f : C_0 \to C_1$ in $\mathcal{C}$, such that:

- $\mathbf{j}^{id_C} = id_{\mathbf{j}^C}$.
- **(Coherence)** For any pair of composable arrow in $\mathcal{C}$, $C_0 \xrightarrow{f} C_1 \xrightarrow{g} C_2$, the following tetrahedron commutes

$$\begin{tikzcd}
 & F(C_2) \arrow[ddl, "F(g)"'] \arrow[d, "\mathbf{j}^{C_2}"] \arrow[ddr, "F(gf)"] & \\
 & \int_{\mathcal{C}} F & \\
F(C_1) \arrow[ur, "\mathbf{j}^{C_1}"] \arrow[rr, "F(f)"'] & & F(C_0) \arrow[ul, "\mathbf{j}^{C_0}"']
\end{tikzcd} \qquad \overset{\mathbf{j}^g}{\Rightarrow} \quad \overset{\mathbf{j}^{gf}}{\Leftarrow} \quad \overset{\mathbf{j}^f}{\Leftarrow}$$

(In this tetrahedron the front face is commutative and the coherence says that $\mathbf{j}^g \mathbf{j}^f F(g) = \mathbf{j}^{gf}$.)

Moreover these functors are universal in the usual colimit sense.

We want to use this universal property to show that the Grothendieck construction commutes with pushouts. More generally, we show that for any small category $\mathcal{D}$, this functor commutes with taking colimits over $\mathcal{D}$. To see this, we can consider the category of functors and weak natural transformations $\mathsf{Cat}^{(\mathcal{C}^{op}\times\mathcal{D})_w}$, and the subcategory $\mathsf{Cat}^{\mathcal{C}^{op}_w\times\mathcal{D}}$ with the same objects and whose arrows are those weak natural transformations $\alpha : F \Rightarrow G : \mathcal{C}^{op} \times \mathcal{D} \to \mathsf{Cat}$ which are strict in the second variable: that is, for any object $C \in \mathcal{C}$, the natural transformation

$$\alpha^{(C,-)} : F(C,-) \Rightarrow G(C,-) : \mathcal{D} \to \mathsf{Cat}$$

is strict.

Clearly we have an isomorphism $\mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}} \cong (\mathsf{Cat}^{\mathcal{C}_w^{op}})^{\mathcal{D}}$ and also an inclusion

$$(\mathsf{Cat}^{\mathcal{D}})^{\mathcal{C}^{op}} \cong \mathsf{Cat}^{\mathcal{C}^{op}\times\mathcal{D}} \subseteq \mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}}.$$

Therefore we have diagonal functors

$$\mathsf{Cat}^{\mathcal{C}_w^{op}} \to \mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}} \qquad \text{and} \qquad \mathsf{Cat}^{\mathcal{D}} \to \mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}},$$

and, as usual, the square of diagonal functors

$$\begin{tikzcd}
\mathsf{Cat} \arrow[r] \arrow[d] & \mathsf{Cat}^{\mathcal{D}} \arrow[d] \\
\mathsf{Cat}^{\mathcal{C}_w^{op}} \arrow[r] & \mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}}
\end{tikzcd}$$

commutes. Therefore the square with the corresponding left adjoint functors is also commutative.

Now the left adjoint to the diagonal functor

$$\mathsf{Cat}^{\mathcal{D}} \to (\mathsf{Cat}^{\mathcal{D}})^{\mathcal{C}^{op}} \subseteq \mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}}$$

takes a functor $F : \mathcal{C}^{op} \times \mathcal{D} \to \mathsf{Cat}$ to the functor in $\mathsf{Cat}^{\mathcal{D}}$ which takes an object $D \in \mathcal{D}$ to the category $\int_{\mathcal{C}} F(-, D)$. So composing through this corner, we see that the adjoint functor of the whole inclusion $\mathsf{Cat} \to \mathsf{Cat}^{\mathcal{C}_w^{op}\times\mathcal{D}}$ is the colimit over $\mathcal{D}$ of the Grothendieck semidirect products. Composing in the other direction tells us that it is also the Grothendieck semidirect product of the $\mathcal{D}$-colimits. Therefore these are equal.

Thus we have proved the following.

Proposition 3.1. *The Grothendieck semidirect product construction commutes with colimits.*

4. Proof of the main theorem

Recall that we are assuming that A is a thorough subset of X such that $A \cap U_i$ is also thorough in each U_i for $i = 1, 2, 3$.

Proof of Theorem 2.3. We first consider the functors of the fundamental groupoids of fixed sets for the various spaces. If $X = U_1 \cup U_2$ then for any subgroup H of G, the fixed set $X^H = U_1^H \cup U_2^H$, and the intersection $U_3 = U_1 \cap U_2$ also satisfies $U_3^H = U_1^H \cap U_2^H$.

Now, since $A \cap U_i$ is thorough in each U_i, $i = 1, 2, 3$, for each H, the fundamental groupoids $\Pi(U_i^H, A)$ and $\Pi(X^H, A)$ are deformation retracts of $\Pi(U_i^H)$ and $\Pi(X^H)$ respectively. So the ordinary van Kampen theorem [1] states that the square

$$\begin{tikzcd}
\Pi(U_3^H, A) \arrow[r, "i_1"] \arrow[d, "i_2"] & \Pi(U_1^H, A) \arrow[d, "j_1"] \\
\Pi(U_2^H, A) \arrow[r, "j_2"] & \Pi(X^H, A)
\end{tikzcd}$$

is a pushout of categories. Then, since pushouts in $\mathsf{Cat}^{\mathcal{O}_G^{op}}$ are computed objectwise and the Grothendieck semidirect product preserves pushouts by Proposition 3.1, the square

$$\begin{tikzcd}
\int_{\mathcal{O}_G} \underline{\Pi}_{U_3,A} = \Pi_G(U_3, A) \arrow[r, "i_1"] \arrow[d, "i_2"] & \int_{\mathcal{O}_G} \underline{\Pi}_{U_1,A} = \Pi_G(U_1, A) \arrow[d, "j_1"] \\
\int_{\mathcal{O}_G} \underline{\Pi}_{U_2,A} = \Pi_G(U_2, A) \arrow[r, "j_2"] & \int_{\mathcal{O}_G} \underline{\Pi}_{X,A} = \Pi_G(X, A)
\end{tikzcd}$$

is also a pushout diagram in Cat, and consequently in the slice category $\mathsf{Cat}/\mathcal{O}_G$ and in $\mathsf{Fib}(\mathcal{O}_G)$ of categories fibre on groupoids over $\mathcal{O}_G$.).

For the discrete version, we note that the homotopy category of a category enriched in topological spaces is created by applying the functor π_0 to the morphisms; and π_0 is the left adjoint of the inclusion of categories *Sets* → *Spaces*, using the discrete topology. So h is the left adjoint of the inclusion of ordinary categories into categories enriched over topological spaces, and so it commutes with pushouts (and more general colimits). Thus the previous diagram implies that

$$\begin{tikzcd}
\Pi^d_G(U_3, A) \arrow[r, "i_1"] \arrow[d, "i_2"] & \Pi^d_G(U_1, A) \arrow[d, "j_1"] \\
\Pi^d_G(U_2, A) \arrow[r, "j_2"] & \Pi^d_G(X, A)
\end{tikzcd}$$

is also a pushout diagram in Cat and in $\mathsf{Fib}(h\mathcal{O}_G)$. $\square$

Let us note that, as in the non-equivariant case, the fundamental groupoid $\Pi_G(X, A)$ is a fibre deformation retract of $\Pi_G(X)$ over $\mathcal{O}_G$. In fact, the inclusions $i_H : \Pi(X^H, A) \to \Pi(X^H)$ fit together to give a natural transformation $\mathbf{i} : \underline{\Pi}_{X,A} \Rightarrow \underline{\Pi}_X$; so, by applying the Grothendieck construction, we have an inclusion

$$\int_{\mathcal{O}_G} \mathbf{i} : \Pi_G(X, A) \to \Pi_G(X).$$

Moreover, we can define retractions $r_H : \Pi(X^H) \to \Pi(X^H, A)$ for each H as follows: given $x \in X^H$, there is an element $a^H_x \in A$ in the same connected component of x in X^H (this is our thoroughness condition). So we can choose a path $\gamma^H_x : x \to a^H_x$; this gives an isomorphism $x \to a^H_x$ in $\Pi(X^H)$. For $a \in A$, we choose the identity path. These paths can be used to define the retraction r_H.

There are many choices involved in the above process, so there is no reason to think that these r_H's fit together to give a natural transformation between $\underline{\Pi}_X$ and $\underline{\Pi}_{X,A}$. We can, however, define a weak natural transformation $\mathbf{r} : \underline{\Pi}_X \Rightarrow \underline{\Pi}_{X,A}$. For any map $\alpha : G/H \to G/K$ in $\mathcal{O}_G$, we just take the natural transformation $\mathbf{r}^\alpha$:

$$\begin{tikzcd}
\Pi(X^H) \arrow[r, "\mathbf{r}_H"] \arrow[dr, phantom, "\Downarrow \mathbf{r}^\alpha"] & \Pi(X^H, A) \\
\Pi(X^K) \arrow[r, "\mathbf{r}_K"'] \arrow[u, "\underline{\Pi}_X(\alpha)"] & \Pi(X^H, A) \arrow[u, "\underline{\Pi}_{X,A}(\alpha)"']
\end{tikzcd}$$

defined on $x \in \Pi(X^K)$ by the composition

$$\mathbf{r}^\alpha_x : a^H_{\alpha x} \xrightarrow{(\gamma^H_{\alpha x})^{-1}} \alpha x \xrightarrow{\alpha \gamma^K_x} \alpha a^K_x .$$

Since the Grothendieck construction takes weak natural transformations to functors of categories, we get a functor

$$\int_{\mathcal{O}_G} \mathbf{r} : \Pi_G(X) \to \Pi_G(X, A),$$

over $\mathcal{O}_G$, such that $\int_{\mathcal{O}_G} \mathbf{r} \int_{\mathcal{O}_G} \mathbf{i} = id$. With a little care, we can fix the natural transformations $i_H r_H \Rightarrow id$ to build a natural transformation $\int_{\mathcal{O}_G} \mathbf{i} \int_{\mathcal{O}_G} \mathbf{r} \Rightarrow id$.

We could use the van Kampen theorem on $\Pi_G(X)$ (coming from the ordinary van Kampen theorem on $\Pi(X^H)$) and the above fibre deformation retract to give an alternate proof of the van Kampen theorem for equivariant fundamental groupoids with restricted basepoints.

5. Application to G-CW complexes

An equivariant G-CW complex X is a colimit of a sequence of intermediate G-spaces, the n-skeletons X^n, where each X^n is created from X^{n-1} by pushouts of G-cells which have the form $G/H \times D^n$, attached along equivariant maps from the boundary $G/H \times S^n$. Every G-space can be approximated by a G-CW complex up to weak equivariant homotopy type, and these spaces possess many properties analogous to ordinary CW complexes, so this is a useful category to work with for equivariant homotopy [9].

We can use Theorem 2.3 to show that the equivariant fundamental groupoids of a G-CW complex depends only on the 2-skeleton for any set of basepoints which is contained in the 2-skeleton. Note that for finite groups, the G-skeleton can be taken to be just the ordinary CW skeleton of the fixed sets X^H, since we can arrange a cellular G-action; and then this fact follows immediately from the fact that each $\Pi(X^H)$ depends only on the 2-skeleton. For more general groups, however, the equivariant skeleton is quite distinct from the non-equivariant one, as it is impossible to get a cellular action; instead the G-skeleton is carrying the topology of G along with it, since the cells $G/H \times D^n$ can have a non-trivial topology coming from the structure of G/H.

Proposition 5.1. *Let X be a G-CW complex, and A be a set of basepoints $\{a_i\}$ which is contained in the 2-skeleton X^2. Then the inclusion $X^2 \to X$ induces isomorphisms $\Pi_G(X^2, A) \simeq \Pi_G(X, A)$ and $\Pi_G^d(X^2, A) \simeq \Pi_G^d(X, A)$.*

Proof. We start by looking at the fundamental groupoids of the cells $G/H \times D^n$ with basepoints restricted to the boundary points $G/H \times S^{n-1}$; note that these boundary points form a thorough set of basepoints.

The objects of $\Pi_G(G/H \times D^n, A)$ are defined by pairs $(G/K, p)$ where $p \in A^K$. Since we are taking A to be $G/H \times S^{n-1}$ here, if $K \subseteq H$ we can take any point $p = (g, x)$ where $g \in G/H$ and $x \in S^{n-1}$; otherwise A^K is empty. Given two such objects $(G/K_1, (g_1, x_1))$ and $(G/K_2, (g_2, x_2))$, the morphisms between them in $\Pi_G(G/H \times D^n, A)$ are given by pairs (α, γ) where $\alpha : G/K_1 \to G/K_2$ is specified by an element $\alpha \in G/K_2$, and γ is a homotopy class of paths in $G/H \times D^n$ from (g_1, x_1) to $(\alpha g_2, x_2)$. Hence $\alpha \in G/K_2$ must be such that $g_1 H$and $\alpha g_2 H$ are in the same connected component of G/H, and γ is determined by a homotopy class of paths from $g_1 H$ to $g_2 \alpha H$ in G/H. For $n \geq 2$, there is a unique homotopy class of paths between x_1 and x_2 for any two points of D^n. So this component does not add any information to the description of the path γ.

We have a similar description of $\Pi_G^d(G/H \times D^n, A)$. Objects are the same as above, and the morphisms are now determined by an equivalence class of the morphisms of $\Pi_G(G/H \times D^n, A)$; so an equivalence class of morphisms is specified by a class $[\alpha] \in \pi_0(G/K_2)^{K_1}$ such that $g_1 H$ and $\alpha g_2 H$ are in the same connected component of G/H.

Both descriptions have no reliance on the space D^n beyond the fact that it is connected and simply connected. In fact, the exact same descriptions give $\Pi_G(G/H \times S^{n-1})$ and $\Pi_G^d(G/H \times S^{n-1})$ for $n \geq 2$; so the inclusion $G/H \times S^{n-1} \to G/H \times D^n$ induces isomorphisms on equivariant fundamental groupoids. Similarly for any set of basepoints A contained in S^{n-1}, the inclusion $G/H \times S^{n-1} \to G/H \times D^n$ induces an isomorphism of the equivariant fundamental groupoids. By Theorem 2.3, the fundamental groupoids of X^n can be calculated from those of X^{n-1} as a pushout along these inclusion maps. Since the pushout of an isomorphism is an isomorphism, we see that the inclusion $X^2 \to X$ gives an isomorphism of equivariant fundamental groupoids for $n \geq 2$.

6. Example

In this section we work through an example of how to compute with the equivariant van Kampen theorem. We will use the group $G = \mathbb{Z}/2\mathbb{Z}$; this is a discrete group, so the discrete and regular equivariant fundamental groupoids are the same. Thus our base category is the orbit category $\mathcal{O}_G$ of G:

$$\begin{tikzcd} G/e \arrow[loop above, "\tau"] \arrow[r, "\rho"] & G/G, \end{tikzcd} \qquad \tau^2 = id, \qquad \tau\rho = \rho.$$

We consider the space $X = S^1$ with the G-action given by reflection through the vertical axis. So X has two fixed points N and S (for north and south pole). We cover X by two open G-sets U_1 and U_2 where U_1 is the upper half and U_2 is the lower half, overlapping in the intersection U_3 given by two small open contractible neighborhoods of the equator points $\{E, W\}$. Then $\tau E = W$, and so $A = \{N, S, E, W\}$ is a G-subset; and $A \cap U_i$ is thorough in each U_i for $i = 1, 2, 3$, and A is thorough in X. We will calculate $\Pi_G(X, A)$ using this cover.

Firstly, $U_3 \cap A = \{E, W\}$ and U_3^G is empty, so $\underline{\Pi}_{U_3,A}(G/G) = \emptyset$ and $\underline{\Pi}_{U_3,A}(G/e)$ has objects E and W. Therefore $\Pi_G(U_3, A)$ has two objects, $(G/e, E)$ and $(G/e, W)$, and two non-identity maps: the 'relabel' maps $(\tau, id_E) : (G/e, E) \to (G/e, W)$, and s $(\tau, id_W) : (G/e, W)$ to $(G/e, E)$. Since $\tau^2 = e$ and $\tau id_E = id_W$, these maps are inverse to each other.

Next, $U_2 \cap A = \{S, E, W\}$. The space U_2 is contractible, so $\underline{\Pi}_{U_2,A}(G/e) = \Pi(U_2, A)$ has a unique arrow between any two objects, and the action of τ switches E and W. The groupoid $\underline{\Pi}_{U_2,A}(G/G) = \Pi(U_2^G, A)$ is the single object S with its identity map, and the map induced by $\rho : G/e \to G/G$ is the inclusion of this subcategory. Thus $\Pi_G(U_2, A)$ has four objects: $(G/e, E)$, $(G/e, W)$, $(G/e, S)$ and $(G/G, S)$, and the resulting category looks like

$$\begin{tikzcd}
(G/e, W) \arrow[r, bend left, "{(\tau,\beta^{-1})}"] \arrow[r, shift left, "{(e,\alpha)}" description] \arrow[dr, "{(\rho,\alpha)}"'] & (G/e, S) \arrow[l, shift left, "{(e,\alpha^{-1})}" description] \arrow[l, bend left, "{(\tau,\beta)}"] \arrow[r, bend left, "{(\tau,\alpha^{-1})}"] \arrow[r, shift left, "{(e,\beta)}" description] \arrow[d, "{(\rho,id_S)}"] & (G/e, E) \arrow[l, shift left, "{(e,\beta^{-1})}" description] \arrow[l, bend left, "{(\tau,\beta^{-1})}"] \arrow[dl, "{(\rho,\beta^{-1})}"] \\
 & (G/G, S) &
\end{tikzcd}$$

where many of the composite maps have been suppressed. The category $\Pi_G(U_1, A)$ looks the same, except that we have N instead of S.

Before computing the pushout, we will reduce our categories a little. Any full representative subcategory $\mathcal{D}'$ of $\mathcal{D}$ induces functor $r : \mathcal{D} \to \mathcal{D}'$ which is a deformation retract, and this property preserved by pushouts over a functor which is injective on objects [2, Theorem 6.7.3]. Moreover, if these categories are all fibred in groupoids over $\mathcal{C}$, and $\mathcal{D}'$ is a *fibre representative subcategory* for which each object $D \in \mathcal{D}$ has an isomorphism within the fibre to an object $D' \in \mathcal{D}$ over $id_{\pi(D)}$ in $\mathcal{C}$, then the retract r and its pushout are fibre deformation retracts over $\mathcal{C}$.

Now in $\Pi_G(U_1, A)$, all the objects $(G/e, P)$ are isomorphic over $e = id_{G/e}$ in $\mathcal{O}_G$. Thus the full subcategory $\mathcal{D}'$ of $\Pi_G(U_1, A)$ on objects $(G/e, E)$, $(G/e, W)$ and $(G/G, N)$ is a fibre representative subcategory. The category $\mathcal{D}'$ has the form

$$\begin{tikzcd}
(G/e, W) \arrow[rr, bend left, "\gamma"] \arrow[rr, shift left, "\phi" description] \arrow[dr, "\rho_W"'] & & (G/e, E) \arrow[ll, shift left, "\phi^{-1}" description] \arrow[ll, bend left, "\gamma^{-1}"] \arrow[dl, "\rho_E"] \\
 & (G/G, N) &
\end{tikzcd}$$

where the abbreviated morphisms stand for: $\gamma = (\tau, id_W)$, $\gamma^{-1} = (\tau, id_E)$, $\phi = (e, \beta\alpha)$, $\phi^{-1} = (e, \alpha^{-1}\beta^{-1})$, $\rho_W = (\rho, \alpha)$ and $\rho_E = (\rho, \beta^{-1})$. Using these full descriptions and the semidirect product rule for composition, we can calculate that $\phi^{-1}\gamma = \gamma^{-1}\phi$ is the non-trivial automorphism of $(G/e, W)$ over τ, and similarly $\phi\gamma^{-1} = \gamma\phi^{-1}$ is an automorphism of $(G/e, E)$; and that $\rho_E\phi = \rho_E\gamma = \rho_E$, and likewise $\rho_W\phi^{-1} = \rho_W\gamma^{-1} = \rho_W$. Since the functor $\Pi_G(U_1, A) \to \Pi_G(X, A)$ is injective on objects, the full representative category $\mathcal{D}'$ of $\Pi_G(U_1, A)$ gives rise to a full representative category $\mathcal{E}'$ of $\Pi_G(X, A)$, via a pushout diagram.

We make a similar reduction on $\Pi_G(U_2, A)$, and define a fibre representative subcategory $\mathcal{D}''$ as the full subcategory on the objects $(G/e, E)$ and $(G/G, S)$. Then $\mathcal{D}''$ can be pictured as:

$$\begin{tikzcd}
(G/e, E) \arrow[loop above, "\gamma"] \arrow[r, "\eta"] & (G/G, S)
\end{tikzcd}$$

where $\gamma = (\tau, \alpha^{-1}\beta^{-1}) = (e, \beta\alpha)(\tau, id_E) = (\tau, id_W)(e, \alpha^{-1}\beta^{-1})$ and $\eta = (\rho, \beta^{-1})$, and the compositions are described by $\gamma^2 = id$ and $\eta\gamma = \eta$; the projection $\mathcal{D}'' \to \mathcal{O}_G$ is an isomorphism of categories. We can again take a pushout over $\Pi_G(U_2, A) \to \Pi_G(X, A)$ to get a fibre representative subcategory $\mathcal{E}''$.

Now we can form the following diagram, where $\mathcal{E}$ is the pushout of the outer square:

$$\begin{tikzcd}
\Pi_G(U_3, A) \arrow[r] \arrow[d, "i"] & \Pi_G(U_2, A) \arrow[r, "r_2"] \arrow[d, "j"] & \mathcal{D}'' \arrow[d] \\
\Pi_G(U_1, A) \arrow[r] \arrow[d, "r_1"'] & \Pi_G(X, A) \arrow[r, "r_2'"'] \arrow[d, "r_1'"] & \mathcal{E}'' \arrow[d] \\
\mathcal{D}' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E}
\end{tikzcd}$$

Both inclusions $\Pi_G(U_3, A) \to \Pi_G(U_i, A)$ for $i = 1, 2$ in the top left are injective on objects; since the pushout category has objects constructed as the set pushout of object sets, and pushouts in sets preserve injections, the inclusions $\Pi_G(U_i, A) \to \Pi_G(X, A)$ are likewise injective on objects for $i = 1, 2$. Similarly, $r_1 i$ is injective on objects, and so $r'_1 j$ is also. Thus the right rectangle from $fg(U_2, A)$ to $\mathcal{E}$ is a pushout in which r''_2 is a retraction; so $\mathcal{E}$ is a retract of $\mathcal{E}'$ and hence of the category $\Pi_G(X, A)$.

Thus we calculate the category $\mathcal{E}$, equivalent to the category $\Pi_G(X, A)$, as a pushout of the following:

$$\begin{tikzcd}
(G/e, W) \arrow[rr, bend left, "\gamma^{-1}"] & {} \arrow[dd, dotted] & (G/e, E) \arrow[ll, bend left, "\gamma"] & {} \arrow[r, dotted] & (G/e, E) \arrow[loop above, "\gamma"] \arrow[d, "\eta"] \\
& & & & (G/G, S) \\
(G/e, W) \arrow[rr, bend left, "\gamma^{-1}"] \arrow[rr, bend left=10, "\phi^{-1}"] \arrow[dr, "\rho_W"'] & & (G/e, E) \arrow[ll, bend left=10, "\phi"] \arrow[ll, bend left, "\gamma"] \arrow[dl, "\rho_E"] & & \\
& (G/G, N) & & &
\end{tikzcd}$$

It is clear that the resulting pushout category has objects $(G/e, E)$, $(G/G, N)$ and $(G/G, S)$. Morphisms are given by strings of composable morphisms from the subcategories, with suitable identifications [12]. Therefore the automorphisms of the object $(G/e, E)$ are strings of the morphisms γ and ϕ, subject to the identifications $\gamma^2 = id$ and $\phi = \gamma\phi^{-1}\gamma$. From this presentation, we see that this automorphism group is exactly the infinite dihedral group D_∞, where morphisms in the subgroup (ϕ) project to $id : G/e \to G/e$ in $\mathcal{O}_G$, and the reflections to the non-trivial map τ. For morphisms $(G/e, E) \to (G/G, S)$, we have a map η and can compose with any automorphism of $(G/e, E)$, subject to the relation $\gamma\eta = \gamma$. So this morphism set is $D_\infty/(\gamma)$. Similarly morphisms $(G/e, E) \to (G/G, N)$ are given by $D_\infty/(\phi^{-1}\gamma)$, since $\phi\rho = \gamma\rho$ in $\Pi_G(U_2, A)$. The graph of the category $\mathcal{E}$ is therefore

$$\begin{tikzcd}
& (G/e, E) \arrow[loop above, "D_\infty"] \arrow[dl, "D_\infty/(\phi^{-1}\gamma)"'] \arrow[dr, "D_\infty/(\gamma)"] & \\
(G/G, N) & & (G/G, S)
\end{tikzcd}$$

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