CORRECTION TO
TRANSLATION GROUPOIDS AND ORBIFOLD COHOMOLOGY

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This note concerns an error in the proof of Lemma 8.1 of the paper *Translation Groupoids and Orbifold Cohomology*, Canadian J. Math Vol 62 (3), pp 614-645 (2010). This was pointed out to the authors by Li Du of the Georg-August-Universität at Gottingen, who also suggested the outline for the following corrected proof.

The lemma in question reads:

Lemma 8.1. *The class of essential equivalences between Lie groupoids satisfies the 3-for-2 property, i.e., if we have homomorphisms $\mathcal{G} \overset{\varphi}{\to} \mathcal{K} \overset{\psi}{\to} \mathcal{H}$ such that two out of $\{\varphi, \psi, \varphi \circ \psi\}$ are essential equivalences, then so is the third.*

The given proof of this lemma is incorrect in the case where $\psi \circ \varphi$ and ψ are essentially equivalences. There it is stated:

It is a standard property of fibre products that if any two out of (A), (B), and the whole square are fibre products, so is the third.

This is incorrect in general; in particular, when φ and $\psi \circ \varphi$ are merely fully faithful it is not necessary that ψ is also, and counter-examples can be created. Below is a corrected proof of the case in question.

Proof. We consider the case where φ and $\psi \circ \varphi$ are essential equivalences. Since $\psi \circ \varphi$ is essentially surjective, the map $G_0 \times_{H_0} H_1 \to H_0$ is a surjective submersion. This map factors as the top arrow in the following diagram,

$$\begin{tikzcd}
G_0 \times_{H_0} H_1 \arrow[r] \arrow[d] & K_0 \times_{H_0} H_1 \arrow[r] \arrow[d] & H_1 \arrow[r, "t"] \arrow[d, "s"] & H_0 \\
G_0 \arrow[r, "\varphi_0"'] & K_0 \arrow[r, "\psi_0"'] & H_0 &
\end{tikzcd}$$

and we see that this implies that the composite of the last two maps, $K_0 \times_{H_0} H_1 \to H_0$ is a surjective submersion.

Next we consider the following diagram

$$\begin{tikzcd}
G_1 \arrow[r, "\varphi_1"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "(A)"] & K_1 \arrow[r, "\psi_1"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "(B)"] & H_1 \arrow[d, "(s,t)"] \\
G_0 \times G_0 \arrow[r, "\varphi_0 \times \varphi_0"'] & K_0 \times K_0 \arrow[r, "\psi_0 \times \psi_0"'] & H_0 \times H_0.
\end{tikzcd}$$

Since φ and $\psi \circ \varphi$ are essential equivalences, the left square (A) and the entire rectangle are both pullbacks. We want to show that the right square has to be a pullback as well. As indicated by the discussion above, the fact that φ is essentially surjective is an important ingredient. In fact, we would like to assume that φ_0 is actually surjective.

If φ_0 is not surjective, then consider the weak pullback groupoid

$$\begin{tikzcd} G' = G \times^w_K K \arrow[r, "\varphi'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\cong"] & K \arrow[d, "1_K"] \\ G \arrow[r, "\varphi"'] & K. \end{tikzcd}$$

Since φ is an essential equivalence, so is φ'. In addition, π is also an essential equivalence, because it is a weak pullback of an identity arrow (which is obviously an essential equivalence).

So we replace (A) by a new square (A'), which is again a pullback:

$$\begin{tikzcd} G'_1 \arrow[r, "\varphi'_1"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "(A')"] & K_1 \arrow[r, "\psi_1"] \arrow[d, "(s,t)"'] \arrow[dr, phantom, "(B)"] & H_1 \arrow[d, "(s,t)"] \\ G'_0 \times G'_0 \arrow[r, "\varphi'_0 \times \varphi'_0"'] & K_0 \times K_0 \arrow[r, "\psi_0 \times \psi_0"'] & H_0 \times H_0. \end{tikzcd}$$

Furthermore, the entire rectangle is again a pullback: Note that $\psi \circ \varphi' \cong (\psi \circ \varphi) \circ \pi$. The latter is an essential equivalence as a composite of essential equivalences and hence so is the former, because it is isomorphic to an essential equivalence. We have also that the map $\varphi' \colon G'_0 = G_0 \times_{K_0} K_1 \times_{K_0} K_0 \to K_0$, defined by $(x, k, t(k)) \mapsto t(k)$, is surjective since φ is essentially surjective.

Now consider the pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d] & H_1 \times_{t,H_0} K_0 \arrow[d, "s\pi_1"] \\ K_0 \arrow[r, "\psi_0"'] & H_0 \end{tikzcd}$$

Since the map $s\pi_1$ is a surjective submersion, this pullback is a smooth manifold, and we get a smooth map $K_1 \to P = K_0 \times_{H_0,s} H_1 \times_{t,H_0} K_0$. Next consider the diagram

$$\begin{tikzcd} G'_1 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & H_1 \arrow[d] \\ G'_0 \times G'_0 \arrow[r] & K_0 \times K_0 \arrow[r] & H_0 \times H_0 \end{tikzcd}$$

We know that the right square is a pullback, and therefore the left square is a pullback if and only if the whole rectangle is a pullback. But the whole rectangle is a pullback as we just observed and so the left square is a pullback.

So now consider

$$\begin{tikzcd}
G_1' \arrow[r] \arrow[d, equal] & K_1 \arrow[d] \\
G_1' \arrow[r] \arrow[d] & P \arrow[d] \\
G_0' \times G_0' \arrow[r] & K_0 \times K_0
\end{tikzcd}$$

The bottom square is a pullback according to the previous argument, and we know that the whole rectangle is a pullback since $\varphi' \colon G' \to K$ is fully faithful. Therefore, the top square is also a pullback.

Now the bottom map is a surjective submersion (it is surjective as argued above and it is submersion because the groupoids are étale), and therefore the pullback map $G_1' \to P$ is also a surjective submersion. Then looking at the top square, we see that the pullback of the map $K_1 \to P$ is the identity map, and hence a diffeomorphism. Therefore the original map must have also been a diffeomorphism. So $K_1 \cong P$ and so the original square (B) is a pullback as required. $\square$