

Orbispaces and their Mapping Spaces via Groupoids: A Categorical Approach

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1. Introduction

Orbifolds were first introduced by Satake (who called them V-manifolds), and later studied by Conway [2], Thurston [12] and others. They were developed as a generalization of manifolds, and the original approach to their study was based on charts and atlases. The difference with the orbifold context is that we allow certain singularities: the local neighbourhoods are homeomorphic to $U = \tilde{U}/G$ where G is a finite group acting on an open set $\tilde{U} \subseteq \mathbb{R}^n$. An orbifold M can then be defined via an orbifold atlas, which is a locally compatible family of charts (\tilde{U}, G) such that the sets \tilde{U}/G give a cover of M . The usual notion of equivalence of atlases through common refinement is used; details can be found in [10, 11]. Note that the original definition required that all group actions be effective, but it is often useful to drop this requirement; we will not require that G acts effectively on \tilde{U} .

Working with orbifold atlases is cumbersome, particularly when dealing with maps between orbifolds. Therefore an alternate way of representing orbifolds using groupoids has been developed. It was shown in [5] that every smooth orbifold can be represented by a Lie groupoid. This representation is not unique, but is determined up to Morita equivalence. This way of representing orbifolds allows for a natural generalization to orbispaces, without a smooth structure, via topological groupoids. It also allows for a groupoid-based definition of orbifold maps (originally called the ‘good’ or ‘generalized’ orbifold maps) which works well for homotopy theory, as noted in [1].

In this paper, we give an accessible introduction to the theory of orbispaces via groupoids. We define a certain class of topological groupoids, which we call orbifold groupoids. Each orbifold groupoid represents an orbifold, but just as with orbifolds and Lie groupoids, this representation is not unique: orbispaces are Morita equivalence classes of orbifold groupoids. The orbifold groupoid category can be used as a basis for developing results about orbispaces. We will discuss the connection between orbifold groupoids and orbispaces, focusing particularly on creating mapping objects for orbispaces which themselves have orbifold structure. Throughout this paper, we illustrate our definitions and results with numerous examples which we hope will be useful in seeing how the categorical point of view is used to study these spaces.

The maps between orbispaces can be defined either in terms of a bicategory of fractions, or in terms of Hilsun-Skandalis maps. This paper takes the bicategory of fractions approach, providing a more concrete description of the mapping space construction. Related work has been done by Chen [3], using a more atlas-based approach to representing orbispaces, by Haefliger [4] for étale groupoids and Hilsun-Skandalis maps and by Noohi [7] for topological stacks. In this paper, we lay the groundwork for the bicategory approach and illustrate how it can be used to define a mapping orbispace. Further results about the properties of this mapping orbispace are given in [9].

This paper begins with background sections. Section 2 gives the definition of orbifoldoids, and illustrates how these represent orbispaces. Section 3 defines homomorphisms between orbifoldoids, and also defines natural transformations, creating the bicategory of orbifoldoids. Section 4 defines the orbispace category in terms of the bicategory of fractions of orbifoldoids, again giving examples of how the orbispace category is represented in this fashion.

Section 5 shows how to use the definitions of Section 3 to create a topological groupoid representing the maps between orbifoldoids. We work through several non-trivial examples of the resulting mapping space, showing how the orbispace structure appears in this approach. We finish this paper by showing in Section 6 that we can recover the inertia groupoid of [6] as a mapping orbispace from a specific orbispace, a one-point space with isotropy.

2. Topological Groupoids and Orbifoldoids

We begin with the standard definition of a topological groupoid: a groupoid in the category of topological spaces, where we have spaces instead of sets and all maps are continuous.

DEFINITION 2.1. A *topological groupoid* \mathcal{G} consists of a space of objects \mathcal{G}_0 and a space of arrows \mathcal{G}_1 . The category structure is defined by the following continuous maps.

- The *source map*, $s : \mathcal{G}_1 \rightarrow \mathcal{G}_0$, which gives the domain of each arrow.
- The *target map*, $t : \mathcal{G}_1 \rightarrow \mathcal{G}_0$, which gives the codomain of each arrow.
- The *unit map*, $u : \mathcal{G}_0 \rightarrow \mathcal{G}_1$, which gives the identity arrow on an object.
- The *composition map*, $m : \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \rightarrow \mathcal{G}_1$ where $m(g_1, g_2) = g_2 \circ g_1 = g_2 g_1$. The pullback over \mathcal{G}_0 ensures that $t(g_1) = s(g_2)$, so that we are only composing the arrows that match up at their ends.
- The *inverse map*, $i : \mathcal{G}_1 \rightarrow \mathcal{G}_1$ defined by $i(g) = g^{-1}$.

These maps need to satisfy the usual category axioms, as well as the expected relationship between an arrow and its inverse. Specifically, we must have:

- (identity) $m(g, ut(g)) = g$ and $m(us(g), g) = g$,
- (associativity) $m(g_1, m(g_2, g_3)) = m(m(g_1, g_2), g_3)$,
- (inverses) $m(g, i(g)) = us(g)$ and $m(i(g), g) = ut(g)$.

We will often use group notation when discussing arrows in \mathcal{G}_1 , writing the identity $u(x)$ as id_x , multiplication $m(g_1, g_2)$ as $g_2 g_1$ and the inverse map $i(g)$ as g^{-1} .

Associated to any topological groupoid we have a topological space defined as a quotient of the object space. We think of the arrows in the groupoid as identifications, and form the quotient \mathcal{G}_0 / \sim where $x \sim y$ if there is an arrow

$g : x \rightarrow y$ in \mathcal{G}_1 . We will denote this quotient space by $\mathcal{G}_0/\mathcal{G}_1$. In order to keep our topologies reasonable and the singularities in this quotient space modeled by quotients of finite groups, we need to put some restrictions on our topological groupoids.

DEFINITION 2.2. An *orbifold* is a topological groupoid \mathcal{G} such that the object space \mathcal{G}_0 and the arrow space \mathcal{G}_1 are compactly generated locally compact, paracompact Hausdorff spaces, and for which the source and target maps are étale (i.e. local homeomorphisms), and the map $(s, t) : \mathcal{G}_1 \rightarrow \mathcal{G}_0 \times \mathcal{G}_0$ is proper (i.e. the preimage of a compact set is compact).

An orbifold defines an orbispace, a topological space with orbifold-type singularities but without the smooth structure. We think of the orbispace as the quotient space of the objects, but with extra structure at the singularities. Philosophically, we want to identify points, but also remember how many times they were identified, and in what way. The orbifold allows us to do this. For any point x in the object space \mathcal{G}_0 , we define the *isotropy group* of x to be $G_x = \{g \in \mathcal{G}_1 | s(g) = t(g) = x\}$. Because we are working in groupoids, if there is an arrow identifying x to y , their isotropy groups will be isomorphic. So the isotropy is well-defined on points of the quotient space, and the structure of the singularities can be encoded via isotropy information.

REMARK 2.3. The joint conditions of being étale and proper immediately ensure that the isotropy groups have to be finite: the étale condition makes them discrete and the proper condition then requires them to be finite. In fact, these conditions give us even more. Let $x \in \mathcal{G}_0$ be any point in the space of objects. Since the groupoid is étale we can find open neighbourhoods U_g of the elements $g \in G_x$ on which both the source and the target maps restrict to homeomorphisms. Then since the group G_x is finite, we can shrink these neighbourhoods so that $s(U_g) = t(U_g)$ for each g , and moreover their images are all the same, i.e., $s(U_g) = s(U_{g'})$ for all $g, g' \in G_x$. Call the common image V_x . It is shown in [5] that properness allows us then to further shrink this neighbourhood V_x (if necessary) so that all arrows in \mathcal{G}_1 which have both their source and target in V_x are in $\bigcup_{g \in G_x} U_g$, i.e., $(s, t)^{-1}(V_x) = \bigcup_{g \in G_x} U_g$ and for each $g \in G_x$, both s and t restrict to a homeomorphism from U_g to V_x . Thus, the quotient space of V_x is a quotient by the group action G_x acting on V_x , and we see that with these orbifolds, we are indeed modelling spaces which are locally the quotients of finite group actions. (The proof of this appears in the proof of the implication 4 \Rightarrow 1 of Theorem 4.1 of [5]. That paper is about orbifolds rather than orbispaces, and the groupoid was required to be effective, but this result does not depend on those conditions.)

Note that orbispaces are represented by orbifolds, but this representation is not unique. Orbispaces will be defined as Morita equivalence classes of orbifolds. Before we consider this equivalence, we give some basic examples of orbifolds and their quotient spaces, illustrating how some standard orbispaces are represented.

The first example is a manifold without singularities.

EXAMPLE 2.4. Consider the sphere S^2 . We will build a topological groupoid \mathcal{G} representing S^2 as follows. Cover S^2 with two open disks, D_1 and D_2 , which intersect each other in an annulus. Let the object space \mathcal{G}_0 be the disjoint union

of the two disks. The arrow space \mathcal{G}_1 needs to encode the identifications along the annular overlap. Hence, we get an annulus of arrows A_1 identifying points along the edge of D_1 with their corresponding points along the edge of D_2 . So the source map takes A_1 to the edge of D_1 , and the target map takes A_1 to the edge of D_2 . Similarly, we have the inverse maps with source and target reversed forming another annulus A_2 . To complete the structure of our topological groupoid, we also include identity arrows, which sit in two disks homeomorphic to D_1 and D_2 . Thus we have the orbifold of Figure 1 below. All the source and target maps are just inclusions, so this is clearly an étale and proper groupoid, hence an orbifold. The quotient space of this groupoid is just our original space S^2 , with no additional non-trivial isotropy information.

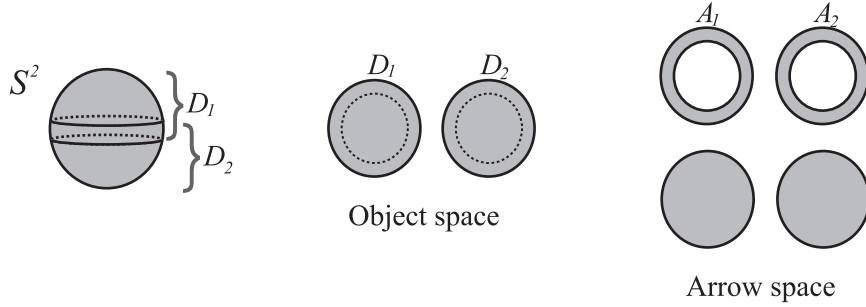


FIGURE 1. 2-sphere groupoid

This example can easily be generalized to create an orbifold that represents any manifold, which is an example of an orbifold with trivial isotropy. Next, we look at how to get non-trivial isotropy points via our groupoid representation. The following example defines a single point with isotropy.

EXAMPLE 2.5. Let G be a finite group. Define the orbifold $*_G$ to have object space consisting of just one point, $*$, and arrow space the discrete space G . Composition and inverses are given by the group structure on G . Source and target maps take any arrow g to the point $*$. Since the object and arrow spaces are finite and discrete, this topological groupoid is clearly étale and proper. The orbifold associated to this orbifold is the quotient of the single point $*$ by the group action, a single point with G -isotropy.

Note that this is an example of a non-effective orbifold. We can create a similar space with effective action by a ‘fat point’ construction as follows.

EXAMPLE 2.6. Consider the open disk D^2 with a $\mathbb{Z}/3$ rotation action that keeps the center point fixed. We create a groupoid with the object space equal to the disk itself. The arrow space encodes the group action. If ν generates $\mathbb{Z}/3$, then we identify points in an orbit via arrows: $x \rightarrow x$, $x \rightarrow \nu x$, and $x \rightarrow \nu^2 x$. Thus the arrow space is three disjoint disks, one for each element of $\mathbb{Z}/3$. The source and target maps are defined by the projection and the action respectively: $s(\nu^i, x) = x$ and $t(\nu^i, x) = \nu^i x$, for $i = 0, 1, 2$. See Figure 2. The inverse arrows are already included, since $\nu^{-1} = \nu^2$. Source and target maps are homeomorphisms

from any component, and there are finitely many components in the arrow space, so this is again étale and proper. The quotient space is topologically a disk with a $\mathbb{Z}/3$ -isotropy point at the center. This orbispace is referred to as an order 3 cone point in Thurston [12]. We will denote it by \mathcal{C}_3 .

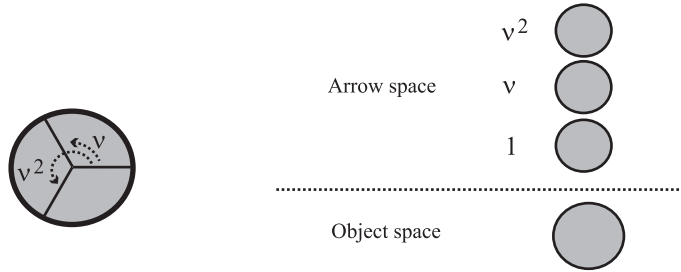


FIGURE 2. Order 3 cone point groupoid \mathcal{C}_3

The previous example is a global quotient, defined by the action of a finite group on a space. Such a global quotient can always be represented by the translation groupoid $G \times X$ where the object space is given by X and the arrow space by $G \times X$, with $s(g, x) = x$ and $t(g, x) = gx$. The following gives another example of a global quotient orbispace and its translation groupoid.

EXAMPLE 2.7. The ‘silvered interval’ is a (closed) interval with $\mathbb{Z}/2$ -isotropy at its endpoints. One way to represent the silvered interval is as the global quotient of a $\mathbb{Z}/2$ action on a circle, with the group acting by reflection and fixing two antipodal points a and b .

We describe the translation groupoid $\mathbb{Z}/2 \times S^1$ associated to this action. The object space of $\mathbb{Z}/2 \times S^1$ is S^1 . The arrow space is $\mathbb{Z}/2 \times S^1$, the disjoint union of two copies of S^1 . The source map is $s(g, x) = x$, and the target map is $t(g, x) = gx$. Thus, an arrow of the form $(1, x)$ is the identity map $x \rightarrow x$, and an arrow of the form (τ, x) is a map from x to its reflection y . The inverse of the arrow (τ, x) is given by $(\tau, \tau x)$. We have non-trivial isotropy maps (τ, a) and (τ, b) creating the isotropy structure on the endpoints of the quotient space. See Figure 3. Once again, it is easy to see that this translation groupoid $\mathbb{Z}/2 \times S^1$ is an orbifold whose orbispace is the silvered interval. We will denote this orbifold by \mathcal{SI} .

The next example is not a global quotient.

EXAMPLE 2.8. The teardrop orbispace is a sphere S^2 , with $\mathbb{Z}/3$ -isotropy at the north pole. We represent it with the orbifold \mathcal{T} , created from an ‘upper hemisphere’ disk D_1 with a $\mathbb{Z}/3$ rotation action (the same as for \mathcal{C}_3 in Example 2.6), and a ‘lower hemisphere’ disk D_2 . The two disks covering the sphere overlap in an annulus around the equator.

The object space \mathcal{T}_0 is the disjoint union of the two upper and lower hemisphere disks, D_1 and D_2 . The arrow space \mathcal{T}_1 needs to encode both the $\mathbb{Z}/3$ action on the upper hemisphere and the identifications along the annular overlap between the two hemispheres. As with \mathcal{C}_3 , there are three disjoint disks in the arrow space with source and target in D_1 , corresponding to the identity, ν , and ν^2 actions on points in D_1 . There is a single disk corresponding to the identity maps on

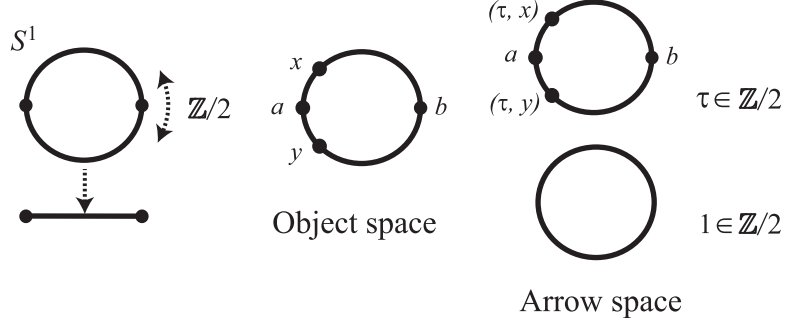
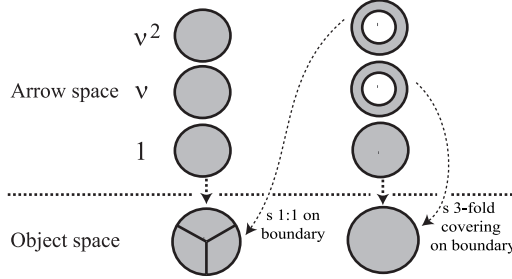


FIGURE 3. Silvered interval

D_2 . For the identification between the disks, the arrow space contains an annulus identifying points in the edge of D_1 to their corresponding points in the edge of D_2 , and a second annulus for the inverse identifications. See Figure 4. These annuli in the arrow space map with the usual inclusion to the upper hemisphere, but via a 3-fold covering to the lower hemisphere (the source map for the first annulus, and the target for the second). This is not a homeomorphism, but it is a local homeomorphism, giving an orbifold.

FIGURE 4. Teardrop groupoid \mathcal{T}

The next example gives a somewhat more complicated orbispace.

EXAMPLE 2.9. Let D be an open disk. The dihedral group D_3 (of order 6) acts on this disk. We set some notation: D_3 is generated by σ and ρ , with $\sigma^2 = 1$, $\rho^3 = 1$, $\sigma\rho = \rho^2\sigma$, and $\sigma\rho^2 = \rho\sigma$. Then $\sigma \in D_3$ acts on the disk by reflection about a chosen line through the center of D , and $\rho \in D_3$ acts by counter-clockwise rotation of D by an angle of $2\pi/3$. See Figure 5. The quotient space under this action is a sector of the disk which we will call a ‘corner’. The edges of the sector become ‘silvered boundaries’ with $\mathbb{Z}/2$ -isotropy, and the center point becomes a corner point with D_3 -isotropy.

We combine three such corners to create a triangular orbispace where each corner has D_3 -isotropy, and the edges have $\mathbb{Z}/2$ -isotropy; see Figure 6. Note that each



FIGURE 5. Disk with D_3 action

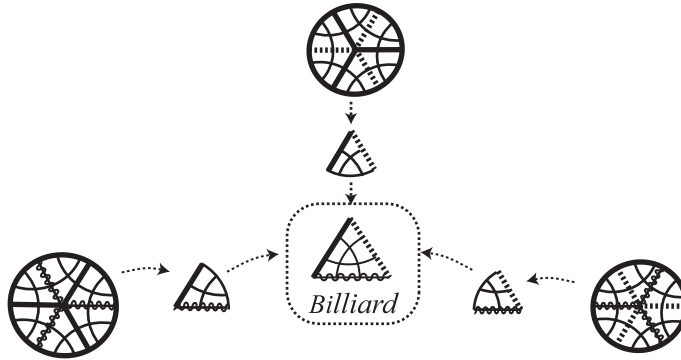


FIGURE 6. Construction of the triangular billiard \mathcal{TB}

sector overlaps each of the other sectors, as shown in Figure 6. Following naming conventions of Thurston [12], we will call this the triangular billiard orbispace \mathcal{TB} .

To create a groupoid representing this orbispace, we start with translation groupoids representing each of the three corners as the global quotient $D_3 \times D$, and adding in the identifications of the overlaps. So the object space is the disjoint union of the three disks, one for each sector. The arrow space contains six disjoint copies of each of these three disks, corresponding to the action of D_3 on each disk, with arrows sending x to gx for each $g \in D_3$. In addition, the arrow space contains copies of the overlap shape, representing the glueing arrows from one disk to another. If we consider a space of arrows glueing points in the overlap from disk A to those in disk B, we see that there are 6 choices of source embeddings of the overlap in disk A (given by three possible images, and a choice of whether to embed with a reflection or not). There are also 6 possible target embeddings of the overlap in a disk B, again 3 with reflections and 3 without. This gives 36 possible copies of the overlap, with source in disk A and target in disk B, but we observe that the map (i, i) (a copy of the overlap with embeddings without reflections) is the same as (σ, σ) (a copy with reflections both into disk A and into disk B), so this reduces the number to 18. See Figure 7. Similarly, there are another 18 copies of the overlap in the arrow space with source in disk B and target in disk A (the inverses of the arrows from disk A to disk B). In total, we have 36 overlaps for each choice of two of the three disks. The orbifold \mathcal{TB} is shown in Figure 8.

We emphasize again that orbifold representations of orbispaces are not unique, and a single orbispace can be represented by many different orbifolds. Our last example gives an illustration of this.

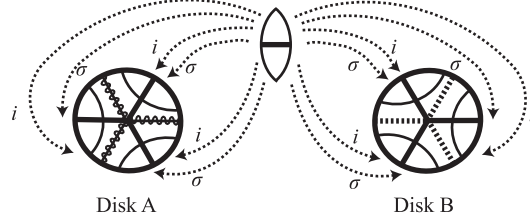
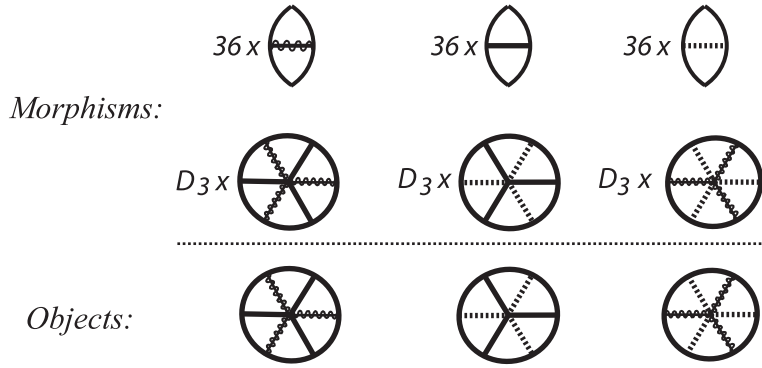
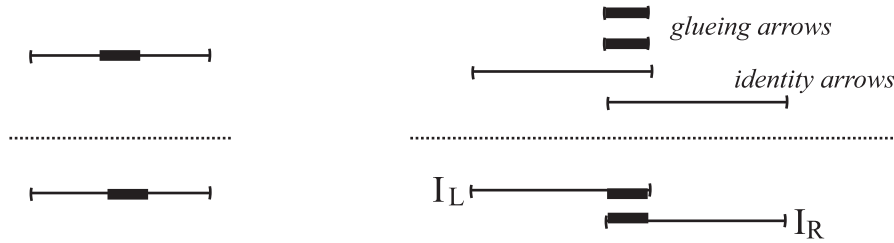


FIGURE 7. Embedding of one of the overlaps

FIGURE 8. Triangular billiard groupoid \mathcal{TB}

EXAMPLE 2.10. Consider the orbispace defined by the interval $I = [0, 1]$, with only trivial isotropy. The simplest representation of this orbispace, \mathcal{I} , has object space $\mathcal{I}_0 = I$ and arrow space $\mathcal{I}_1 = I$, with the only arrows being identity arrows. Alternately, we can represent the interval with a ‘broken’ groupoid \mathcal{I}^2 . The object space \mathcal{I}_0^2 is the disjoint union of two intervals, $I_L = [0, \frac{2}{3})$ and $I_R = (\frac{1}{3}, 1]$. We glue the subinterval $(\frac{1}{3}, \frac{2}{3})$ (in bold in Figure 9) on the right end of I_L to the subinterval $(\frac{1}{3}, \frac{2}{3})$ (in bold) on the left end of I_R to form I . Hence the arrow space consists of the identity arrows represented by copies of I_L and I_R , and two copies of the interval $(\frac{1}{3}, \frac{2}{3})$, one representing the glueing arrows from the subinterval of I_L to the subinterval of I_R , and the other representing their inverses from I_R to I_L .

FIGURE 9. Orbifold groupoids \mathcal{I} and \mathcal{I}^2

This can easily be generalized to an orbifold \mathcal{I}^n representing I in which the interval is broken into n pieces. For example, we will later use \mathcal{I}^3 , where the interval is broken into three pieces. The object space is $\mathcal{I}_0^3 = [0, \frac{1}{2}] \amalg (\frac{1}{4}, \frac{3}{4}) \amalg (\frac{1}{2}, 1]$. The arrow space consists of intervals representing the identity arrows, two intervals $(\frac{1}{4}, \frac{1}{2})$ glueing $[0, \frac{1}{2})$ to $(\frac{1}{4}, \frac{3}{4})$ and vice versa, and two intervals $(\frac{1}{2}, \frac{3}{4})$ glueing $(\frac{1}{4}, \frac{3}{4})$ to $(\frac{1}{2}, 1]$ and vice versa.

These examples illustrate how the groupoid keeps track of the ways that points are identified, both to each other and to themselves. The arrow space encodes both identifications between points in different components of the object space and local isotropy structure in the same categorical language, with the isotropy structure coming from the ways in which a point is identified to itself. Orbifold groupoids allow us to represent orbispaces by keeping track not only of what is identified, but the actual number and structure of those identifications. It is this extra structure that makes orbispaces different from ordinary spaces.

3. Homomorphisms and Natural Transformations of Orbifold Groupoids

Next we want to consider maps of orbispaces as represented by homomorphisms of orbifold groupoids.

DEFINITION 3.1. Let \mathcal{G} and \mathcal{H} be topological groupoids. A *homomorphism* $f: \mathcal{G} \rightarrow \mathcal{H}$ is a continuous functor between topological groupoids. Specifically, f is defined by two continuous maps, $f_0: \mathcal{G}_0 \rightarrow \mathcal{H}_0$, and $f_1: \mathcal{G}_1 \rightarrow \mathcal{H}_1$, which satisfy the functor relations:

- If $x \in \mathcal{G}_0$, then f_1 takes the identity map on x in \mathcal{G} to the identity map on $f_0(x)$ in \mathcal{H} :

$$f_1(u(x)) = u(f_0(x)).$$

- If $g \in \mathcal{G}_1$, then f preserves the source and target of g :

$$f_0(s(g)) = s(f_1(g)),$$

$$f_0(t(g)) = t(f_1(g)).$$

- If g_1 and g_2 are two arrows in \mathcal{G}_1 such that $t(g_1) = s(g_2)$, then the arrows $f_1(g_1)$ and $f_1(g_2)$ can be composed in \mathcal{H}_1 since f preserves the source and target. Moreover, f_1 respects the composition:

$$m(f_1(g_1), f_1(g_2)) = f_1(m(g_1, g_2)).$$

It follows from the above that f preserves inverses as well.

A homomorphism of orbifold groupoids induces a map between the represented orbispaces: if two points are identified in \mathcal{G}_0 via some map g , then $f(g)$ identifies their images in \mathcal{H} . So we get a continuous map on the underlying quotient spaces. The homomorphism also carries information about the isotropy, since an identification in \mathcal{G} is mapped to a specific identification in \mathcal{H} . The following examples illustrate how the homomorphisms encode information about how the map behaves on the isotropy.

EXAMPLE 3.2. We examine possible homomorphisms from $*_G$ to the silvered interval $\mathbb{Z}/2 \times S^1$, our orbispaces from Examples 2.5 and 2.7. We will compare the results for $G = \mathbb{Z}/3$ and $G = \mathbb{Z}/4$.

For each element $g \in G$, denote the corresponding arrow in $*_G$ by g as well. Note that if g has order n in the group, then the arrow g^n is the identity arrow in $*_G$. Since a homomorphism $f : *_G \rightarrow \mathbb{Z}/2 \times S^1$ respects composition, f_1 gives a group homomorphism from G to the isotropy group $I_{f(*)}$. In particular, if $g^n = 1$, then $(f_1(g))^n$, the composition of $f_1(g)$ with itself n times, must be an identity arrow.

Suppose that $G = \mathbb{Z}/3$, and consider a homomorphism $f : *_G \rightarrow \mathbb{Z}/2 \times S^1$. Let $y = f_0(*) \in S^1$. If g is a non-zero element of $\mathbb{Z}/3$, then g has order 3, and $[f(g)]^3 = f(g^3)$ is the identity arrow $y \rightarrow y$. The only arrows h with the property that $h^3 = 1$ have the form $(1, y)$, and it follows that f must take all arrows to identity arrows.

We get more interesting homomorphisms when G has elements of even order. Suppose now that $G = \mathbb{Z}/4$ generated by σ with $\sigma^4 = 1$, and that $f : *_G \rightarrow \mathbb{Z}/2 \times S^1$ is a homomorphism with $y = f_0(*) \in S^1$. If we want a non-trivial map on arrows, we must have y be one of our isotropy points a or b , and $f(\sigma) = (\tau, y)$. Since $\sigma^3 = \sigma^{-1}$, we also have $f(\sigma^3) = (\tau, y)$. Thus, we get two homomorphisms which are non-trivial on arrows, one with $y = a$ and the other with $y = b$, in addition to the maps that take all arrows to the identity.

In general, for any finite group G and any orbifold \mathcal{H} , the homomorphisms $f : *_G \rightarrow \mathcal{H}$ correspond to a point $f(*) \in \mathcal{H}_0$ with a group homomorphism into the isotropy group H_x of x , $f_1 : G \rightarrow H_x$.

Next we look at how to represent a path in an orbifold, that is, a map from an interval to the orbifold, via a groupoid homomorphism.

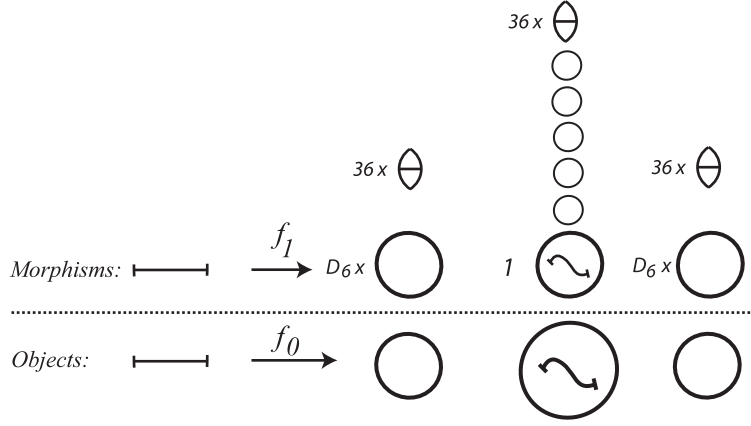
EXAMPLE 3.3. Recall \mathcal{I} , an orbifold representing an interval I from Example 2.10, and \mathcal{TB} , the triangular billiard groupoid from Example 2.9. We can consider paths in \mathcal{TB} which can be represented by homomorphisms $f : \mathcal{I} \rightarrow \mathcal{TB}$. Since the object space is connected and f_0 is continuous, f will send the entire interval I into one of the disks in the object space of \mathcal{TB} . Additionally, as the arrow space is also connected, and f_1 must be continuous and preserve the identities, f_1 can only send the interval $I = \mathcal{I}_1$ to the identity component for the disk that \mathcal{I}_0 is mapped into. So not all paths can be represented by homomorphisms between these particular orbifolds. We will return to this idea in the next section.

We have shown that the representation of an orbifold by an orbifold is not unique; the same thing is true for maps of orbifolds. In order to define the category of orbifolds, we will need to look at identifications of homomorphisms.

DEFINITION 3.4. Given two homomorphisms $f, f' : \mathcal{G} \rightarrow \mathcal{H}$, a 2-cell $\alpha : f \Rightarrow f'$ is a continuous natural transformation between these functors. Specifically, α is given by a continuous function $\alpha : G_0 \rightarrow H_1$ such that $s \circ \alpha = f_0$, $t \circ \alpha = f'_0$ and, for every arrow $g : x \rightarrow y$ in \mathcal{G}_1 , the following naturality square of arrows in \mathcal{H} is commutative:

$$\begin{array}{ccc} f_0(x) & \xrightarrow{\alpha(x)} & f'_0(x) \\ f_1(g) \downarrow & & \downarrow f'_1(g) \\ f_0(y) & \xrightarrow{\alpha(y)} & f'_0(y) \end{array}$$

In other words, $m(\alpha(x), f'_1(g)) = m(f_1(g), \alpha(y))$.


 FIGURE 10. Homomorphism $f : \mathcal{I} \rightarrow \mathcal{TB}$

Two homomorphisms with a natural transformation between them represent the same map between the quotient spaces, since there is an identification arrow between $f_0(x)$ and $f'_0(x)$ for any x .

EXAMPLE 3.5. We define two homomorphisms, f and f' , from the ‘broken’ interval \mathcal{I}^2 (Example 2.10) to the silvered interval \mathcal{SI} (Example 2.7). The first one, f is defined so that f_0 takes I_L and I_R into the upper portion of the circle in \mathcal{SI}_0 , and f_1 takes all arrows to the corresponding identity arrows. The second, f' is defined so that f'_0 is the same as f_0 on I_L , but takes I_R to the reflection of $f_0(I_R)$ in the lower portion of the circle: if $f_0(x) = y$ for some $y \in \mathcal{SI}_0$, then $f'_0(x) = f_0(x)$ for $x \in I_L$, and $f'_0(x) = \tau y = \tau f_0(x)$ for $x \in I_R$. Then f'_1 takes the identity arrows to the identity component, as usual, but takes the glueing arrows in \mathcal{I}_1^2 to the τ component of \mathcal{SI}_1 : if $g : x \rightarrow y$ is a glueing (non-identity) arrow in \mathcal{I}_0^2 , then $f_1(g) = (1, f_0(x))$ and $f'_1(g) = (\tau, f'_0(x))$. See Figure 11. It is easy to check that these maps satisfy the functorial requirements for being homomorphisms.

Now we define a natural transformation $\alpha : f \Rightarrow f'$. Let $\alpha : \mathcal{I}_0^2 \rightarrow \mathcal{SI}_1$ be such that $\alpha(x) = (1, f_0(x))$ for $x \in I_L$ and $\alpha(x) = (\tau, f_0(x))$ for $x \in I_R$. Then for $x \in I_L$, $s \circ \alpha(x) = s(1, f_0(x)) = f_0(x)$ and $t \circ \alpha(x) = t(1, f_0(x)) = f_0(x) = f'_0(x)$. For $x \in I_R$, $s \circ \alpha(x) = s(\tau, f_0(x)) = f_0(x)$ and $t \circ \alpha(x) = t(\tau, f_0(x)) = \tau f_0(x) = f'_0(x)$. So $\alpha(x)$ is indeed an arrow $f_0(x) \rightarrow f'_0(x)$ for each x .

Finally, we need to check that the naturality square commutes for each $g \in \mathcal{I}_1^2$. This is clear for any identity arrow of I_L , since all arrows are the identity. So we can consider the commutative squares for an identity arrow id_x with $x \in I_R$, on the left, and a glueing arrow $g : x \rightarrow y$ with $x \in I_L$ and $y \in I_R$, on the right:

$$\begin{array}{ccc}
 f_0(x) \xrightarrow{(\tau, f_0(x))} \tau f_0(x) & & f_0(x) \xrightarrow{(1, f_0(x))} f_0(x) \\
 (1, f_0(x)) \downarrow & \searrow & \downarrow (\tau, f_0(x)) \\
 f_0(x) \xrightarrow{(\tau, f_0(x))} \tau f_0(x) & & f_0(x) \xrightarrow{(\tau, f_0(x))} \tau f_0(x)
 \end{array}$$

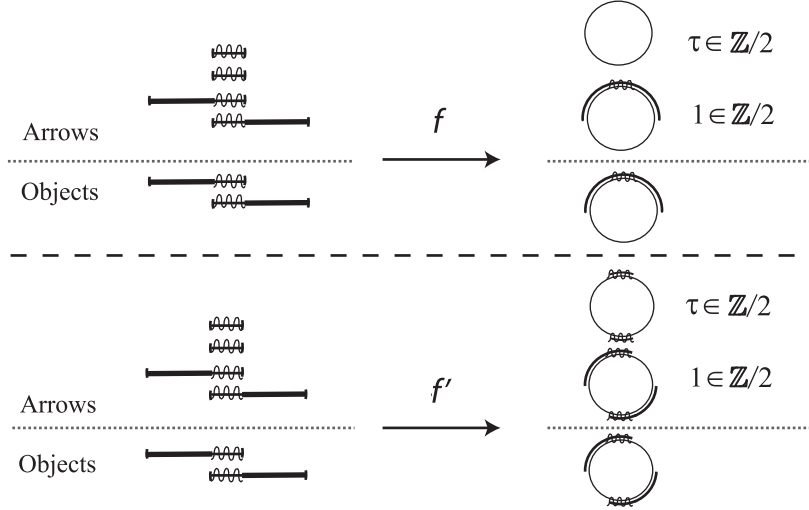


FIGURE 11. Homomorphisms f and f' from I^2 to SI

and see that both these squares commute. The final case, for a glueing arrow $g : x \rightarrow y$ with $x \in I_R$ and $y \in I_L$ is similar to the right square. Notice that both f and f' map the interval continuously onto the same path in the quotient space.

Even though homomorphisms with a natural transformation between them can be thought of as ‘the same’ map of orbispaces, we do not want to simply identify them. In examining the structure of maps between orbispaces, we will also want to take into consideration how many ways homomorphisms are identified. The ways in which a homomorphism can be identified to itself via a natural transformation can be used to give the maps themselves isotropy structure. In order to retain this singularity structure, we will be remembering identifications and working with the 2-category of orbifoldoids, together with their homomorphisms and natural transformations.

4. Representing Orbispaces with Orbifoldoids

Now we begin to build the category of orbispaces, which is our primary interest. As demonstrated above, every orbifoldoid defines an orbispaces, but this definition is not unique: it is possible to have the same quotient space and the same local isotropy structure with two different groupoid representations. This ambiguity is made precise via an equivalence of categories, again suitably topologized.

DEFINITION 4.1. An *essential equivalence* of topological groupoids is a homomorphism $f : \mathcal{G} \rightarrow \mathcal{H}$ satisfying the two conditions:

- E1 f is *essentially surjective on objects* in the sense that $t \circ \pi_2$ is an open surjection:

$$\begin{array}{ccc}
 \mathcal{G}_0 \times_{\mathcal{H}_0} \mathcal{H}_1 & \xrightarrow{\pi_2} & \mathcal{H}_1 \xrightarrow{t} \mathcal{H}_0 \\
 \pi_1 \downarrow & & \downarrow s \\
 \mathcal{G}_0 & \xrightarrow{f_0} & \mathcal{H}_0
 \end{array}$$

i.e., for each object $y \in \mathcal{H}_0$, there is an object $x \in \mathcal{G}_0$ and an arrow (actually, an isomorphism since we are working with groupoids) $h \in \mathcal{H}_1$ from $f_0(x)$ to y .

E2 f is *fully faithful* in the sense that the following diagram is a pullback:

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{f_1} & \mathcal{H}_1 \\ (s,t)\downarrow & & \downarrow (s,t) \\ \mathcal{G}_0 \times \mathcal{G}_0 & \xrightarrow{f_0 \times f_0} & \mathcal{H}_0 \times \mathcal{H}_0 \end{array}$$

i.e., for any two objects of \mathcal{H}_0 the identifications between them in \mathcal{H}_1 are isomorphic to those between any pre-images in \mathcal{G} . In particular, f is an isomorphism on isotropy groups for all objects.

Note that condition E1 ensures that an essential equivalence between groupoids produces a homeomorphism on their quotient spaces, and condition E2 means that the isotropy information about the singularities is also preserved. Therefore an essential equivalence represents an isomorphism on orbispaces. Any two orbifold groupoids \mathcal{G}, \mathcal{H} that can be connected with a zig-zag of essential equivalences $\mathcal{G} \leftarrow \mathcal{K}_1 \rightarrow \mathcal{K}_2 \leftarrow \mathcal{K}_3 \rightarrow \cdots \leftarrow \mathcal{K}_n \rightarrow \mathcal{H}$ are called Morita equivalent, and represent the same orbifold. It was shown in [8] that in fact two étale groupoids \mathcal{G} and \mathcal{H} are Morita equivalent if and only if there is a zig-zag of essential equivalences $\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}$; we only need one zig-zag, and any longer list of zig-zags can be shortened.

The next example shows that our various representations of the interval $I = [0, 1]$ from Example 2.10 are Morita equivalent.

EXAMPLE 4.2. Recall that we have defined two different orbifold groupoids, the ‘unbroken’ \mathcal{I} and the ‘broken’ \mathcal{I}^2 , representing the closed unit interval $I = [0, 1]$ (Example 2.10). Here we show that there is an essential equivalence between them. Let $f : \mathcal{I}^2 \rightarrow \mathcal{I}$ be the homomorphism such that f_0 maps I_L in \mathcal{I}_0^2 to the left side of the interval \mathcal{I}_0 , mapping the bold subinterval of I_L to the bold subinterval of \mathcal{I}_0 , and mapping I_R to the right side of \mathcal{I}_0 in a similar way. Since \mathcal{I}_1 consists only of identity arrows, every arrow in \mathcal{I}_1^2 is mapped to the corresponding identity arrow. To see that f is an essential equivalence, first note that f is surjective, and hence essentially surjective on objects. To see that f is fully faithful, let $y, y' \in \mathcal{I}_0$. Suppose x and x' are preimages of y and y' , respectively, under f_0 . If $y \neq y'$, then there are no arrows either $y \rightarrow y'$ or $x \rightarrow x'$. If $y = y'$, then there is only the identity arrow from y to y' . If $x = x'$, then there is again only the identity arrow; if $x \neq x'$, then x and x' are in different components I_L and I_R of \mathcal{I}_0^2 , and again there is exactly one arrow $g : x \rightarrow x'$ which identifies x to x' . In all cases, the sets of arrows in \mathcal{I}_1^2 and \mathcal{I}_1 are isomorphic via f_1 , since there is always only one.

It is easy to create a similar essential equivalence from \mathcal{I}^n to \mathcal{I} , so all of these orbifold groupoids are Morita equivalent.

We create a category of orbispaces from the category of orbifold groupoids by inverting the Morita equivalences between orbifold groupoids. The way to do this is via a bicategory of fractions $\text{Orbigrpds}(W^{-1})$ where W is the class of essential equivalences [8]. In this construction, an arrow from \mathcal{G} to \mathcal{H} is given by a span of groupoid homomorphisms

$$\mathcal{G} \xleftarrow{v} \mathcal{K} \xrightarrow{\varphi} \mathcal{H}$$

where v is an essential equivalence. Such a span is also called a *generalized map* from \mathcal{G} to \mathcal{H} .

This definition reflects the fact that there are certain maps between orbispaces that can only be carried by certain representing orbifoldoids. So even though two orbifoldoids may be Morita equivalent, the homomorphisms out of them are not the same. In order to represent a map from the orbispace represented by \mathcal{G} to the orbispace represented by \mathcal{H} , we may need to replace the original representing orbifoldoid \mathcal{G} with an alternate \mathcal{K} representing the same orbispace, but which can be used to define the desired map via a span as above. The following gives an example of such a map.

EXAMPLE 4.3. We would like a map from the interval orbispace I with trivial isotropy to the teardrop, whose image is a path crossing from the lower portion of the teardrop to the upper portion, as in Figure 12.

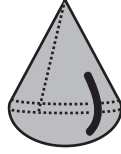


FIGURE 12. Path in the teardrop orbispace

We represent I by the orbifoldoid \mathcal{I} , and the teardrop by the orbifoldoid \mathcal{T} from Example 2.8. However, no homomorphism $\mathcal{I} \rightarrow \mathcal{T}$ can produce the desired path in the teardrop, since the image in \mathcal{T}_0 is partially in one connected component and partially in the other and so the map from objects \mathcal{I}_0 would fail to be continuous. On the other hand, we can create a homomorphism representing this path if we represent the interval orbispace I by the orbifoldoid \mathcal{I}^2 instead; the desired homomorphism $\mathcal{I}^2 \rightarrow \mathcal{T}$ to the teardrop is illustrated in Figure 13, where the overlap in \mathcal{I}^2 allows us to move between components of \mathcal{T}_0 . Note that the interval in the arrow space \mathcal{I}^2 that represents the glueing of the two components gets sent by the homomorphism to arrows in \mathcal{T} which identify the images of the overlap. Thus, this map is represented by a span $\mathcal{I} \leftarrow \mathcal{I}^2 \rightarrow \mathcal{T}$, where the map $\mathcal{I} \leftarrow \mathcal{I}^2$ is the one described in Example 4.2.

The choice of representing orbifoldoid matters in the codomain as well, as the next example shows.

EXAMPLE 4.4. In Example 2.7, we represented the silvered interval by the translation groupoid $\mathcal{SI} = \mathbb{Z}/2 \ltimes S^1$. We can also represent the silvered interval by a ‘broken’ version similar to the ‘broken’ versions of the interval. Specifically, \mathcal{SI}_0^2 is the disjoint union of two open intervals, and \mathcal{SI}_1^2 consists of the identity arrows, arrows representing a $\mathbb{Z}/2$ -folding action on each interval in the object space, and glueing arrows from the ends of one interval in the object space to the other. See Figure 14.

Now, consider the path $f : \mathcal{I} \rightarrow \mathcal{SI}$ in the silvered interval given by mapping the entire interval in \mathcal{I}_0 to the entire upper half of $S^1 = \mathcal{SI}_0$. Notice that the path in the orbispace includes both endpoints of the orbispace (the two points with $\mathbb{Z}/2$ -isotropy). This path cannot be achieved by a map $f' : \mathcal{I} \rightarrow \mathcal{SI}^2$. If such a map

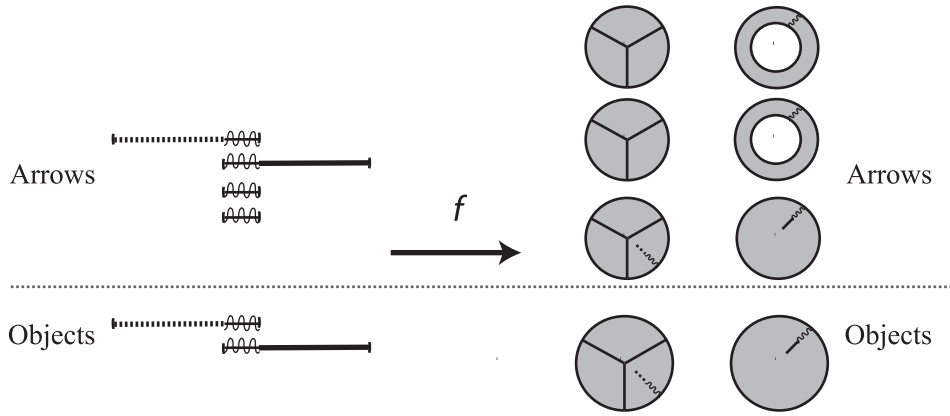


FIGURE 13. Map from \mathcal{I}^2 to \mathcal{T}

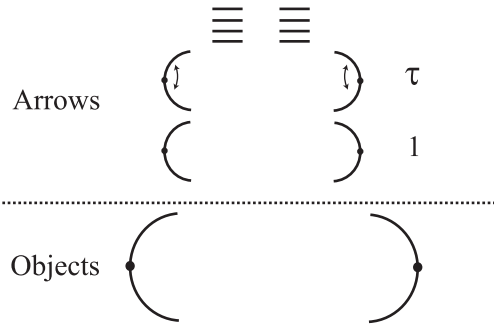
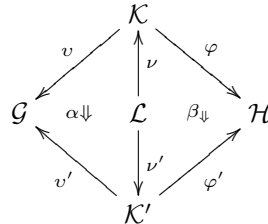


FIGURE 14. Orbifold \mathcal{SI}^2 representing the silvered interval

existed, the continuous map $f'_0 : I \rightarrow \mathcal{SI}_0^2$ would have to map the connected interval I to only one connected component of \mathcal{SI}_0^2 . The resulting path in the orbispace would include only one of the endpoints. Thus, there is no such path $f' : \mathcal{I} \rightarrow \mathcal{SI}^2$.

Next we need to consider identifications between generalized maps. A 2-cell between generalized maps is represented by an equivalence class of diagrams of the following form



The upcoming example gives two generalized maps with a 2-cell between them.

EXAMPLE 4.5. We start with the generalized maps, given by spans. We saw above in Example 4.3 that creating a span $\mathcal{I} \leftarrow \mathcal{I}^2 \rightarrow \mathcal{T}$ allows us to make a

generalized map from \mathcal{I} with image in two different charts, using the overlap in \mathcal{I}^2 to jump between charts. Here we show two spans which put the jump in different places, which have a 2-cell between them and represent the same map on the quotient spaces.

Recall that in Example 4.2 we glued together the two intervals in the object space of \mathcal{I}^2 to create an essential equivalence $f : \mathcal{I}^2 \rightarrow \mathcal{I}$. We do the same thing to \mathcal{I}^3 in two different ways. We can glue the middle and right intervals in the object space together giving a map that we will call $\nu : \mathcal{I}^3 \rightarrow \mathcal{RI}^2$. Alternatively, we can glue the left and middle intervals together, producing a map $\nu' : \mathcal{I}^3 \rightarrow \mathcal{LI}^2$. This gives two representations of the interval, \mathcal{RI}^2 and \mathcal{LI}^2 , which are broken in different places, with essential equivalences $\nu : \mathcal{RI}^2 \rightarrow \mathcal{I}$ and $\nu' : \mathcal{LI}^2 \rightarrow \mathcal{I}$ defined by mapping down to the ‘unbroken’ interval as in Example 4.2.

Now we define two maps $\varphi : \mathcal{RI}^2 \rightarrow \mathcal{TB}$ and $\varphi' : \mathcal{LI}^2 \rightarrow \mathcal{TB}$. On objects, φ maps the two ‘broken’ intervals in \mathcal{RI}^2 to paths in two different disks, labeled disk A and disk B in Figure 15, in the object space of \mathcal{TB} . (Only the pertinent parts of \mathcal{TB} are shown in the Figure.) The glueing arrows are mapped to the appropriate overlaps (of the form $i \circ \sigma$ to account for the necessary reflection) in \mathcal{TB}_0 with domain disk A and codomain disk B , or the inverse with domain disk B and codomain disk A (see Example 2.9). The overlaps will then glue the ends of the two paths together in the resulting orbispace.

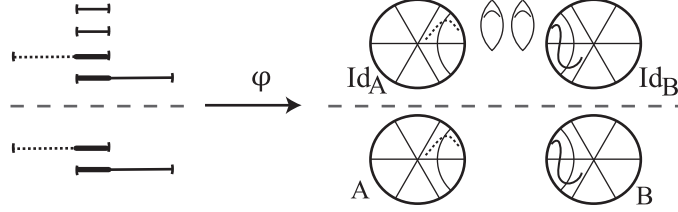


FIGURE 15. $\varphi : \mathcal{RI}^2 \rightarrow \mathcal{TB}$

The map φ' is similar to φ except that the jump between charts occurs in a different place, to match the different overlap in \mathcal{LI}^2 . See Figure 16. This gives us

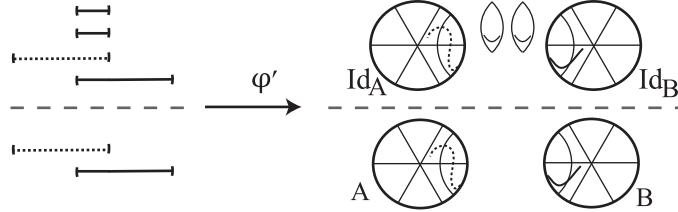


FIGURE 16. $\varphi' : \mathcal{LI}^2 \rightarrow \mathcal{TB}$

two generalized maps $\mathcal{I} \rightarrow \mathcal{TB}$ defined by $\mathcal{I} \xleftarrow{\nu} \mathcal{RI}^2 \xrightarrow{\varphi} \mathcal{TB}$ and

$\mathcal{I} \xleftarrow{\nu'} \mathcal{LI}^2 \xrightarrow{\varphi'} \mathcal{TB}$. These produce the same path in the orbispace for \mathcal{TB} , shown in Figure 17.

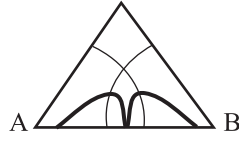
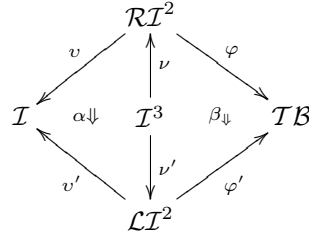


FIGURE 17. Path in \mathcal{TB}

We create a 2-cell between these two spans to identify them.



First, note that $\nu\nu' = \nu'\nu' : \mathcal{I}_0^3 \rightarrow \mathcal{I}_1$, and so we can fill in the left hand side of the above diagram with the identity natural transformation α . The natural transformation $\beta : \varphi\nu \Rightarrow \varphi'\nu'$ is given by the continuous map $\mathcal{I}_0^3 \rightarrow \mathcal{TB}_1$ illustrated in Figure 18. The overlap shown is the one with domain disk B and codomain disk A , the same as in the previous two Figures.

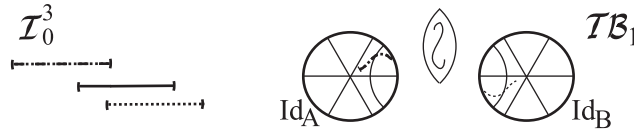
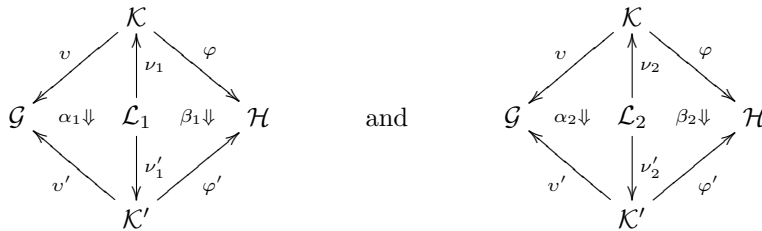


FIGURE 18. Natural transformation β

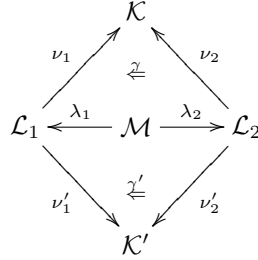
We see in this example that a 2-cell between two generalized maps requires a groupoid representation for the domain orbispace with essential equivalences into the groupoid representations used as the middle of the the two generalized maps. This situation is analogous to maps between manifolds, where one can define the maps on conveniently chosen atlases, but if we want to compare the maps, we need a common refinement.

We consider orbispaces via a 2-category of orbigroupoids with morphisms given by generalized maps and 2-cells as described above. In order to avoid higher structure, we put the following equivalence on the 2-cells. The two diagrams

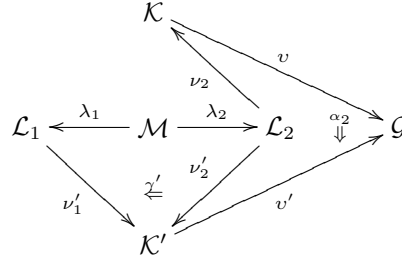


represent the same 2-cell when there are essential equivalences and 2-cells as in the following:

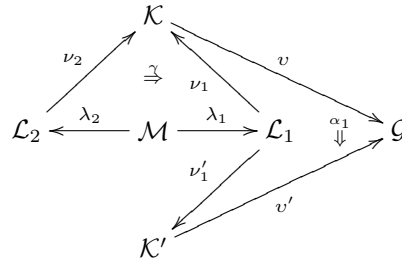
(1)



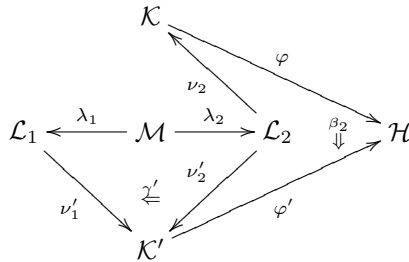
such that the composite of the pasting diagram



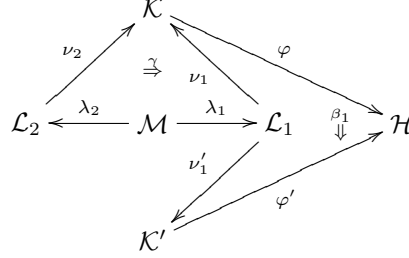
is equal to the composite of the pasting diagram



and the composite of the pasting diagram



is equal to the composite of the pasting diagram



Intuitively, two 2-cell diagrams represent the same 2-cell when there is a further Morita equivalent groupoid for the domain so that the original 2-cells agree on this further groupoid representation. For instance, if two paths into an orbifold \mathcal{H} are defined using two distinct subdivisions of the unit interval, giving rise to orbifold groupoids \mathcal{I}^3 and \mathcal{I}^4 say, a 2-cell diagram could represent a subdivision which is a common refinement of the two subdivisions given. In this case, two 2-cell diagrams would represent the same 2-cell if there is a further subdivision on which the 2-cells become the same.

We define a 2-category of orbispaces with objects the orbifold groupoids, morphisms given by generalized maps and 2-cells given by the equivalence classes of the 2-cell diagrams as described here. Two generalized maps are considered to represent the same map of orbispaces if they have a 2-cell between them, but when we use these to create a mapping space which itself has the structure of an orbifold space, we will need to retain information about the ways in which these generalized maps were identified. We will illustrate this process in the next section.

5. Mapping Spaces for Orbifold Groupoids: Defining $\text{GMap}(\mathcal{G}, \mathcal{H})$

In this section, we construct a mapping groupoid $\text{GMap}(\mathcal{G}, \mathcal{H})$ between two orbifold groupoids. This will be based on the definitions from Section 3, with the maps defined by homomorphisms and identifications between maps given by natural transformations; we explain how to topologize this structure so that we obtain a topological groupoid. We will show examples where this again becomes an orbifold groupoid, hence representing an orbifold space structure on the mapping object. Actually constructing the correct mapping space for any orbispaces using these mapping groupoids takes some technical work which is not addressed here, but is detailed in the paper [9]. In this paper, we merely illustrate how the orbifold space structure comes into existence in these basic building blocks.

The topology we will put on our mapping groupoid is based on topologies defined for various mapping spaces. Topological spaces as a category are not Cartesian closed, and so various substitute categories are commonly used instead. One standard approach is to work with compactly generated spaces. However, if we use the compact open topology to define a topology on a mapping space $\text{Top}(X, Y)$, this may not be compactly generated even when X and Y are. This is commonly fixed by applying the k -functor, which adds in some open and closed sets to the topology and makes $k\text{Top}(X, Y)$ compactly generated again. We will use this standard dodge in defining the topology on our orbifold groupoids. Therefore for any of the spaces that

follow, $\text{Map}(X, Y)$ will denote the topological space $k\text{Top}(X, Y)$ obtained by taking the compact open topology and then applying the k -functor. See [13, 14] for further details.

The mapping groupoid $\text{GMap}(\mathcal{G}, \mathcal{H})$ is defined as follows.

Objects The objects of $\text{GMap}(\mathcal{G}, \mathcal{H})_0$ are the homomorphisms $\mathcal{G} \rightarrow \mathcal{H}$. As defined in Section 3, such a functor f is defined by a map on objects $f_0 : \mathcal{G}_0 \rightarrow \mathcal{H}_0$ and a map on arrows $f_1 : \mathcal{G}_1 \rightarrow \mathcal{H}_1$. We topologize these as a subspace of $\text{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \text{Map}(\mathcal{G}_1, \mathcal{H}_1)$ with the k -ified compact open topology. The subspace taken will be those pairs of maps which make all the necessary diagrams commute: $sf_1 = f_0s$, $tf_1 = f_0t$, $m(f_1 \times f_1) = f_1m$, and $uf_0 = f_1u$. So,

$$\begin{aligned} \text{GMap}(\mathcal{G}, \mathcal{H})_0 &= \{(f_0, f_1) \in \text{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \text{Map}(\mathcal{G}_1, \mathcal{H}_1) | \\ &\quad sf_1 = f_0s, tf_1 = f_0t, m(f_1 \times f_1) = f_1m, uf_0 = f_1u\} \end{aligned}$$

This is a closed subspace of $\text{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \text{Map}(\mathcal{G}_1, \mathcal{H}_1)$. Notice that in this subspace, f_0 is completely determined by f_1 , since we can view \mathcal{G}_0 as a subspace of \mathcal{G}_1 via the map to the identity arrows u . So we can think of a point in this space as being defined by just an f_1 which preserves composition and maps $u(\mathcal{G}_0)$ to $u(\mathcal{H}_0)$, and represent this same subspace as

$$\text{GMap}(\mathcal{G}, \mathcal{H})_0 = \{f \in \text{Map}(\mathcal{G}_1, \mathcal{H}_1) | m(f \times f) = fm, f(u(\mathcal{G}_0)) \subseteq u(\mathcal{H}_0)\},$$

equipped with the subspace topology. In this representation, we can see that this is a closed subspace because \mathcal{H} is étale.

Arrows The arrows of the mapping groupoid $\text{GMap}(\mathcal{G}, \mathcal{H})_1$ are defined by natural transformations between the functors which are its objects. Explicitly, a natural transformation $\alpha : f \rightarrow f'$ is given by $\alpha \in \text{Map}(\mathcal{G}_0, \mathcal{H}_1)$ making the following diagram commute: for $g : x \rightarrow y$ in \mathcal{G}_1 ,

$$(2) \quad \begin{array}{ccc} f_0(x) & \xrightarrow{\alpha(x)} & f'_0(x) \\ f_1(g) \downarrow & & \downarrow f'_1(g) \\ f_0(y) & \xrightarrow{\alpha(y)} & f'_0(y) \end{array}$$

We can think of this as a triple (f, α, f') where $f = (f_0, f_1)$ and $f' = (f'_0, f'_1)$ satisfy the functor relations, and in addition, (f, α, f') satisfies $m(\alpha(x), f'_1(g)) = m(f_1(g), \alpha(y))$ (and this equation makes sense, i.e. $s\alpha = f_0$ and $t\alpha = f'_0$). So we are looking at a subspace of the space

$$(3) \quad \begin{array}{ccc} [\text{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \text{Map}(\mathcal{G}_1, \mathcal{H}_1)] & \times_{\text{Map}(\mathcal{G}_0, \mathcal{H}_0)} & \text{Map}(\mathcal{G}_0, \mathcal{H}_1) \\ & \times_{\text{Map}(\mathcal{G}_0, \mathcal{H}_0)} & [\text{Map}(\mathcal{G}_0, \mathcal{H}_0) \times \text{Map}(\mathcal{G}_1, \mathcal{H}_1)] \end{array}$$

with the k -ified compact open topology.

Now we look closer at the space of arrows of $\text{GMap}(\mathcal{G}, \mathcal{H})$. Explicitly, the elements of the subspace of natural transformations can be written as $(f_0, f_1, \alpha, f'_0, f'_1)$ such that the following holds:

- $s\alpha = f_0$ and $t\alpha = f'_0$
- $m(\alpha s, f'_1) = m(f_1, \alpha t)$
- $sf_1 = f_0s, tf_1 = f_0t, m(f_1 \times f_1) = f_1m, uf_0 = f_1u$ (i.e. f is a functor)
- $sf'_1 = f'_0s, tf'_1 = f'_0t, m(f'_1 \times f'_1) = f'_1m, uf'_0 = f'_1u$ (i.e. f' is a functor)

Note that $\text{GMap}(\mathcal{G}, \mathcal{H})_1$ is a closed subspace of (3). Again, we notice that parts of such 5-tuples completely determine the rest, just as was the case for the space of objects.

Structure Maps Now we look at the structure maps of the groupoid $\text{GMap}(\mathcal{G}, \mathcal{H})$.

- The *source map* $s: \text{GMap}(\mathcal{G}, \mathcal{H})_1 \rightarrow \text{GMap}(\mathcal{G}, \mathcal{H})_0$ is defined by

$$s(f_0, f_1, \alpha, f'_0, f'_1) = (f_0, f_1).$$

- Similarly, the *target map* $t: \text{GMap}(\mathcal{G}, \mathcal{H})_1 \rightarrow \text{GMap}(\mathcal{G}, \mathcal{H})_0$ is defined by

$$t(f_0, f_1, \alpha, f'_0, f'_1) = (f'_0, f'_1).$$

- The *unit map* $u: \text{GMap}(\mathcal{G}, \mathcal{H})_0 \rightarrow \text{GMap}(\mathcal{G}, \mathcal{H})_1$ is defined by

$$u(f_0, f_1) = (f_0, f_1, u, f_0, f_1).$$

- The *composition map*

$$m: \text{GMap}(\mathcal{G}, \mathcal{H})_1 \times_{\text{GMap}(\mathcal{G}, \mathcal{H})_0} \text{GMap}(\mathcal{G}, \mathcal{H})_1 \rightarrow \text{GMap}(\mathcal{G}, \mathcal{H})_1$$

is defined by

$$m((f_0, f_1, \alpha, f'_0, f'_1), (f'_0, f'_1, \beta, f''_0, f''_1)) = (f_0, f_1, m(\alpha, \beta), f''_0, f''_1).$$

- The *inverse map* $i: \text{GMap}(\mathcal{G}, \mathcal{H})_1 \rightarrow \text{GMap}(\mathcal{G}, \mathcal{H})_1$ is defined by

$$i(f_0, f_1, \alpha, f'_0, f'_1) = (f'_0, f'_1, i\alpha, f_0, f_1).$$

It is clear from the definitions that each of these will be continuous, since they are defined from continuous maps on the various components. Hence, $\text{GMap}(\mathcal{G}, \mathcal{H})$ is a topological groupoid.

This groupoid may not be an orbifold, but in many cases it is. So we have a way of defining a mapping orbifold which carries the isotropy structure on the maps. This may not be the correct mapping space in the 2-category of orbispaces because of the need to consider generalized maps. However, the basic orbifold structure carried by these mapping groupoids $\text{GMap}(\mathcal{G}, \mathcal{H})$ is what is used to create the mapping orbifold of [9]. Therefore we finish this section by carefully working through several examples.

The first example comes from the following observation.

PROPOSITION 5.1. *There is an isomorphism of topological groupoids*

$$\text{GMap}(*_1, \mathcal{G}) \cong \mathcal{G}.$$

This result follows from the fact that the category of topological groupoids in compactly generated spaces is Cartesian closed, but we opt to include a proof to illustrate the process of creating these mapping groupoids, and as a first example of the more general case of calculating $\text{GMap}(*_G, \mathcal{G})$ where G is an arbitrary finite group.

PROOF. The objects of $\text{GMap}(*_1, \mathcal{G})_0$ are homomorphisms $f: *_1 \rightarrow \mathcal{G}$ defined on the objects, $f_0: * \rightarrow \mathcal{G}_0$, and on the arrows, $f_1: 1 \rightarrow \mathcal{G}_1$. Now f_0 can map $*$ into any point $y \in \mathcal{G}_0$, and this determines the map on the arrow, $f_1(1) = id_y$. Thus the objects f in $\text{GMap}(*_1, \mathcal{G})$ are in one-to-one correspondence with the objects y of \mathcal{G} . Moreover, the topology is defined using the subspace topology on the mapping space $\text{Map}(*, \mathcal{G}_0) \times \text{Map}(1, \mathcal{G}_1)$ with $uf_0 = f_1u$, which means that the correspondence $f \rightarrow y$ is a homeomorphism.

The arrows of $\text{GMap}(*_1, \mathcal{G})_1$ consist of natural transformations between homomorphisms. If $f, f' \in \text{GMap}(*_1, \mathcal{G})_0$, a natural transformation $\alpha : f \Rightarrow f'$ is given by $\alpha(*) = g \in \mathcal{G}_1$ such that $s \circ \alpha(*) = y = f_0(*)$, $t \circ \alpha(*) = y' = f'_0(*)$, and the following diagram commutes:

$$\begin{array}{ccc} y & \xrightarrow{g} & y' \\ f_1(1)=id_y \downarrow & & \downarrow f'_1(1)=id_{y'} \\ y & \xrightarrow{g} & y' \end{array}$$

Since this diagram always commutes, the existence of α depends only on the existence of an arrow $g : f_0(*) \rightarrow f'_0(*)$ in \mathcal{G}_1 . Hence, the arrows in $\text{GMap}(*_1, \mathcal{G})$ are in one-to-one correspondence with the arrows in \mathcal{G} . Again, looking at the topology of $\text{GMap}(*_1, \mathcal{G})_1$, it is easy to see that this correspondence is a homeomorphism. \square

In the next examples, we will consider the mapping spaces from the one-point space $*_G$ (with isotropy group G) to the triangular billiard \mathcal{TB} of Example 2.9. We will compare the results for isotropy groups G taken to be the trivial group, $\mathbb{Z}/2$, $\mathbb{Z}/3$, and $\mathbb{Z}/6$.

In Proposition 5.1 we proved that for any orbifold, $\text{GMap}(*_1, \mathcal{G}) \simeq \mathcal{G}$. Thus $\text{GMap}(*_1, \mathcal{TB}) \simeq \mathcal{TB}$.

EXAMPLE 5.2. Here we describe $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})$. We denote the elements of $\mathbb{Z}/2$ by 1 and τ , and use the same notation for the corresponding arrows in $*_{\mathbb{Z}/2}$. We will denote the component disks of the object space \mathcal{TB}_0 by disks A , B , and C . Recall that the arrow space \mathcal{TB}_1 includes six disjoint copies of each disk, one for each element of D_3 . We will denote the arrow disk that represents the rotation element ρ of D_3 acting on disk A by ρ_A . So arrows in the disk ρ_A have source $x \in A$ and target $\rho x \in A$. The other arrow disks will be denoted similarly. The arrow space \mathcal{TB}_1 also contains overlaps representing the arrows with source in one disk and target in another.

The object space of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})$ consists of homomorphisms from $*_{\mathbb{Z}/2}$ to \mathcal{TB} . Any such homomorphism will map the object $*$ to a point y in one of the disks A , B , or C , and the arrow 1 to the corresponding point in 1_A , 1_B , or 1_C . The arrow τ must map to an arrow $y \rightarrow y$, so the choices for the image of the arrow τ will depend on the point y . We will consider what happens when y is in disk A ; the cases with y in disks B and C are similar. If $f_0(*) = y$ has trivial isotropy, so y is neither the center point nor a point on one of the reflections lines in disk A , then $f_1(\tau)$ must map to the point representing id_y in disk 1_A ; see map f in Figure 19. If $f'_0(*) = y$ is a point on one of the reflection lines, say the one for the reflection σ on disk A , but not the center point, then $f'_1(\tau)$ can be either in disk 1_A or disk σ_A ; see map f' in Figure 19. Similarly for y fixed by $\sigma\rho$ or $\sigma\rho^2$. Finally, if $f''_0(*) = y$ is the center point of disk A , then every element of D_3 fixes y . However, since τ has order 2, $f''_1(\tau)$ must be the center point of one of the arrow disks 1_A , σ_A , $\sigma\rho_A$, or $\sigma\rho_A^2$; see map f'' in Figure 19.

As a topological space, $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$ is defined as a subspace of $\text{Map}(*, \mathcal{TB}_0) \times \text{Map}(\mathbb{Z}/2, \mathcal{TB}_1)$ with the compatibility conditions making f_0 and f_1 into a functor of groupoids. So if $f_1(\tau)$ and $f'_1(\tau)$ land in a different components of \mathcal{TB}_1 as in the above case, then f and f' will be in different components of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$. On the other hand, the example homomorphisms f' and f'' from above are in the same

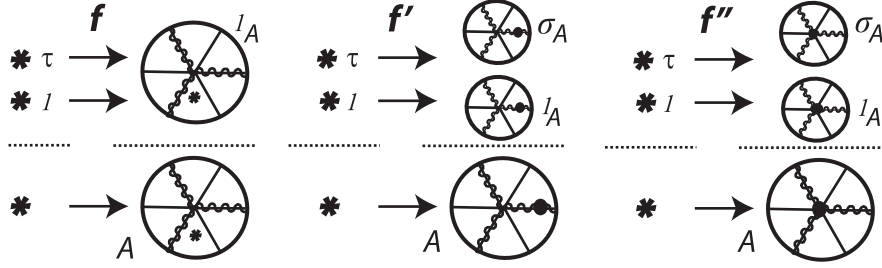


FIGURE 19. Some objects in $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$

component, the component which takes y to the points fixed by σ and takes τ to the arrow disk σ_A , and this component is homeomorphic to the line segment that is the subspace of disk A fixed by σ . Similarly we get line segments corresponding to $f_1(\tau) = \sigma\rho$ and $f_1(\tau) = \sigma\rho^2$. So each disk A , B and C contributes one disk and three line segments to the space of objects $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$. Note that there are 4 homomorphisms with image $y = c$ the center point, showing up in each of the 4 components. See Figure 20.

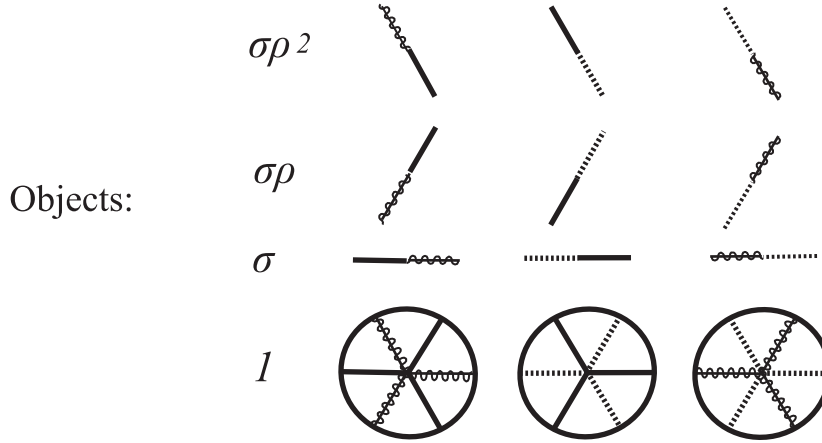


FIGURE 20. Space of objects $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$

Now we consider the space of arrows $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_1$, given by natural transformations $\alpha: f \rightarrow f'$ between functors f and f' . These are defined by a map $\alpha \in \text{Map}(*, \mathcal{TB}_1)$ with the properties that $\alpha(*) : f_0(*) \rightarrow f'_0(*)$, and for any map $g : * \rightarrow *$ in the arrow space of $*_{\mathbb{Z}/2}$, the following diagram commutes:

$$\begin{array}{ccc}
 f_0(*) & \xrightarrow{\alpha(*)} & f'_0(*) \\
 f_1(g) \downarrow & & \downarrow f'_1(g) \\
 f_0(*) & \xrightarrow{\alpha(*)} & f'_0(*)
 \end{array}$$

Note that for commutativity to hold, we must have the conjugacy relation

$$(4) \quad \alpha(*)f_1(g)\alpha(*)^{-1} = f'_1(g).$$

If $g = 1$, then $f_1(g)$ and $f'_1(g)$ are identity maps and the diagram always commutes. So suppose $g = \tau$. If $f_1(\tau)$ is an identity, then the conjugacy relation becomes $\alpha(*)\alpha(*)^{-1} = f'_1(\tau)$, and so $f'_1(\tau)$ must also be an identity. In this case, the natural transformations $\alpha : f \Rightarrow f'$ correspond exactly to maps $f_0(*) \rightarrow f'_0(*)$ in \mathcal{TB}_1 . Thus these arrows form a subspace $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_1$ consisting of a copy of \mathcal{TB}_1 . This is not surprising since these homomorphisms factor through $*_1$, and by Proposition 5.1, homomorphisms from $*_1$ and their natural transformations form a copy of \mathcal{TB} .

Now, consider the case where $f_1(\tau)$ is a non-identity arrow. The conjugacy relation (4) implies that $f'_1(\tau)$ must also be a non-identity arrow. Let $y = f_0(*)$ and $y' = f'_0(*)$. By our analysis of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$, we know that y and y' are points on reflection lines in disks A , B , or C (they may be in different disks).

First, suppose that y and y' are in the same disk, say disk A , and $f_1(\tau)$ is the corresponding arrow in one of the disks σ_A , $\sigma\rho_A$, or $\sigma\rho_A^2$. Note that $\alpha(*) : y \rightarrow y'$, so y' must be one of y , ρy , or $\rho^2 y$, and $\alpha f_1(\tau)\alpha^{-1} = f'(\tau)$. So for example, if $y' = \rho y$ and $f(\tau) = \sigma$, there are two possible choices for α that will satisfy (with abuse of notation) $\alpha\sigma\alpha^{-1} = \sigma\rho$: $\alpha(*)$ must be the arrow corresponding to y in either disk ρ_A or disk $\sigma\rho_A^2$. Similarly, for each such pair y, y' we have exactly two natural transformations from f to f' . Note that this includes the case when $f = f'$. Again, the topology is based on a subspace of the mapping space $\text{Map}(*, \mathcal{TB}_1)$, which means that we have components corresponding to the α 's that map τ into the various components of \mathcal{TB}_1 . So the arrow space $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_1$ contains 6 copies of each reflection line, with two copies mapping to each of itself and the other 2 lines.

Now consider the case where y and y' are in different disks, say y is in disk A and y' is in disk C . Then if $\alpha(*) : y \rightarrow y'$, both y and y' must be in the overlap area of disks A and C . In this case, there will be exactly two such $\alpha(*)$ arrows glueing an end of the line segment associated with disk A to an end of the line segments associated with disk C , one with a reflection and one without (see Example 2.9); for each of these choices, the conjugacy relation 4 will hold. Note that topologically, these are subspaces of the $3 \times 36 = 108$ components of the overlap arrows in \mathcal{TB}_1 , now arranged to glue together the overlaps of the $3 \times 3 = 9$ line segments, with 2 glueings in each direction for each choice of pairs that overlap. We illustrate the effect of all of the arrows (glueings) on the line segments in $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$ in Figure 21; note that we have not drawn the multiplicity of the glueing arrows. As shown in the figure, the quotient space is a circle with $\mathbb{Z}/2$ -isotropy at each point.

Putting all of the natural transformations together, we have that the arrow space $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_1$ contains a copy of \mathcal{TB}_1 , 6 line segments for each of the 3 reflection lines in each of the disks A , B , and C , and one shorter line segment for each overlap in \mathcal{TB}_1 . The line segments of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$ wind up glued together into a circle with $\mathbb{Z}/2$ -isotropy at each point, and the three disks in $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$ are glued together into a copy of \mathcal{TB} , as in Figure 22.

This example shows how the mapping groupoid inherits both topology and isotropy structure from the domain and codomain groupoids, giving a rather interesting structure. For comparison, we will now construct $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})$, which

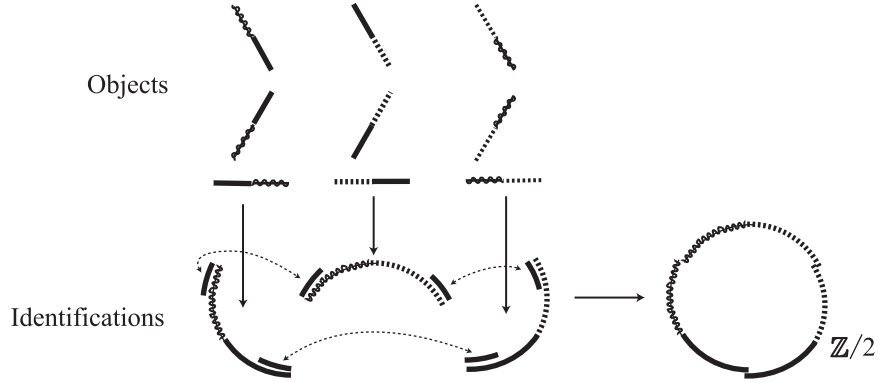


FIGURE 21. Glueings of the line segment components of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})_0$

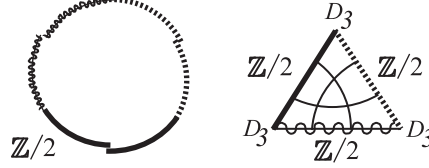


FIGURE 22. The quotient space of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})$

we will see is a lot simpler since there is only one point with an isotropy group whose order is a multiple of 3.

EXAMPLE 5.3. Now we consider $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})$. We denote the elements of $\mathbb{Z}/3$ by 1 , ν , and ν^2 , and hence also the arrows of $*_{\mathbb{Z}/3}$. The analysis of $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})$ is very similar to that of Example 5.2. The difference is that $\mathbb{Z}/3$ has elements of order 3 instead of order 2, and so we will be looking at points in \mathcal{TB}_0 with order 3 isotropy elements. As before, any homomorphism $f \in \text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_0$ will map the object $*$ to a point y in one of the disks A , B , or C , and the arrow 1 to the corresponding point in 1_A , 1_B , or 1_C . If $f_0(*) = y$ does not have order 3 isotropy, i.e., if y is not the center point of a disk, then $f_1(\nu)$ and $f_1(\nu^2)$ must also map to the corresponding identity point in 1_A , 1_B , or 1_C . If $f_0(*) = y$ is the center point of a disk, say disk A , then $f_1(\nu)$ can be any arrow of order 3, namely the center point of one of the disks 1_A , ρ_A , or ρ_A^2 . Since ν^2 is the inverse of ν , $f_1(\nu^2)$ is determined by $f_1(\nu)$ and must be the center point of 1_A , ρ_A^2 , or ρ_A , respectively. So we get 3 homomorphisms with $f_0(*) = y$ the center of disk A , one of which is the centre point for an identity component disk, and two isolated points which are their own components. The other two disks are similar, and each disk A , B and C contributes one disk and two points to $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_0$. See Figure 23.

For the space of arrows $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_1$, consider the natural transformations α between functors $f, f' \in \text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_0$. As before, $\alpha \in \text{Map}(*, \mathcal{TB}_1)$ is such that $\alpha(*) : f_0(*) \rightarrow f'_0(*)$, and for any arrow g of $*_{\mathbb{Z}/3}$, $\alpha(*)f_1(g)\alpha(*)^{-1} = f'_1(g)$. The conjugacy relation always holds for $g = 1$, and if it holds for $g = \nu$, then it will

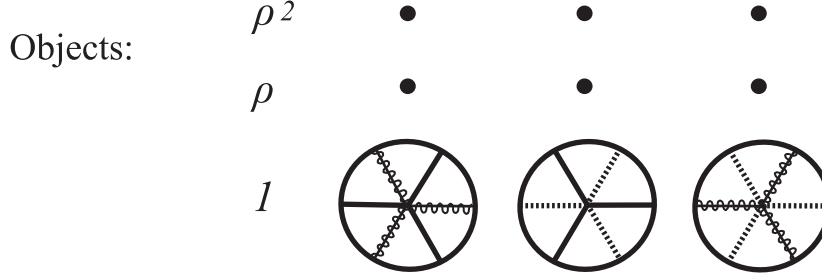


FIGURE 23. Space of objects $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_0$

also hold for $g = \nu^2$, so we only need to check that the conjugacy relation holds for $g = \nu$.

Suppose $g = \nu$. If $f_1(\nu)$ is an identity, then the conjugacy relation becomes $\alpha(*)\alpha(*)^{-1} = f'_1(\nu)$, and $f'_1(\nu)$ must also be an identity, and As in the previous example, the natural transformations $\alpha : f \Rightarrow f'$ correspond exactly to maps $f_0(*) \rightarrow f'_0(*)$ in \mathcal{TB}_1 , and these arrows form a subspace $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_1$ consisting of a copy of \mathcal{TB}_1 .

Now, consider the case where $f_1(\nu)$ and $f'_1(\nu)$ are non-identity arrows. Let $y = f_0(*)$ and $y' = f'_0(*)$; then y and y' are center points of disks A, B , or C . They must both be in the same disk for there to be an arrow $\alpha(*) : y \rightarrow y'$ in \mathcal{TB}_1 . If $y = y'$ is the center point of disk A , there are three possible α 's for each choice of f_1 and f'_1 . For example, if $f_1(\nu) \in \rho_A$ and $f'_1(\nu) \in \rho_A^2$, then $\alpha(*)$ is the center point of any one of $\sigma_A, \sigma_{\rho_A}$ and $\sigma_{\rho_A^2}$. These arrows glue together the pairs of object points in $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_0$ for disk A , resulting in one point with $\mathbb{Z}/3$ -isotropy in the quotient space. The result is similar if y is the center point of disk B or C .

All together, we have that the arrow space $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})_1$ contains a copy of \mathcal{TB}_1 and 12 disjoint arrow points associated to each of the disks A, B , and C . The quotient space for the mapping groupoid $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})$ is composed of a copy of \mathcal{TB} and three disjoint points, each with $\mathbb{Z}/3$ -isotropy. See Figure 24.

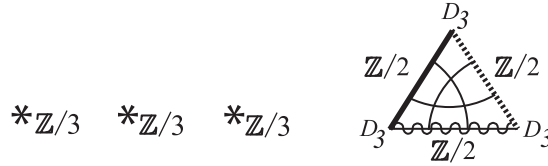


FIGURE 24. The quotient space of $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})$

We have seen how maps from $*_{\mathbb{Z}/2}$ pick up information about isotropy of order 2, and maps from $*_{\mathbb{Z}/3}$ pick up information about isotropy of order 3. If we want to get all of it combined, we consider $G = \mathbb{Z}/6$.

EXAMPLE 5.4. Finally we consider $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})$, generated by γ with $\gamma^6 = 1$. Note that $\mathbb{Z}/6$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/3$, where the order two generator $\tau = \gamma^3$

and the order three generator $\nu = \gamma^2$. Again, we use the same notation for the arrows of $*_{\mathbb{Z}/6}$. The analysis is a combination of the analyses above for $*_{\mathbb{Z}/2}$ and $*_{\mathbb{Z}/3}$.

Any homomorphism $f \in \text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})_0$ maps $*$ to a point y in one of the disks of A , B , or C and arrow 1 to the corresponding point in 1_A , 1_B , or 1_C . We have to consider points with isotropy of order 2 or 3 (or trivial isotropy). Since a homomorphism respects composition and γ generates $\mathbb{Z}/6$, the image $f_1(\gamma)$ determines the rest of f_1 .

If $f_0(*) = y$ does not have isotropy of order 2 or 3, then γ (and hence all of the other arrows in $\mathbb{Z}/6$) maps to id_y , and we get a homeomorphic copy of each of the disks. If y has order 2 isotropy (and hence is on a reflection line) then f_1 can map γ to any of the arrows determined in the $*_{\mathbb{Z}/2}$ case, that is, to reflection arrows. If, for example, $f_1(\gamma)$ is in the σ arrow disk, then $f_1(\gamma^3) = f_1(\gamma^5)$ are also the arrow corresponding to y in the σ disk, and $f_1(\gamma^2)$ and $f_1(\gamma^4)$ map to the identity arrow on y . As before, this contributes three line segments for each disk to the object space. If y has order 3 isotropy (and hence is a center point) then $f_1(\gamma)$ can be any of the order 3 rotation arrows. So we also have two isolated points for each disk.

Overall, we get a disk, three line segments, and two points in $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})_0$ associated to each disk. See Figure 25.

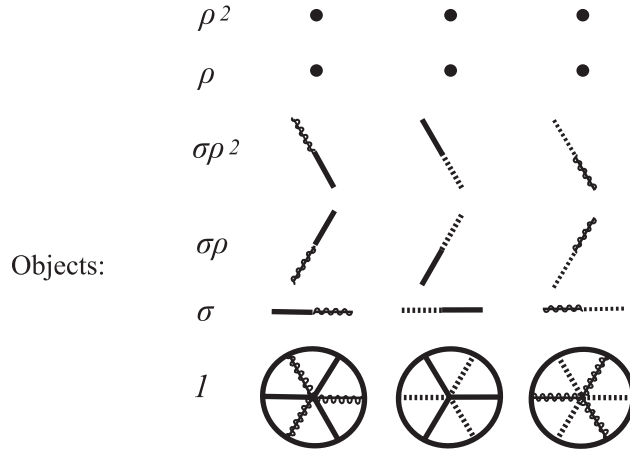
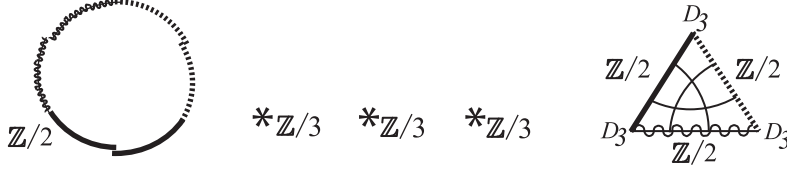


FIGURE 25. Space of objects $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})_0$

The space of arrows $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})_1$ also correspond to the arrows considered in Examples 5.2 and 5.3, since they are based on conjugacy relations of arrows in \mathcal{TB} . We get 6 line segments in $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})_1$ for each reflection line in each of the disks in \mathcal{TB}_0 , and 12 disjoint arrow points in $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})_1$ for the isolated points of each of the disks. The quotient space of the mapping groupoid $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})$ is given in Figure 26. Notice that it includes each of the components of the quotient spaces of $\text{GMap}(*_{\mathbb{Z}/2}, \mathcal{TB})$ and $\text{GMap}(*_{\mathbb{Z}/3}, \mathcal{TB})$, as expected.

EXAMPLE 5.5. We now describe the mapping space $\text{GMap}(\mathcal{I}, \mathcal{C}_3)$ from the interval to the order three cone point of Example 2.6. For any $f \in \text{GMap}(\mathcal{I}, \mathcal{C}_3)_0$, we must have $f_0 : \mathcal{I}_0 \rightarrow D^2$ and $f_1 : \mathcal{I}_1 \rightarrow (\mathcal{C}_3)_1 = \mathbb{Z}/3 \times D^2$, compatible with the functor conditions. Any continuous map from the interval to the disk will work for

FIGURE 26. The quotient space of $\text{GMap}(*_{\mathbb{Z}/6}, \mathcal{TB})$

f_0 . Then, to be compatible, f_1 must be of the form $f_1(id_x) = (id_{f_0(x)}, f_0(x))$ for each $x \in \mathcal{I}_0$. Thus, $\text{GMap}(\mathcal{I}, \mathcal{C}_3)_0$ is homeomorphic to $\text{Map}(I, D^2)$, the space of continuous maps $I \rightarrow D^2$ with the k -ified compact open topology.

Next, to understand $\text{GMap}(\mathcal{I}, \mathcal{C}_3)_1$, let $f \in \text{GMap}(\mathcal{I}, \mathcal{C}_3)_0$. We will characterize the natural transformations α by looking at the possible homomorphisms $f' \in \text{GMap}(\mathcal{I}, \mathcal{C}_3)_0$ for which there is a natural transformation $\alpha : f \rightarrow f'$. By the definition of a natural transformation, for each $x \in \mathcal{I}_0$, we must have $\alpha(x) \in (\mathcal{C}_3)_1$ such that $\alpha(x) : f_0(x) \rightarrow f'_0(x)$. It follows that $f'_0(x)$ must be in the orbit of $f_0(x)$ under the action of $\mathbb{Z}/3$. Additionally, since α must be continuous, it lands in one component of the arrows of \mathcal{C}_3 , and so f'_0 must be $1f$, νf or $\nu^2 f$ for all $x \in \mathcal{I}_0$. For each such f' , we get exactly one $\alpha : f \rightarrow f'$ where α maps into the component of the appropriate rotation. The topology is given by a subspace of $\text{Map}(I, D^2) \times \text{Map}(I, \mathbb{Z}/3 \times D^2)$, and the compatibility in this case means we are really just looking at $\text{Map}(I, \mathbb{Z}/3 \times D^2)$ which is homeomorphic to $\mathbb{Z}/3 \times \text{Map}(I, D^2)$, with source and target maps defined as in a translation groupoid.

So the mapping groupoid is just the translation groupoid $\text{GMap}(\mathcal{I}, \mathcal{C}_3) \simeq \mathbb{Z}/3 \times \text{Map}(I, D^2)$. Note that each point in $\text{GMap}(\mathcal{I}, \mathcal{C}_3)_0 = \text{Map}(I, D^2)$ has trivial isotropy, except the point given by the map taking the entire interval I to the cone point at the center of D^2 , which has $\mathbb{Z}/3$ -isotropy.

This example can be generalized as follows.

PROPOSITION 5.6. *If X is a connected orbispace with trivial isotropy, represented as an orbifold X with object space X and arrow space X (the identity maps), and G is a finite group acting on a topological space Y , then*

$$\text{GMap}(X, G \times Y) \simeq G \times \text{Map}(X, Y).$$

PROOF. The argument is the same as given in Example 5.5 above. Let $f \in \text{GMap}(X, G \times Y)_0$. Then f is determined by $f_0 : X \rightarrow Y$ since X_1 consists only of identity arrows. So $\text{GMap}(X, G \times Y)_0$ is homeomorphic to $\text{Map}(X, Y)$. Arrows $\alpha : f \rightarrow f'$ in $\text{GMap}(X, G \times Y)_1$ are given by continuous maps $\alpha : X \rightarrow G \times Y$ such that for each $x \in X$, $\alpha(x) : f_0(x) \rightarrow f'_0(x)$. Thus, such an arrow exists only when $f'_0(x)$ is in the orbit of $f_0(x)$ under the action of G , and continuity requires that α land in one component of the arrows $G \times Y$. Therefore f'_0 is of the form gf_0 for some $g \in G$, and α is completely determined by f and g ; so $\text{GMap}(X, G \times Y)_1$ is homeomorphic to $G \times \text{Map}(X, Y)$, with source and target maps as given by the translation groupoid $G \times \text{Map}(X, Y)$. \square

In each of these cases, we produced an orbispace. However, $\text{GMap}(\mathcal{G}, \mathcal{H})$ is not necessarily an orbifold (i.e., étale and proper) in general. Even when it is, we have examples showing that it is not invariant under Morita equivalence in

the sense that when \mathcal{G} and \mathcal{G}' are Morita equivalent orbifold groupoids, and \mathcal{H} and \mathcal{H}' are Morita equivalent orbifold groupoids, $\mathrm{GMap}(\mathcal{G}, \mathcal{H})$ and $\mathrm{GMap}(\mathcal{G}', \mathcal{H}')$ are not necessarily Morita equivalent; see Examples 4.3 and 4.4. Therefore the mapping groupoids we have created are not sufficient to be mapping objects for the category of orbispaces. However, they do form the basic foundation for defining a mapping orbifold space which encodes the generalized maps; see [9].

6. Inertia Groupoids and Mapping Groupoids

In this section we will show that the inertia groupoid of [6] can be obtained as a mapping groupoid.

DEFINITION 6.1. Given an orbifold groupoid \mathcal{G} , its *inertia groupoid* $\wedge\mathcal{G}$ is defined as follows:

- **Objects** $(\wedge\mathcal{G})_0$: objects in $(\wedge\mathcal{G})_0$ are loops,

$$\begin{aligned} (\wedge\mathcal{G})_0 &= \{g \in \mathcal{G}_1 \mid s(g) = t(g)\} \\ &= \{(x, g) \in \mathcal{G}_0 \times \mathcal{G}_1 \mid x = s(g) = t(g)\} \end{aligned}$$

Note that since $s(g) = t(g)$, g is an element of the isotropy group of $x = s(g)$, i.e. $g \in G_x$

- **Arrows** $(\wedge\mathcal{G})_1$: Let $x, y \in \mathcal{G}_0$, and $h : x \rightarrow y$ be an arrow $h \in \mathcal{G}_1$. Then h defines arrows $(x, g) \rightarrow (h(x) = y, hgh^{-1})$ for all $g \in G_x$.

The examples we have seen thus far indicate that we might be able to describe the inertia groupoid $\wedge\mathcal{G}$ as a mapping groupoid of the form $\mathrm{GMap}(*_G, \mathcal{G})$. This is indeed the case, but if we want to do it as a map from $*_G$ with finite isotropy group G (so that $*_G$ is an orbifold groupoid) we need to require that there is an upper bound on the orders of the isotropy groups of \mathcal{G} . A natural way to ensure this is to require that \mathcal{G} is orbit compact.

DEFINITION 6.2. An orbifold groupoid \mathcal{G} is *orbit compact* if the quotient space $\mathcal{G}_0/\mathcal{G}_1$ is compact.

We will prove that for orbit compact groupoids, the inertia groupoid can be obtained as a mapping space $\mathrm{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$.

REMARK 6.3. This condition is rather natural. In [9] we require the condition of orbit compactness on \mathcal{G} in order to show that the mapping groupoid produced there gives an orbifold space. This condition is equivalent to Haefliger's condition [4] that \mathcal{G}_0 have a relatively compact open subset which meets every orbit (used there in proving that the 2-category of étale groupoids and Hilsum-Skandalis maps is Cartesian closed.)

PROPOSITION 6.4. *There is a functor $\Phi : \mathrm{GMap}(*_{\mathbb{Z}/n}, \mathcal{G}) \rightarrow \wedge(\mathcal{G})$.*

PROOF. Let \mathbb{Z}/n be generated by an element σ of order n . We define the functor Φ as follows. Objects of $\mathrm{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$ are defined by functors, so consider a functor $f : *_{\mathbb{Z}/n} \rightarrow \mathcal{G}$. Then $f_0(*) = x \in \mathcal{G}_0$ and $f_1(\sigma) = g \in \mathcal{G}_1$ such that $s(g) = t(g) = x$. So define

$$\Phi(f) = (x, g) = (f(*), f(\sigma)) \in (\wedge\mathcal{G})_0$$

Arrows of $\text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$ are defined by natural transformations between functors. So suppose that $\alpha : f \Rightarrow f'$ is a natural transformation in $\text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})_1$. Then α is defined by a map $* \rightarrow h \in \mathcal{G}_1$ such that the following diagram commutes:

$$\begin{array}{ccc} f(*) & \xrightarrow{f(\sigma)} & f(*) \\ h \downarrow & & \downarrow h \\ f'(*) & \xrightarrow{f'(\sigma)} & f'(*) \end{array}$$

Therefore we can define $\Phi(\alpha) = \alpha(*) = h$. The above commutative diagram says that $hf(\sigma)h^{-1} = f'(\sigma)$, and so h defines a morphism in $(\Lambda\mathcal{G})_1$,

$$(f(*), f(\sigma)) \xrightarrow{h} (f'(*), f'(\sigma)) = (h(f(*)), hf(\sigma)h^{-1}).$$

To show that $\Phi : \text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G}) \rightarrow (\Lambda\mathcal{G})_1$ is a functor we need to verify that the composition is respected. Let $\alpha_1 : f \Rightarrow f'$ and $\alpha_2 : f' \Rightarrow f''$ be natural transformations and h_1 and h_2 be the corresponding morphisms in $(\Lambda\mathcal{G})_1$. We need to check that $\Phi(\alpha_2 \circ \alpha_1) = \Phi(\alpha_2) \circ \Phi(\alpha_1)$:

$$\begin{aligned} \Phi(\alpha_2) \circ \Phi(\alpha_1) &= (f(*), f(\sigma)) \xrightarrow{h_2 \circ h_1} (h_2(h_1 f(*)), h_2(h_1 f(\sigma)h_1^{-1})h_2^{-1}) \\ &= (f(*), f(\sigma)) \xrightarrow{h_2 \circ h_1} ((h_2 h_1) f(*), (h_2 h_1) f(\sigma) (h_2 h_1)^{-1}) \\ &= (f(*), f(\sigma)) \xrightarrow{h_2 \circ h_1} (f''(*), f''(\sigma)) \\ &= \Phi(\alpha_1 \circ \alpha_2) \end{aligned}$$

□

PROPOSITION 6.5. *For any n , the functor Φ is injective on objects, and full and faithful. So Φ is an inclusion of $\text{GMap}(*_n, \mathcal{G})$ as a full subcategory of $\Lambda(\mathcal{G})$.*

PROOF. The map $\Phi_0 : \text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})_0 \rightarrow \Lambda(\mathcal{G})_0$ is injective on objects, since it sends a functor f to $(f(*), f(\sigma))$ and a functor from $*_{\mathbb{Z}/n}$ is determined by this information. To prove that Φ is full, we consider arbitrary f and f' , and suppose there is a morphism $\Phi(f) \xrightarrow{h} \Phi(f')$. So h satisfies

$$\begin{aligned} h(\Phi(f)) &= h(f(*), f(\sigma)) \\ &= (hf(*), hf(\sigma)h^{-1}) \\ &= (f'(*), f'(\sigma)) \end{aligned}$$

If $hf(*) = f'(*)$ and $hf(\sigma)h^{-1} = f'(\sigma)$ this means that the diagram

$$\begin{array}{ccc} f(*) & \xrightarrow{f(\sigma)} & f(*) \\ h \downarrow & & \downarrow h \\ f'(*) & \xrightarrow{f'(\sigma)} & f'(*) \end{array}$$

commutes. So $\alpha(*) = h$ defines a natural transformation between f and f' with $\Phi(\alpha) = h$. Hence Φ is surjective on arrow sets so it is full.

Now we show that Φ is faithful. So suppose that $\Phi(\alpha) = \Phi(\alpha')$ where $\alpha, \alpha' : f \Rightarrow f'$ are natural transformations between functors f and f' . This means that if $\alpha(*) = h_1$ and $h_2 = \alpha(*) \in (\Lambda\mathcal{G})_1$ then $h_1 = h_2$ and

$$(h_1 f(*), h_1 f(\sigma) h_1^{-1}) = (f'(*), f'(\sigma)) = (h_2 f(*), h_2 f(\sigma) h_2^{-1})$$

So both α and α' correspond to the same commutative diagram, that is to the same natural transformation between f and f' . Hence the functor Φ is faithful and the proposition follows. \square

PROPOSITION 6.6. *For each orbit compact orbifold \mathcal{G} , there is an n such that*

$$\wedge(\mathcal{G}) \simeq \text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$$

PROOF. In proposition 6.5 we have seen that for any n ,

$$\Phi : \text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G}) \rightarrow \wedge(\mathcal{G})$$

is an inclusion of $\text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$ as a full subcategory of $\wedge(\mathcal{G})$. To find an isomorphism $\text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G}) \xrightarrow{\cong} \wedge(\mathcal{G})$ we need to establish conditions on n that will guarantee that Φ surjection on objects as well.

For each $x \in \mathcal{G}_0$ let V_x be a neighbourhood of x as in Remark 2.3. Then the quotients of these V_x form an open cover of the quotient space $\mathcal{G}_1/\mathcal{G}_0$. Since \mathcal{G} is orbit compact, there is a finite subset V_{x_1}, \dots, V_{x_m} such that their quotients cover the quotient space. This means that each point in \mathcal{G}_0 is in the orbit of a point in $\bigcup_{i=1}^m V_{x_i}$. So all points have isotropy groups that are conjugate to the isotropy groups of points in $\bigcup_{i=1}^m V_{x_i}$. Now for each $i = 1, \dots, m$ all points in V_{x_i} have an isotropy group which is a subgroup of G_{x_i} , so we only need consider the groups G_{x_1}, \dots, G_{x_m} and all these groups are finite.

A map f in $\text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G})$ is given by a point $x = f(*)$ and an arrow $g = f(*) \in G_x$, and we have such a pair for any $g \in G_x$ such that $g^n = 1$. Hence $\Phi : \text{GMap}(*_{\mathbb{Z}/n}, \mathcal{G}) \rightarrow \wedge(\mathcal{G})$ will be surjective on objects if every isotropy element $h \in \mathcal{G}$ has order dividing n . Since each isotropy subgroup is conjugate to a subgroup of a structure group G_{x_i} , we can define n to be the smallest common multiple of the orders of the structure groups G_{x_1}, \dots, G_{x_m} . \square

7. Conclusion and Future Work

Although the GMap construction described in this paper is very useful in understanding how the isotropy structure comes about on the mapping space, it is not sufficient as a mapping object for orbispaces. We have mentioned that $\text{GMap}(\mathcal{G}, \mathcal{H})$ is not necessarily an orbifold (i.e., étale and proper), and that even when it is, it is not invariant under Morita equivalence.

In a forthcoming paper, two of the authors continue this project and show how to define a mapping groupoid which is Morita invariant, and use this to show that the bicategory of orbit compact orbispaces is Cartesian closed. Producing this groupoid requires some careful considerations of how to create a small category which can be inverted, rather than working with all essential equivalences, and in how to put the topology in place on top of the generalized maps and 2-cells. The groupoids obtained this way are homotopy colimits of the $\text{GMap}(\mathcal{G}, \mathcal{H})$ groupoids described in this paper, where the homotopy colimit is taken over a diagram of groupoids representing the same orbifold as \mathcal{G} . We will even see that the inclusions

of the $\text{GMap}(\mathcal{G}, \mathcal{H})$ groupoids into the larger groupoid are all fully faithful and all connected components of the object and arrow spaces of the mapping spaces as orbispaces are open subsets of the $\text{GMap}(\mathcal{G}, \mathcal{H})$ groupoids. So all the local structure on the true orbispace is obtained from the $\text{GMap}(\mathcal{G}, \mathcal{H})$ groupoids as illustrated here. Approaching orbispaces via groupoids allows us to get a more concrete understanding of what is going on with this category and its maps.

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