# Bicategories of Action Groupoids

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**Abstract**

We prove that the 2-category of action Lie groupoids localised in the following three different ways yield equivalent bicategories: localising at equivariant weak equivalences à la Pronk, localising using surjective submersive equivariant weak equivalences and anafunctors à la Roberts, and localising at all weak equivalences. This generalises the known result for representable orbifold groupoids. As an application, we show that any weak equivalence between action Lie groupoids is isomorphic to the composition of two special equivariant weak equivalences, again extending a result known for representable orbifold groupoids.

**Keywords:** action groupoid, bicategory of fractions

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# 1 Introduction

In many contexts, we consider the study of topological or Lie groupoids up to Morita equivalence, an equivalence relation generated by a topologised version of a categorical equivalence referred to as a weak equivalence. One standard way of working with this equivalence relation is to follow the method of Pronk and define a bicategory of fractions of Lie groupoids $\mathbf{LieGpoid}[W^{-1}]$ in which all of the Morita equivalences become invertible 1-cells (or arrows) [21]. In this bicategory, a 1-cell $\mathcal{G} \to \mathcal{H}$ is defined by a span of functors $\mathcal{G} \leftarrow \mathcal{K} \to \mathcal{H}$ where the functor $\mathcal{K} \to \mathcal{G}$ is a weak equivalence, and a 2-cell is defined as an equivalence class of diagrams with natural transformations. There is also a "smaller" type of localisation using anafunctors à la Roberts [25, 27]. By additionally requiring the elements of the class of weak equivalences under consideration to be surjective submersive on objects, we obtain a bicategory with similar 1-cells, but whose 2-cells are optimised representatives of the equivalence classes defining the 2-cells in the more general bicategory of fractions. Thus using anafunctors, a 2-cell is a natural transformation instead of a class of 2-commutative diagrams. This allows for easier computations when working with these action groupoids. Roberts proves that this bicategory is equivalent to the bicategory of fractions obtained using the recipe of Pronk.

Understanding conditions under which a class of 1-cells of a general bicategory admits a localisation is a current field of study; besides the references above, see also [1, 24]. The localisation of topological groupoids is made explicit in [6], where the authors discuss Lie groupoids but details are only provided for the topological case. [30] provides a detailed development of localisation for diffeological groupoids using Pronk's and Roberts' approaches, comparing them as well as connecting them to the theory of bibundles in the diffeological groupoid context established in [29], as well as to stacks over diffeological spaces.

An extremely important class of groupoids comes from actions of groups on nice spaces, such as Lie groups on smooth manifolds. When we have such a groupoid, we have the possibility of defining traditional functorial equivariant invariants, such as a fundamental group or cohomology theories. The issue that arises then is of respecting Morita equivalence. We can use functoriality to check that our chosen invariant is unchanged under *equivariant* weak equivalences, but *a priori* there is no mechanism for checking more general weak equivalences. Therefore, it becomes necessary to know whether Morita equivalent action groupoids are also Morita equivalent via equivariant weak equivalences.

Thus we consider the following question: if we consider the full sub-2-category of action groupoids can we create a bicategory of fractions inverting the equivariant weak equivalences? If so, is this equivalent to the full sub-bicategory we get by inverting more general weak equivalences and then restricting to action groupoids? If both questions have affirmative answers, then we attain the following possible template for transferring equivariant techniques to groupoids: consider a groupoid that is Morita equivalent to an action groupoid, the latter of which admits some equivariant invariant. If this invariant respects Morita equivalences between action groupoids, meaning that equviariant weak equivalences induce isomorphisms between the invariants, then this equivariant invariant is well-defined.

This strategy has been successfully applied in special cases. One such case is the study of orbifolds; originally defined by Satake [28] using charts, these have more recently been described using certain topological and Lie groupoids up to Morita equivalence. For representable orbifold groupoids, those Morita equivalent to an action groupoid, Pronk and Scull define Bredon cohomology in the case when coefficient systems satisfy a certain condition [22, Proposition 5.13]. Recent progress in the representability of orbifolds [20] suggests that this is a mild condition on orbifolds in general.

In this paper, we consider more general Lie groupoids and show that in many cases, we can restrict to the full sub-2-category of action groupoids and equivariant functors and still capture the desired notion of Morita equivalence. Instead of focusing on orbifolds, we allow our group actions to have any subset of the following properties: free, locally free, transitive, effective, compact, discrete, proper, and being Morita equivalent to a proper étale Lie groupoid. (There is, of course, much redundancy when multiple properties are taken together.) We show that for any selection of these properties, we can create a bicategory of fractions which inverts equivariant weak equivalences, and that this bicategory is equivalent to that in which general weak equivalences are inverted, the full sub-bicategory whose objects are the action groupoids admitting the chosen properties. This yields affirmative answers to the two questions above for many special cases of action groupoid. Moreover, we show that we can create this localisation using the method of Roberts, giving us a smaller and more concrete category to work with. This is a first step towards generalising [22] to more general action groupoids, and defining Bredon cohomology as a Morita invariant in a more general context.

As an additional step towards understanding when equivariant invariants might be Morita equivalent, we also show that the decomposition of equivariant weak equivalences used in [22, Proposition 3.5] also applies in our more general setting of action groupoids satisfying the reader's choice of properties. This allows us to break down equivariant weak equivalences into two specific types: projections and inclusions. This decomposition has already proved useful in other contexts such as topological complexity [2] and should be similarly useful for computing other equivariant invariants.

Throughout, we refer to [4, 13] for categorical definitions such as bicategory, pseudofunctor, 2-category, and 2-commutative, and all of their constituent parts. Also, modern terminology is introduced by the [19], and so we utilise that here as well.

The paper is structured as follows: In Section 2, we introduce our perspective on how to work with smooth maps inspired by diffeology; namely, we use the so-called local curve lifting (LCL) condition to show that various smooth maps are submersive or diffeomorphisms. Section 3 contains background information on Lie groupoids. In Section 4, we review the constructions mentioned above for the two methods of localisation of **LieGpoid** at the class of weak equivalences. Sections 5 and 6 contains our main results about action groupoids, answering the two questions posed above. In particular, in Section 5, we construct the localisation of action groupoids with the reader's choice of properties (see Proposition 44), and in Section 6 we show that this is equivalent to the full sub-bicategory of such action groupoids (see Theorem 55). In Section 7, as an application we show that equivariant weak equivalences can be decomposed into an equivariant projection and an equivariant inclusion (see Theorem 59), and thus by Proposition 44, any Morita equivalence between action groupoids can be replaced with one in which the weak equivalences are equivariant and decomposable.

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# 2 Preliminaries: A Diffeological Perspective on Smooth Maps

We begin by emphasising our perspective on differential geometry for this paper, and illustrate the technique of proof for the differential geometric aspects of the results. First, we clarify our setting: this paper is about Lie groupoids, and involves smooth manifolds. That is:

**Assumptions 1**

- By "smooth" we mean $C^\infty$; that is, infinitely differentiable.
- By "manifold" we mean a smooth manifold without boundary.
- By "curve" we mean a smooth map $p\colon I \to M$ where $M$ is a manifold and $I$ denotes an open interval $I = (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.
- Given the curve above, by "shrinking $I$" we mean taking some sufficiently smaller $\varepsilon' \in (0, \varepsilon]$ and redefining $I$ to be $(-\varepsilon', \varepsilon')$.

Throughout this work, our perspective on diffeomorphisms and smooth maps is inspired by looking at what happens to curves. We start with the following.

**Lemma 2.** *Let $M$ and $N$ be manifolds. A function $f\colon M \to N$ is a smooth map if and only if for any curve $p\colon I \to M$, the composite $f \circ p$ is a curve.*

*Proof* This is an immediate consequence of Boman's Lemma [5], which is the same statement, but where $M$ is a Euclidean space and $N$ is $\mathbb{R}$. □

We use this as inspiration for developing a condition that will allow us to check that maps are submersions or diffeomorphisms by looking only at liftings of curves, without having to explicitly deal with tangent bundles. This will be used repeatedly throughout this paper. This may be considered the "diffeological perspective" of smooth maps. Diffeological spaces generalise manifolds, and their diffeological structure is defined by specifying which maps are considered smooth. Smooth manifolds form a full subcategory of diffeological spaces, and diffeological spaces are closed under subsets, quotients, and fibred products. Although we do not explicitly work with diffeological spaces, we take advantage of the nice properties of this larger category. For instance, a fibred product of manifolds is itself a manifold; this can be established by constructing a (diffeological) diffeomorphism from the fibred product to a manifold without checking that the fibred product is a manifold *a priori*. (In classical differential geometry, one would need to establish that both spaces are manifolds *before* constructing a diffeomorphism between them.) More succinctly, the language of diffeology allows one to separate the property of a space being smooth from the property of a map being smooth. See [12] for more details on diffeological spaces.

**Definition 3** (LCL Condition) A function $f\colon M \to N$ satisfies **the local curve lifting (LCL) condition** if for any curve $p\colon I \to N$ and $x \in M$ satisfying $f(x) = p(0)$, after possibly shrinking $I$, there exists a (smooth) lift $q\colon I \to M$ of $p$ (restricted to the redefined $I$) through $x$ with respect to $f$. Explicitly, $q$ satisfies $f \circ q = p$ and $q(0) = x$.

The LCL condition allows us to identify submersions and diffeomorphisms as follows.

**Lemma 4.** *Let $M$ and $N$ be manifolds.*

1. *A smooth surjection $f\colon M \to N$ is a surjective submersion if and only if it satisfies the LCL condition.*
2. *A smooth map $f\colon M \to N$ is a diffeomorphism if and only if $f$ is bijective and satisfies the LCL condition.*

*Proof* For Claim 1, suppose $f$ is a surjective submersion. Fix a curve $p\colon I \to N$ and let $x \in M$ with $f(x) = p(0)$. Since $f$ is submersive, by the Rank Theorem [14, Theorem 4.12], there exist open coordinate neighbourhoods $U$ of $x$ and $V$ of $p(0)$ such that

1. $U$ is identified with $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$, where $m = \dim M$;

2. $x$ is identified with $0 \in \mathbb{R}^m$;
3. $V$ is identified with $\mathbb{R}^n$, where $n = \dim N$; and
4. $f$ is identified with the projection $\mathbb{R}^m \to \mathbb{R}^n \colon (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_n)$.

Shrink $I$ so that $p(I) \subseteq V$. Define $q\colon I \to \mathbb{R}^m$ by $q(t) = (p(t), 0)$. Then $q$ satisfies $f \circ q = p$ and $q(0) = x$ as required, and the LCL condition is satisfied.

Conversely, suppose that $f$ satisfies the LCL condition and fix $y \in N$ and $v \in T_y N$. Let $p\colon I \to N$ be a curve such that $p(0) = y$ and $\dot{p}(0) = v$. The LCL condition ensures that after shrinking $I$, there is a curve $q\colon I \to M$ such that $f \circ q = p$ and $q(0) = x$. Then $f_*(\dot{q}(0)) = \dot{p}(0) = v$. It follows that $f$ is a surjective submersion.

For Claim 2 suppose $f$ is a diffeomorphism. Then it is injective, surjective, and submersive, and the LCL condition follows from Claim 1.

Conversely, suppose $f$ is bijective and satisfies the LCL condition. Then $f$ is also a surjective submersion by Claim 1, hence a bijective submersion and thus a diffeomorphism. □

Note that in Claim 2 of Lemma 4, the LCL condition can be relaxed to showing that any curve $p\colon I \to N$ lifts to a curve in $M$; showing that it lifts to a curve through a specified point is not necessary since injectivity will uniquely determine which point it goes through.

In what follows, we will repeatedly make use of the LCL condition and Lemma 4 in proving that various constructions are suitably smooth. We will also make use of the following well-known fact about fibred products of manifolds, which follows from the Transversality Theorem [14, Theorem 6.30].

**Lemma 5.** *Let $f\colon M \to P$ and $g\colon N \to P$ be smooth maps between manifolds, in which $f$ is a surjective submersion. Then $M\,{}_f{\times}_g N$ is a manifold and $\mathrm{pr}_2$ is a surjective submersion.*

# 3 The 2-Category of Lie Groupoids

In this section we discuss the 2-category of Lie groupoids and its properties, with special attention to the notion of weak equivalence, which gives rise to the ubiquitous notion of Morita equivalence.

Our basic context is groupoids. To set notation, we will assume that a groupoid $\mathcal{G}$ consists of objects $\mathcal{G}_0$ and arrows $\mathcal{G}_1$, together with structure maps:

- source and target maps $s, t\colon \mathcal{G}_1 \to \mathcal{G}_0$;
- a unit map $u\colon \mathcal{G}_0 \to \mathcal{G}_1$;
- a multiplication map $\cdot\colon \mathcal{G}_1\,{}_s{\times}_t \mathcal{G}_1 \to \mathcal{G}_1$; and
- an inverse map $\mathrm{inv}\colon \mathcal{G}_1 \to \mathcal{G}_1$ where we indicate the inverse $\mathrm{inv}(g)$ by $g^{-1}$.

## 3.1 The 2-Category of Lie Groupoids

**Definition 6** (The 2-Category **LieGpoid**) The 2-category of **Lie groupoids**, denoted **LieGpoid**, is defined by:

0. Objects: A **Lie groupoid** $\mathcal{G}$ is a groupoid in which both objects $\mathcal{G}_0$ and arrows $\mathcal{G}_1$ are manifolds, and all structure maps are smooth, with the source and target maps surjective submersions. We will assume that all Lie groupoids have Hausdorff object and arrow spaces.
1. Arrows: A **functor** $\varphi \colon \mathcal{G} \to \mathcal{H}$ between two Lie groupoids is a functor $(\varphi_0, \varphi_1)$ in which $\varphi_0 \colon \mathcal{G}_0 \to \mathcal{H}_0$ and $\varphi_1 \colon \mathcal{G}_1 \to \mathcal{H}_1$ are smooth maps.
2. 2-cells: A **natural transformation** $\eta \colon \varphi \Rightarrow \psi$ between functors $\varphi, \psi \colon \mathcal{G} \to \mathcal{H}$ is a natural transformation in which the defining map $\eta \colon \mathcal{G}_0 \to \mathcal{H}_1$ is smooth.

For the remainder of this section, we assume that all of our groupoids are Lie groupoids and that all functors and natural transformations are smooth.

We now consider the notion of Morita equivalence on the 2-category **LieGpoid**. The idea of Morita equivalence is ubiquitous throughout the study of groupoids in areas such as orbifolds, actions of groupoids, and stacks. This will be defined using the following.

**Definition 7** (Weak Equivalence) A functor $\varphi \colon \mathcal{G} \to \mathcal{H}$ is a **weak equivalence** (sometimes called an **equivalence** or an **essential equivalence** in the literature) if it satisfies the following two conditions:

1. **Smooth Essential Surjectivity:** The induced map

$$\mathbf{ES}_\varphi \colon \mathcal{G}_{0\,\varphi_0}\!\times_t \mathcal{H}_1 \to \mathcal{H}_0 \colon (x,h) \mapsto s(h)$$

   is a surjective submersion.
2. **Smooth Fully Faithfulness:** The induced map

$$\mathbf{FF}_\varphi \colon \mathcal{G}_1 \to \mathcal{G}_0^2\,{}_{\varphi_0^2}\!\times_{(s,t)} \mathcal{H}_1 \colon g \mapsto (s(g), t(g), \varphi(g))$$

   is a diffeomorphism.

We will denote a weak equivalence with the symbol $\xrightarrow{\simeq}$, and denote the class of all weak equivalences in **LieGpoid** by $W$. We will also consistently use the notation $\mathbf{ES}_\varphi$ and $\mathbf{FF}_\varphi$ for the essential surjectivity and fully faithful maps induced by a functor $\varphi$ as defined above.

Weak equivalences satisfy the so-called 3-for-2 property, as proved in [23].

**Lemma 8.** *Let $\varphi\colon \mathcal{G} \to \mathcal{H}$ and $\psi\colon \mathcal{H} \to \mathcal{K}$ be functors in* **LieGpoid**. *If any two of $\varphi$, $\psi$, or $\psi \circ \varphi$ are weak equivalences, so is the third.*

The following helps us identify weak equivalences in **LieGpoid**.

**Lemma 9.** *Let $\varphi\colon \mathcal{G} \to \mathcal{H}$ be smoothly fully faithful and suppose $\varphi_0$ is a surjective submersion. Then $\varphi$ is a weak equivalence.*

*Proof* To check smooth essential surjectivity, we will check that $\mathbf{ES}_\varphi$ satisfies the LCL condition and apply Lemma 4. Since $\varphi_0$ is surjective, for any $y \in \mathcal{H}_0$, there is some $x \in \varphi_0^{-1}(y)$, for which $\mathbf{ES}_\varphi(x, u_y) = y$; thus $\mathbf{ES}_\varphi$ is surjective. Fix a curve $p\colon I \to \mathcal{H}_0$ defined by $p(t) = y_\tau$. Choose $(x_0, h_0) \in \mathcal{G}_0{}_\varphi\times_t \mathcal{H}_1$ such that $\mathbf{ES}_\varphi(x_0, h_0) = s(h_0) = y_0$. Since the source map of $\mathcal{H}$ is a surjective submersion, it satisfies the LCL condition and so after shrinking $I$, there is a lift $h_\tau$ of $y_\tau$ through $h_0$ with $s(h_\tau) = y_\tau$. Similarly, since $\varphi_0$ is a surjective submersion, after shrinking $I$, there is a lift $x_\tau$ of $t(h_\tau)$ through $x_0$ such that $\varphi(x_\tau) = t(h_\tau)$. Then $(x_\tau, h_\tau)$ is a lift of $y_\tau$ through $(x_0, h_0)$, and it follows that $\mathbf{ES}_\varphi$ is a surjective submersion. □

A fully faithful functor has the following lifting property with respect to natural transformations, the 2-cells of **LieGpoid**.

**Lemma 10.** *A functor $\varphi\colon \mathcal{G} \to \mathcal{H}$ in* **LieGpoid** *is smoothly fully faithful if and only if for any functors $\psi, \psi'\colon \mathcal{K} \to \mathcal{G}$ and natural transformation $\eta\colon \varphi \circ \psi \Rightarrow \varphi \circ \psi'$, there exists a unique natural transformation $\eta'\colon \psi \Rightarrow \psi'$ such that $\eta = \varphi\eta'$.*

*Proof* Suppose $\varphi$ is smoothly fully faithful. Fix functors $\psi, \psi'\colon \mathcal{K} \to \mathcal{G}$ and natural transformation $\eta\colon \varphi \circ \psi \Rightarrow \varphi \circ \psi'$. Define $\eta'\colon \mathcal{K}_0 \to \mathcal{G}_1$ by $\eta'(z) := {\mathbf{FF}_\varphi}^{-1}(\psi(z), \psi'(z), \eta(z))$; this is well-defined and smooth. The fact that $\varphi\eta' = \eta$ follows from the construction, and uniqueness follows from smooth fully faithfulness of $\varphi$.

Conversely, suppose for any functors $\psi, \psi'\colon \mathcal{K} \to \mathcal{G}$ and natural transformation $\eta\colon \varphi \circ \psi \Rightarrow \varphi \circ \psi'$, there exists a unique natural transformation $\eta'\colon \psi \Rightarrow \psi'$ such that $\eta = \varphi\eta'$. Let $g_1, g_2 \in \mathcal{G}_1$ such that $\mathbf{FF}_\varphi(g_1) = \mathbf{FF}_\varphi(g_2)$. Then $s(g_1) = s(g_2)$, $t(g_1) = t(g_2)$, and $\varphi(g_1) = \varphi(g_2)$. Let $\mathcal{K}$ be the trivial Lie groupoid of a point, $\psi\colon \mathcal{K} \to \mathcal{G}$ be the functor sending the point to $s(g_1)$, $\psi'\colon \mathcal{K} \to \mathcal{G}$ the functor sending the point to $t(g_1)$, and $\eta\colon \varphi \circ \psi \Rightarrow \varphi \circ \psi'$ sending the point to the arrow $\varphi(g_1)$. There is a unique $\eta'\colon \psi \Rightarrow \psi'$ such that $\varphi\eta' = \eta$; that is, $\eta' = g_1 = g_2$. Thus $\mathbf{FF}_\varphi$ is injective. Similarly, if $(x_1, x_2, h) \in \mathcal{G}_0^2{}_{\varphi^2}\times_{(s,t)} \mathcal{H}_1$, then setting $\psi$, $\psi'$, and $\eta$ to send the object of $\mathcal{K}$ to $x_1$, $x_2$, and $h$, resp., the unique $\eta'$ such that $\varphi\eta' = \eta$ evaluated at the object of $\mathcal{K}$ is sent by $\mathbf{FF}_\varphi$ to $(x_1, x_2, h)$. Thus $\mathbf{FF}_\varphi$ is bijective.

Now we show that $\mathbf{FF}_\varphi$ satisfies the LCL condition. Fix a curve $p = (x_\tau, x'_\tau, h_\tau)\colon I \to \mathcal{G}^2_{0\ \varphi^2}\times_{(s,t)}\mathcal{H}_1$. Set $\mathcal{K}$ to be the trivial Lie groupoid of $I$, the functors $\mathcal{K} \to \mathcal{G}$ defined by $\psi_0 = x_\tau$, $\psi'_0 = x'_\tau$, and $\eta : \varphi \circ \psi \Rightarrow \varphi \circ \psi'$ defined by the natural transformation $h_\tau$. There is a unique $\eta'\colon \psi \Rightarrow \psi'$ defined by $g_\tau$ such that $\varphi\eta' = \eta$, so $\varphi(g_\tau) = h_\tau$. Thus $\mathbf{FF}_\varphi(g_\tau) = p$, showing that $g_\tau$ gives the required lift. By Item 2 of Lemma 4, it follows that $\mathbf{FF}_\varphi$ is a diffeomorphism, establishing smooth fully faithfulness of $\varphi$. □

## 3.2 Pullbacks of Lie Groupoids

In this section we define the strict and weak pullbacks of Lie groupoids, and consider conditions that ensure that these will again be Lie groupoids.

**Definition 11** (Strict Pullback) Let $\varphi\colon \mathcal{G} \to \mathcal{K}$ and $\psi\colon \mathcal{H} \to \mathcal{K}$ be functors. The **strict pullback** of $\varphi$ and $\psi$ is the groupoid $\mathcal{G}_\varphi\times_\psi\mathcal{H}$, whose object and arrow spaces are the corresponding fibred products of the object and arrow spaces of $\mathcal{G}$ and $\mathcal{H}$, resp. The structure maps are restrictions of those from the product Lie groupoid $\mathcal{G}\times\mathcal{H}$. The strict pullback comes equipped with two projection functors $\mathrm{pr}_1$ and $\mathrm{pr}_2$, the restrictions of those from the product Lie groupoid.

The strict pullback may not be a Lie groupoid in general. The following proposition provides a sufficient condition for when it is a Lie groupoid.

**Proposition 12.** *Let $\varphi\colon \mathcal{G} \to \mathcal{K}$ and $\psi\colon \mathcal{H} \to \mathcal{K}$ be functors in which $\varphi$ is a surjective submersive weak equivalence. Then $\mathcal{G}_\varphi\times_\psi\mathcal{H}$ is a Lie groupoid and* $\mathrm{pr}_2$ *is a surjective submersive weak equivalence.*

*Proof* By Lemma 5, the object and arrow spaces of $\mathcal{G}_\varphi\times_\psi\mathcal{H}$ are manifolds. We verify that the source map of the pullback groupoid is a surjective submersion using the LCL condition. Fix a curve $p = (x_\tau, y_\tau)\colon I \to \mathcal{G}_{0\,\varphi_0}\times_{\psi_0}\mathcal{H}_0$ and let $(g_0, h_0) \in \mathcal{G}_{1\,\varphi_1}\times_{\psi_1}\mathcal{H}_1$ such that $s(g_0, h_0) = (x_0, y_0)$. After shrinking $I$, there is a lift $h_\tau\colon I \to \mathcal{H}_1$ of $y_\tau$ through $h_0$ such that $s(h_\tau) = y_\tau$. Then $t(\psi_1(h_\tau))$ defines a curve $I \to \mathcal{K}_0$, and since $\varphi_0$ is a surjective submersion, after shrinking $I$ again there is a lift $x'_\tau\colon I \to \mathcal{G}_0$ of this curve through $t(g_0)$ with $\varphi(x'_\tau) = t(\psi(h_\tau))$. The curve $g_\tau := {\mathbf{FF}_\varphi}^{-1}(x_\tau, x'_\tau, \psi_1(h_\tau))\colon I \to \mathcal{G}_1$ is a lift of $x_\tau$ through $g_0$ such that $s(g_\tau) = x_\tau$. Thus $(g_\tau, h_\tau)\colon I \to \mathcal{G}_{1\,\varphi_1}\times_{\psi_1}\mathcal{H}_1$ is well-defined, and is the desired lift of $(x_\tau, y_\tau)$ through $(g_0, h_0)$ such that $s(g_\tau, h_\tau) = (x_\tau, y_\tau)$ verifying the LCL condition for the source map of the pullback groupoid. By Item 1 of Lemma 4, the source map is a surjective submersion, from which it follows that the target map is as well. Thus $\mathcal{G}_\varphi\times_\psi\mathcal{H}$ is a Lie groupoid.

Next we show that $\mathrm{pr}_2$ is a surjective submersive weak equivalence. By Lemma 5, the map $(\mathrm{pr}_2)_0\colon (\mathcal{G}_\varphi\times_\psi\mathcal{H})_0 \to \mathcal{H}_0$ is a surjective submersion. Since $\mathbf{FF}_\varphi$ is a diffeomorphism, it follows that $\mathbf{FF}_{\mathrm{pr}_2}$ is bijective. Let

$$p = ((x_\tau, y_\tau), (x'_\tau, y'_\tau), h_\tau)\colon I \to (\mathcal{G}_\varphi\times_\psi\mathcal{H})^2_{0\ \varphi^2_0}\times_{(s,t)}\mathcal{H}_1$$

be a curve. Define the curve $g_\tau := \mathbf{FF}_\varphi{}^{-1}((x_\tau, x'_\tau), \psi_1(h_\tau))$. Then $(g_\tau, h_\tau)$ defines the desired lift of the curve $p$ and $\mathbf{FF}_{\mathrm{pr}_2}$ satisfies the LCL condition. Hence $\mathbf{FF}_{\mathrm{pr}_2}$ is a surjective submersion. By Item 2 of Lemma 4, $\mathbf{FF}_{\mathrm{pr}_2}$ is a diffeomorphism. By Lemma 9, $\mathrm{pr}_2$ is in $\mathsf{sW}$. This proves the second statement. □

We also consider the weak pullback of groupoids.

**Definition 13** (Weak Pullback) Let $\varphi\colon \mathcal{G} \to \mathcal{K}$ and $\psi\colon \mathcal{H} \to \mathcal{K}$ be functors in **LieGpoid**. The **weak pullback** of $\varphi$ and $\psi$ is the groupoid $\mathcal{G}{}_\varphi\overset{\mathrm{w}}{\times}{}_\psi\mathcal{H}$, whose object space is

$$(\mathcal{G}{}_\varphi\overset{\mathrm{w}}{\times}{}_\psi\mathcal{H})_0 := \left\{(x,k,y) \in \mathcal{G}_0 \times \mathcal{K}_1 \times \mathcal{H}_0 \mid \varphi(x) \overset{k}{\curvearrowright} \psi(y)\right\},$$

and arrow space is

$$(\mathcal{G}{}_\varphi\overset{\mathrm{w}}{\times}{}_\psi\mathcal{H})_1 := \left\{(g,k,h) \in \mathcal{G}_1 \times \mathcal{K}_1 \times \mathcal{H}_1 \mid \varphi(s(g)) \overset{k}{\curvearrowright} \psi(s(h))\right\}.$$

The structure maps are as follows:

- the source map is given by $(g,k,h) \mapsto (s(g), k, s(h))$,
- the target map is given by $(g,k,h) \mapsto (t(g), \psi(h)k\varphi(g)^{-1}, t(h))$,
- the unit map is given by $(x,k,y) \mapsto (u_x, k, u_y)$,
- the multiplication is given by $(g_1, h_2k_2g_2^{-1}, h_1)\cdot(g_2,k_2,h_2) = (g_1g_2, k_2, h_1h_2)$,
- and the inverse of $(g,k,h)$ is given by $(g^{-1}, hkg^{-1}, h^{-1})$.

The weak pullback comes equipped with two projection functors $\mathrm{pr}_1$ and $\mathrm{pr}_3$ to $\mathcal{G}$ and $\mathcal{H}$, respectively, and the natural transformation $\mathrm{PR}_2\colon \varphi\circ\mathrm{pr}_1 \Rightarrow \psi\circ\mathrm{pr}_3$.

In general, the weak pullback may not be a Lie groupoid. The following proposition, which is [17, Proposition 5.12(iv)], gives a sufficient condition for when it is a Lie groupoid.

**Proposition 14.** *Let $\varphi\colon \mathcal{G} \to \mathcal{K}$ and $\psi\colon \mathcal{H} \to \mathcal{K}$ be functors in which $\varphi$ is a weak equivalence. Then $\mathcal{G}{}_\varphi\overset{\mathrm{w}}{\times}{}_\psi\mathcal{H}$ is a Lie groupoid and $\mathrm{pr}_3$ is a surjective submersive weak equivalence.*

## 3.3 Surjective Submersive Weak Equivalences

We will make use of the special properties of weak equivalences which are also surjective submersions on objects. We start with the following standard result on surjective submersions.

**Lemma 15.** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be smooth maps between manifolds such that $f$ and $g\circ f$ are surjective submersions. Then $g$ is a surjective submersion.*

We now define an important type of weak equivalence.

**Definition 16** (Surjective Submersive Weak Equivalence) We call a weak equivalence between Lie groupoids that is a surjective submersion on objects **surjective submersive**, and will often denote a surjective submersive weak equivalence using an arrow $\xrightarrow{\simeq}\!\!\!\!\rightarrow$. We refer to the class of surjective submersive weak equivalences as $\mathsf{sW}$.

The following lemma shows how surjective submersive weak equivalences interact with natural transformations. This first property is called "co-fully faithfulness" by Roberts [27, Definition 2.12] and Pronk-Scull [24, Definition 5.1].

**Lemma 17.** *Let $\varphi\colon \mathcal{G} \to \mathcal{H}$ be in $\mathsf{sW}$. Then for any functors $\psi, \psi'\colon \mathcal{H} \to \mathcal{K}$ and natural transformation $\eta\colon \psi \circ \varphi \Rightarrow \psi' \circ \varphi$, there exists a unique natural transformation $\eta'\colon \psi \Rightarrow \psi'$ such that $\eta = \eta'\varphi$.*

*Proof* Fix functors $\psi, \psi'\colon \mathcal{H} \to \mathcal{K}$ and natural transformation $\eta\colon \psi \circ \varphi \Rightarrow \psi' \circ \varphi$. Define $\eta'\colon \mathcal{H}_0 \to \mathcal{K}_1$ by $\eta'(y) := \eta(x)$, where $x \in \varphi_0^{-1}(y)$. Since $\varphi_0$ is surjective, $\varphi_0^{-1}(y)$ is non-empty. Suppose $\varphi_0(x_1) = \varphi_0(x_2)$. Since $\varphi$ is a weak equivalence, there exists an arrow $g = \mathbf{FF}_\varphi{}^{-1}(x_1, x_2, u_{\varphi(x_1)})$ from $x_1$ to $x_2$. Naturality gives the following commutative diagram

$$\begin{tikzcd}
\psi \circ \varphi(x_1) \arrow[r, "\eta_{x_1}"] \arrow[d, "\psi\circ\varphi(g)"'] & \psi' \circ \varphi(x_1) \arrow[d, "\psi'\circ\varphi(g)"] \\
\psi \circ \varphi(x_2) \arrow[r, "\eta_{x_2}"'] & \psi' \circ \varphi(x_2).
\end{tikzcd}$$

Since $\varphi(g) = u_{\varphi(x_1)}$, we have $\eta_{x_1} = \eta_{x_2}$, and so $\eta'$ is well-defined. By construction $\eta = \eta'\varphi$.

To show that $\eta'$ is smooth, fix a curve $p = y_\tau\colon I \to \mathcal{H}_0$. After shrinking $I$, there exists a curve $x_\tau\colon I \to \mathcal{G}_0$ such that $y_\tau = \varphi(x_\tau)$, since $\varphi$ is a surjective submersion. Since $\eta'(y_\tau) = \eta(x_\tau)$, we conclude that $\eta'(y_\tau)$ is a curve in $\mathcal{K}_1$. By Lemma 2, $\eta'$ is smooth. The naturality of $\eta'$ follows from that of $\eta$. Finally, uniqueness follows from the construction. $\square$

The following identifies weak equivalences using a property called "locally split" in [27, Definition 3.22].

**Lemma 18.** *A functor $\varphi\colon \mathcal{G} \to \mathcal{H}$ is a weak equivalence if and only if it is smoothly fully faithful and there exist a $\psi\colon \mathcal{K} \to \mathcal{H}$ in $\mathsf{sW}$, a functor $\sigma\colon \mathcal{K} \to \mathcal{G}$, and a natural transformation $\eta\colon \varphi \circ \sigma \Rightarrow \psi$.*

*Proof* Suppose $\varphi$ is a weak equivalence. Choose $\mathcal{K} := \mathcal{G}_\varphi \overset{\mathrm{w}}{\times}_{\mathrm{id}_\mathcal{H}} \mathcal{H}$ and $\psi = \mathrm{pr}_3$, $\sigma = \mathrm{pr}_1$ and $\eta = \mathrm{PR}_2$. Then $\mathcal{K}$ is a Lie groupoid and $\psi \in \mathsf{sW}$ by Proposition 14.

Conversely, suppose $\varphi$ is smoothly fully faithful, and there exist $\psi\colon \mathcal{K} \to \mathcal{H}$ in $\mathsf{sW}$, a functor $\sigma\colon \mathcal{K} \to \mathcal{G}$, and a natural transformation $\eta\colon \varphi \circ \sigma \Rightarrow \psi$. We will verify that $\varphi$ is a weak equivalence by checking the necessary conditions on the induced maps $\mathbf{ES}_\varphi$.

Let $y \in \mathcal{H}_0$. Since $\psi_0$ is surjective, there exists $z \in \mathcal{K}_0$ such that $\psi(z) = y$. Then $(\sigma(z), \eta(z)^{-1}) \in \mathcal{G}_{0\,\varphi}\times_t \mathcal{H}_1$ and $\mathbf{ES}_\varphi(\sigma(z), \eta(z)^{-1}) = y$. Thus $\mathbf{ES}_\varphi$ is surjective.

To show $\mathbf{ES}_\varphi$ is a surjective submersion, fix a curve $p = y_\tau\colon I \to \mathcal{H}_0$. Let $(x_0, h_0) \in \mathcal{G}_{0\,\varphi}\times_t \mathcal{H}_1$ such that $\mathbf{ES}_\varphi(x_0, h_0) = y_0$, so $s(h_0) = y_0$ and $t(h) = x_0$. Since $\psi_0$ is a surjective submersion, after shrinking $I$, there is a curve $z_\tau\colon I \to \mathcal{K}_0$ such that $\psi(z_\tau) = y_\tau$. Since $\varphi$ is smoothly fully faithful, $\mathbf{FF}_\varphi$ is a diffeomorphism and so we define $g_0 \in \mathcal{G}_1$ by ${\mathbf{FF}_\varphi}^{-1}(\sigma(z_0), x_0, h_0 \cdot \eta(z_0))$. Since source maps of Lie groupoids are surjective submersions, we can lift the curve $\sigma(z_\tau)$ after shrinking $I$ to get a curve $g_\tau\colon I \to \mathcal{G}_1$ through $g_0$ such that $s(g_\tau) = \sigma(z_\tau)$. Then the desired lift of the curve $p$ through $(x_0, h_0)$ is defined by $(t(g_\tau), \varphi(g_\tau) \cdot \eta(z_\tau)^{-1})$. Thus $\mathbf{ES}_\varphi$ satisfies the LCL condition and is a surjective submersion, completing the verification that $\varphi$ is a weak equivalence. □

# 4 Localising Lie Groupoids at Weak Equivalences

In this section we discuss how to construct from the 2-category **LieGpoid** a localised bicategory which inverts weak equivalences. Localising at the class $W$ of weak equivalences gives us a formal mechanism for working with Morita equivalence classes of groupoids. We will use a bicategory of fractions construction in which the objects are still the Lie groupoids of **LieGpoid**, but the 1- and 2-cells are adjusted. In particular, the arrows of the bicategory of fractions will be given by so-called "spans" of arrows of **LieGpoid**, with the result that we will add inverse arrows for any weak equivalence, making all weak equivalences into invertible 1-cells and hence making all Morita equivalences into isomorphisms of the localised category.

In this section, we outline two constructions of this localisation: the first is the bicategory of fractions defined by [21], and a second related but smaller construction based on so-called anafunctors by [27]. Throughout this section, we will assume that we work in **LieGpoid**, so all groupoids are Lie groupoids and all functors and natural transformations are smooth.

## 4.1 The Localised Bicategory LieGpoid$[W^{-1}]$

In this section we discuss the following result.

**Proposition 19.** *[21] There is a bicategory* **LieGpoid**$[W^{-1}]$*, the* ***bicategory of fractions*** *of the* 2*-category* **LieGpoid** *localised at the class* $W$*, which satisfies the universal property of a localisation: any functor from* **LieGpoid** *to another bicategory which takes* 1*-cells in* $W$ *to invertible* 1*-cells will factor through this localised bicategory* **LieGpoid**$[W^{-1}]$*.*

We begin by explaining the construction and properties of this bicategory, and then give an outline of the proof of Proposition 19.

The objects of **LieGpoid**$[W^{-1}]$ are the same as the objects of **LieGpoid**, and the arrows are defined by "spans" of arrows:

**Definition 20** (Generalised Morphism of **LieGpoid**$[W^{-1}]$) A **generalised morphism** between Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ is a Lie groupoid $\mathcal{K}$ and two functors $\varphi\colon \mathcal{K} \to \mathcal{G}$ and $\psi\colon \mathcal{K} \to \mathcal{H}$ in which $\varphi$ is a weak equivalence. Denote the generalised morphism by $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$.

The **identity generalised morphism** of $\mathcal{G}$ in **LieGpoid**$[W^{-1}]$ is given by $\mathcal{G} \xleftarrow{=} \mathcal{G} \xrightarrow{=} \mathcal{G}$.

A generalised morphism $\mathcal{G} \xleftarrow{\simeq} \mathcal{K} \to \mathcal{H}$ can be thought of as replacing $\mathcal{G}$ with a Lie groupoid $\mathcal{K}$ which is weakly equivalent to it, which admits the left functor $\mathcal{K} \to \mathcal{H}$. In this way, we consider weakly equivalent groupoids to be interchangeable. We can make this more precise with the following.

**Definition 21** (Morita Equivalence) We say that $\mathcal{G}$ and $\mathcal{H}$ are **Morita equivalent** if there is a generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{\simeq} \mathcal{H}$ where *both* $\varphi$ and $\psi$ are weak equivalences.

Weakly equivalent groupoids are always Morita equivalent, as we can create a generalised morphism between them using an identity morphism as one leg. It turns out that Morita equivalence is the equivalence relation generated by the weak equivalences. A Morita equivalence $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{\simeq} \mathcal{H}$ will be invertible in **LieGpoid**$[W^{-1}]$, with inverse defined by the generalised morphism $\mathcal{H} \xleftarrow[\psi]{\simeq} \mathcal{K} \xrightarrow[\varphi]{\simeq} \mathcal{G}$.

The composition of two generalised morphisms makes use of the weak pullback of Definition 13.

**Definition 22** (Composition of Generalised Morphisms in **LieGpoid**$[W^{-1}]$) Let $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ and $\mathcal{H} \xleftarrow[\chi]{\simeq} \mathcal{L} \xrightarrow[\omega]{} \mathcal{I}$ be generalised morphisms. Define their

**composition** to be the generalised morphism

$$\mathcal{G} \underset{\varphi \circ \mathrm{pr}_1}{\overset{\simeq}{\longleftarrow}} \mathcal{K}_{\psi} \overset{\mathrm{w}}{\times}_{\chi} \mathcal{L} \underset{\omega \circ \mathrm{pr}_3}{\longrightarrow} \mathcal{I}.$$

It follows from Proposition 14 and Lemma 8 that the composition of two generalised morphisms is a generalised morphism.

*Remark 23* The composition of generalised morphisms is an associative operation; that is, the associator is trivial.

Our generalised morphisms allow us to treat Morita equivalent groupoids as equivalent. Of course, there may be many different choices of groupoids weakly equivalent to $\mathcal{G}$, and we want to recognise when two choices of generalised morphism carry the same geometric information. Thus we define the following 2-cells.

**Definition 24** (2-Cell Between Generalised Morphisms in **LieGpoid**$[W^{-1}]$) Given two generalised morphisms $\mathcal{G} \underset{\varphi}{\overset{\simeq}{\longleftarrow}} \mathcal{K} \underset{\psi}{\longrightarrow} \mathcal{H}$ and $\mathcal{G} \underset{\varphi'}{\overset{\simeq}{\longleftarrow}} \mathcal{K}' \underset{\psi'}{\longrightarrow} \mathcal{H}$ we define a 2-cell from the first generalised morphism to the second as follows. Consider a generalised morphism of the form $\mathcal{K} \underset{\alpha}{\overset{\simeq}{\longleftarrow}} \mathcal{L} \underset{\alpha'}{\overset{\simeq}{\longrightarrow}} \mathcal{K}'$ in which both functors are weak equivalences, along with two natural transformations $\eta_1 \colon \varphi \circ \alpha \Rightarrow \varphi' \circ \alpha'$ and $\eta_2 \colon \psi \circ \alpha \Rightarrow \psi' \circ \alpha'$ making the following diagram 2-commute:

$$\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] & \\
\mathcal{G} & \mathcal{L} \arrow[u, "\alpha", "\simeq"'] \arrow[d, "\alpha'"', "\simeq"] \arrow[l, phantom, "\Downarrow \eta_1"] \arrow[r, phantom, "\Downarrow \eta_2"] & \mathcal{H} \\
 & \mathcal{K}'. \arrow[ul, "\varphi'", "\simeq"'] \arrow[ur, "\psi'"'] &
\end{tikzcd} \tag{1}$$

We will often denote such a diagram (1) by the quadruple $(\alpha, \alpha', \eta_1, \eta_2)$. We define an equivalence relation on these diagrams as follows: Suppose $\mathcal{K} \underset{\beta}{\overset{\simeq}{\longleftarrow}} \mathcal{M} \underset{\beta'}{\overset{\simeq}{\longrightarrow}} \mathcal{K}'$ with $\mu_1 \colon \varphi \circ \beta \Rightarrow \varphi' \circ \beta'$ and $\mu_2 \colon \psi \circ \beta \Rightarrow \psi' \circ \beta'$ make up another diagram $(\beta, \beta', \mu_1, \mu_2)$ of the form (1). We will say that $(\alpha, \alpha', \eta_1, \eta_2)$ is equivalent to $(\beta, \beta', \mu_1, \mu_2)$ if there exists yet another generalised morphism $\mathcal{L} \underset{\gamma}{\overset{\simeq}{\longleftarrow}} \mathcal{N} \underset{\gamma'}{\overset{\simeq}{\longrightarrow}} \mathcal{M}$ and natural transformations $\nu_1 \colon \alpha \circ \gamma \Rightarrow \beta \circ \gamma'$ and $\nu_2 \colon \alpha' \circ \gamma \Rightarrow \beta' \circ \gamma'$ such that

$$(\mu_1 \gamma') \circ (\varphi \nu_1) = (\varphi' \nu_2) \circ (\eta_1 \gamma) \quad \text{and} \quad (\mu_2 \gamma') \circ (\psi \nu_1) = (\psi' \nu_2) \circ (\eta_2 \gamma). \tag{2}$$

A **2-cell** from $\mathcal{G} \underset{\varphi}{\overset{\simeq}{\longleftarrow}} \mathcal{K} \underset{\psi}{\longrightarrow} \mathcal{H}$ to $\mathcal{G} \underset{\varphi'}{\overset{\simeq}{\longleftarrow}} \mathcal{K}' \underset{\psi'}{\longrightarrow} \mathcal{H}$ is an equivalence class of these diagrams as in (1); for the above diagram we denote this $[\alpha, \alpha', \eta_1, \eta_2]$. The **identity 2-cell** of a generalised morphism $\mathcal{G} \overset{\simeq}{\longleftarrow} \mathcal{K} \rightarrow \mathcal{H}$ is given by $[\mathrm{id}_{\mathcal{K}}, \mathrm{id}_{\mathcal{K}}, \mathrm{ID}_{\varphi}, \mathrm{ID}_{\psi}]$

(where ID represents the identity natural transformation) and represented by the diagram

$$\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] \arrow[d, equal] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowright"] & \mathcal{K} \arrow[r, phantom, "\circlearrowright"] \arrow[d, equal] & \mathcal{H} \\
 & \mathcal{K}. \arrow[ul, "\varphi", "\simeq"'] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Further details on the unitor and composition of 2-cells in this category are given in Appendix A. We will not need the details of these constructions for most of our results, but they will be needed for a few results in Section 6.

*Outline of Proof of Proposition 19* In order to form the bicategory of fractions of **LieGpoid** with respect to the class of weak equivalences $W$, the class of weak equivalences must satisfy conditions (BF1)-(BF5) of [21, Section 2.1]. The first is that $W$ includes all identity generalised morphisms, which is immediate. The second is that $W$ is closed under composition, which is implied by Lemma 8. The third is the statement of Lemma 14. The fourth is that weak equivalences must satisfy a property called “representably fully faithfulness” by Roberts [27, Definition 2.2], and this is the content of Lemma 10. Finally, the fifth property is that if there is a natural transformation between two functors in which if one is a weak equivalence, then they both are; this is [17, Proposition 5.12(i)]. □

The bicategory $\mathbf{LieGpoid}[W^{-1}]$ inverts all weak equivalences and satisfies the universal property of a localization. On a practical level, this bicategory can be hard to work with on the 2-cell level, since the 2-cells are defined as equivalence classes of diagrams.

## 4.2 The Localised Bicategory AnaLieGpoid

We now describe an alternate localised bicategory **AnaLieGpoid** developed in [25–27] which is “smaller” than $\mathbf{LieGpoid}[W^{-1}]$ described in the previous section. This has the following properties:

- the 1-cells are generalised morphisms whose left side *is a surjective submersive* weak equivalence,
- the composition is created using a *strict* (not weak) pullback,
- the 2-cells are defined by natural transformations (not equivalence classes of diagrams).

The fact that this smaller construction applies in the category of Lie groupoids is well-known to experts but tracking down exact references has proved difficult. Therefore in this section, we will describe the construction and give an outline of the proof of the following:

**Proposition 25.** *There is a bicategory* **AnaLieGpoid**, *the* ***anafunctor bicategory*** *of* **LieGpoid** *localised at* sW.

The starting point to understanding the difference between **LieGpoid**$[W^{-1}]$ and **AnaLieGpoid** is in looking at a subset of the weak equivalences, the surjective submersive weak equivalences of Definition 16. This is a sub-class sW of weak equivalences. *A priori* we will create a localisation which inverts only this subclass sW. It will turn out that the resulting localised bicategories are equivalent.

Thus we start creating **AnaLieGpoid** using objects of **LieGpoid** as before, but look only at generalised morphisms which use a surjective submersive weak equivalence as their left leg. These are called anafunctors in [27].

**Definition 26** (Anafunctor) An **anafunctor** is a generalised morphism $\mathcal{G} \underset{\varphi}{\overset{\simeq}{\longleftarrow}} \mathcal{K} \underset{\psi}{\longrightarrow} \mathcal{H}$ such that $\varphi \in \mathsf{sW}$, so the map $\mathcal{K} \overset{\simeq}{\twoheadrightarrow} \mathcal{G}$ is a surjective submersive weak equivalence. The identity generalised morphism of Definition 20 is an anafunctor, and so it defines the **identity anafunctor** of $\mathcal{G}$.

The composition of anafunctors is defined using the **strict** pullback of Definition 11:

**Definition 27** (Composition of Anafunctors) Let $\mathcal{G} \underset{\varphi}{\overset{\simeq}{\longleftarrow}} \mathcal{K} \underset{\psi}{\longrightarrow} \mathcal{H}$ and $\mathcal{H} \underset{\chi}{\overset{\simeq}{\longleftarrow}} \mathcal{L} \underset{\omega}{\longrightarrow} \mathcal{I}$ be anafunctors. Define their **composition** to be the anafunctor

$$\mathcal{G} \underset{\varphi\circ\mathrm{pr}_1}{\overset{\simeq}{\longleftarrow}} \mathcal{K}_\psi\times_\chi\mathcal{L} \underset{\omega\circ\mathrm{pr}_3}{\longrightarrow} \mathcal{I}.$$

It follows from Proposition 12, Lemma 8, and the fact that the composition of surjective submersions is again a surjective submersion that the composition of two anafunctors is an anafunctor.

*Remark 28* Similar to the case of generalised morphisms, composition of anafunctors is an associative operation.

As in the case for generalised morphisms, we have a choice of replacement for the left side $\mathcal{G}$ in an anafunctor $\mathcal{G} \underset{\varphi}{\overset{\simeq}{\longleftarrow}} \mathcal{K} \underset{\psi}{\longrightarrow} \mathcal{H}$ and must identify when two anafunctors are encoding the same geometry. Viewed as generalised morphisms, there are multiple representatives of 2-cells between anafunctors. Here, we choose a canonical representative of a 2-cell which will be defined by a particular natural transformation. This representative natural transformation will *be* the 2-cell in our new bicategory **AnaLieGpoid**. So when we define our bicategory with anafunctors a 1-cells, our 2-cells will be actual natural transformations rather than equivalence classes as in **LieGpoid**$[W^{-1}]$.

**Definition 29** (2-cells in **AnaLieGpoid**) Given two anafunctors $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ and $\mathcal{G} \xleftarrow[\varphi']{\simeq} \mathcal{K}' \xrightarrow[\psi']{} \mathcal{H}$ a **2-cell** between them is a natural transformation $\eta$ making the following diagram 2-commute.

$$\begin{tikzcd} & \mathcal{K} \arrow[dr, "\psi"] & \\ \mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}' \arrow[ur, "\mathrm{pr}_1", "\simeq"'] \arrow[dr, "\mathrm{pr}_2"', "\simeq"] \arrow[rr, phantom, "\overset{\eta}{\Rightarrow}"] & & \mathcal{H} \\ & \mathcal{K}' \arrow[ur, "\psi'"'] & \end{tikzcd}$$

The **identity 2-cell** of an anafunctor $\mathcal{G} \xleftarrow{\simeq} \mathcal{K} \to \mathcal{H}$ is given by the natural transformation

$$\iota_{\mathcal{G}\leftarrow\mathcal{K}\to\mathcal{H}} \colon (\mathcal{K}_\varphi\times_\varphi\mathcal{K})_0 \to \mathcal{H}_1 \colon (y_1, y_2) \mapsto \psi(\mathbf{FF}_\varphi{}^{-1}((y_1, y_2), u_{\varphi(y_1)})).$$

Observe that these anafunctor 2-cells can be drawn in the diagram form (1) of the 2-cells of **LieGpoid**$[W^{-1}]$ of Definition 24, by adding the left side of the diagram with the trivial natural transformation. Thus we can think of an anafunctor 2-cell as a canonical representative of our earlier 2-cells, in which the left natural transformation is just the trivial one. In **AnaLieGpoid**, we work with actual natural transformations as 2-cells, although the vertical and horizontal compositions are *not* the usual composition of natural transformations. We again defer the precise definitions of vertical composition, horizontal composition, and unitors to Appendix A, as the details are only used for the results of Section 6.

*Outline of Proof of Proposition 25* Roberts in [27] works in a general context to construct a bicategory whose 1-cells are given by anafunctors, with their compositions given by strict pullbacks. Applying this construction to our setting of Lie groupoids requires $\mathsf{sW}$ to be a “bi-fully faithful singleton strict pretopology” on **LieGpoid**. This translates into the following requirements for $\mathsf{sW}$: All identity arrows must be contained in $\mathsf{sW}$; this is immediate. $\mathsf{sW}$ must be closed under strict pullback, proved in Proposition 12. $\mathsf{sW}$ must be closed under composition, which follows from Lemma 8 and the fact that surjective submersions are closed under composition. Finally, elements of $\mathsf{sW}$ are required to be representably fully faithful and co-fully faithful, the

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content of Lemmas 10 and 17. See [27, Definitions 2.9, 2.12] for more details. Thus by [27, Proposition 3.20] we have that **AnaLieGpoid** is a bicategory. □

## 4.3 Comparison of AnaLieGpoid and LieGpoid$[W^{-1}]$

Both the constructions we have just outlined create localisations of **LieGpoid** at $W$. Here we present the result that the two bicategories are equivalent, and detail the key step in its proof.

To compare **AnaLieGpoid** to **LieGpoid**$[W^{-1}]$, we again apply a result of Roberts; be warned that what is called a "weak equivalence" in his paper [27] is defined there to be a functor that is $\mathsf{sW}$-locally split and representably fully faithful. However, by Lemmas 10 and 18, this is equivalent to smooth essential surjectivity and smooth fully faithfulness, and so our notion of weak equivalence coincides with his. Thus we have [27, Theorem 3.24]:

**Proposition 30.** *The inclusion* **AnaLieGpoid** $\to$ **LieGpoid**$[W^{-1}]$ *is an equivalence of bicategories, where this inclusion takes a* 2*-cell to its equivalence class.*

Proposition 30 implies that any generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ admits a 2-cell from itself to an anafunctor, such as $\mathcal{G} \xleftarrow[\mathrm{pr}_1]{\simeq} \mathcal{G}_{\mathrm{id}_\mathcal{G}} \overset{\mathrm{w}}{\times}_\varphi \mathcal{K} \xrightarrow[\psi \circ \mathrm{pr}_3]{} \mathcal{H}$, where the weak equivalence is surjective submersive. It also implies that we can replace the composition in **LieGpoid**$[W^{-1}]$, defined by the weak pullback, by the composition of **AnaLieGpoid** using the strict pullback, and obtain a 2-cell between the two compositions. Below we construct the required 2-cell between the two compositions explicitly, which we will need in Section 6.

**Proposition 31.** *Let* $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ *and* $\mathcal{H} \xleftarrow[\chi]{\simeq} \mathcal{L} \xrightarrow[\omega]{} \mathcal{I}$ *be generalised morphisms where* $\chi \in \mathsf{sW}$*, so that the second generalised morphism is also an anafunctor. Then the composition defined using the weak pullback in* **LieGpoid**$[W^{-1}]$ *of Definition 22 is equivalent to the composition defined using the strict pullback in* **AnaLieGpoid** *defined in Definition 27.*

*Proof* We need create a 2-cell between the two compositions. We define the 2-cell using the following strictly commutative diagram:

$$\begin{tikzcd}
& & \mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{L} \arrow[dll, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[drr, "\omega\circ\mathrm{pr}_3"] & & \\
\mathcal{G} & \circlearrowleft & \mathcal{K}_{\psi}\times_{\chi}\mathcal{L} \arrow[u, "\mathrm{inc}", "\simeq"'] \arrow[d, equal] & \circlearrowleft & \mathcal{I} \\
& & \mathcal{K}_{\psi}\times_{\chi}\mathcal{L}. \arrow[ull, "\varphi\circ\mathrm{pr}_1", "\simeq"'] \arrow[urr, "\omega\circ\mathrm{pr}_2"'] & &
\end{tikzcd}$$

Here, inc is the inclusion functor defined on objects by $\mathrm{inc}_0\colon (x,y)\mapsto (x,u_{\psi(x)},y)$ and on arrows by $\mathrm{inc}_1\colon (k,\ell)\mapsto (k,u_{s\circ\psi(k)},\ell)$. We check that inc is indeed a weak equivalence. It is straightforward to check that $\mathbf{FF}_{\mathrm{inc}}$ is bijective. To show it is a diffeomorphism, we will check the LCL condition: a curve

$$p\colon I\to (\mathcal{K}_{\psi}\times_{\chi}\mathcal{L})^2_0{}_{\mathrm{inc}^2}\times_{(s,t)}(\mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{L})_1$$

must have the form

$$p(t)=((x_\tau,y_\tau),(x'_\tau,y'_\tau),(k_\tau,u_{x_\tau},\ell_\tau)$$

and so $(k_\tau,\ell_\tau)$ gives the desired lift, since $p=\mathbf{FF}_{\mathrm{inc}}(k_\tau,\ell_\tau)$. By Item 2 of Lemma 4, $\mathbf{FF}_{\mathrm{inc}}$ is a diffeomorphism.

Let $(x,h,y)\in(\mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{L})_0$. Since $\chi_0$ is surjective, there exists $y'\in\mathcal{L}_0$ such that $\chi(y')=\psi(x)$. Define $\ell:=\mathbf{FF}_{\chi}{}^{-1}(y,y',h^{-1})$. Then $\mathbf{ES}_{\mathrm{inc}}((x,y'),(u_x,h,\ell))=(x,h,y)$, and so $\mathbf{ES}_{\mathrm{inc}}$ is surjective.

To show that $\mathbf{ES}_{\mathrm{inc}}$ is a surjective submersion, we again use the LCL condition: let $p=(x_\tau,h_\tau,y_\tau)\colon I\to(\mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{L})_0$ be a curve and suppose we have a lift $((x'_0,y'_0),(k_0,h_0,\ell_0))\in((\mathcal{K}_{\psi}\times_{\chi}\mathcal{L})_0)_{\mathrm{inc}}\times_t(\mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{L})_1$. By definition, we must have that $t(k_0)=x'_0$, $t(\ell_0)=y'_0$, and $\chi(\ell_0)h_0\psi(k_0)^{-1}=u_{\chi(y'_0)}$. Since $\mathcal{K}$ is a Lie groupoid, its source map is a surjective submersion and so after shrinking $I$, there is a curve $k_\tau\colon I\to\mathcal{K}_1$ through $k_0$ with $s(k_\tau)=x_\tau$; denote the target of this $t(x_\tau)=x'_\tau$, and note that $\psi(x'_0)=t(\psi(k_0))=t(\chi(\ell_0)h_0)=t\chi(\ell_0)$. Next, since $\chi_0$ is a surjective submersion we can lift $t(\psi(k_\tau))=\psi(x'_\tau)$ through $t(\ell_0)$ to get $y'_\tau$. Thus we have $(y_\tau,y'_\tau,\psi(k_\tau)h_\tau^{-1})\in\mathcal{L}^2_0{}_{\chi^2}\times_{(s,t)}\mathcal{H}_1$ and so we can define $\ell_\tau=\mathbf{FF}_{\chi}{}^{-1}(y_\tau,y'_\tau,\psi(k_\tau)h_\tau^{-1})$. Then $q=(x'_\tau,y'_\tau,(k_\tau,h_\tau,\ell_\tau))$ is the desired lift of $p$ with $\mathbf{ES}_{\mathrm{inc}}(q)=p$.

By Lemma 8, since $\varphi$ is a weak equivalence, $\varphi\circ\mathrm{pr}_1$ is a weak equivalence (in both instances it appears in the diagram above) provided $\mathrm{pr}_1$ is. But by Proposition 14, $\mathrm{pr}_1\colon\mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{L}\to\mathcal{K}$ is a weak equivalence since $\chi$ is, and by Proposition 12 $\mathrm{pr}_1\colon\mathcal{K}_{\psi}\times_{\chi}\mathcal{L}$ is a weak equivalence (in fact, it is in $\mathsf{sW}$) since $\chi$ is in $\mathsf{sW}$. Thus the diagram above represents an equivalence between the two generalised morphisms. □

# 5 Action Groupoids

Our main interest in this paper is in Lie groupoids which come from the action of a Lie group on a manifold. For the rest of the paper, we will focus on these. Consequently,

**Assumptions 32**

- All group actions will be assumed to be (smooth) Lie group actions of a Lie group on a manifold.

We define action groupoids as follows.

**Definition 33** (Action Groupoid) The **action groupoid** of a group action of a Lie group $G$ on a manifold $X$, denoted by $G \ltimes X$, is defined by the following data:

- the object space is $X$,
- the arrow space is $G \times X$, where the pair $(g, x)$ is the arrow $x \to gx$ (for clarity, we sometimes denote this $g \cdot x$),
- multiplication is given by $(g_1, g_2 x)(g_2, x) = (g_1 g_2, x)$,
- the unit at $x$ is given by $(e, x)$, where $e$ is the identity of $G$, and
- the inverse of $(g, x)$ is $(g^{-1}, gx)$.

We are interested in looking at action groupoids with various special properties which commonly come up in contexts such as the study of orbifolds, symplectic geometry, and bundle theory. These are all standard conditions, but we include definitions here for convenience. The first type of these gives conditions on actions.

**Definition 34** (Action Types) A group action of $G$ on $X$ is

1. **free** if all stabilisers are trivial,
2. **locally free** if there is a neighbourhood $U$ of $e$ in $G$ such that the restriction of the action to $U$ is free,
3. **transitive** if for each pair $x, y \in X$ there exists a $g \in G$ such that $gx = y$, and
4. **effective** if for each $g \neq e \in G$ there exists $x \in X$ such that $gx \neq x$.

We will apply these adjectives to both the action and the corresponding action groupoid.

The second type are topological conditions. We can require our Lie group $G$ to be compact or discrete, in which case we refer to the action (and action groupoid) as being **compact** or **discrete**, resp. Or we can consider topological conditions on the action map, as with:

**Definition 35** (Proper Groupoid) A Lie groupoid $\mathcal{G}$ is **proper** if the map $\mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0 \colon g \mapsto (s(g), t(g))$ is a proper map. In particular, if $\mathcal{G}$ is an action groupoid, then we say that the corresponding action is **proper**.

We can also require the source of a Lie groupoid to be a local diffeomorphism. For an action groupoid this implies the group is discrete, so this is not useful for us. Instead, we consider action groupoids that are Morita equivalent to such groupoids (see Definition 20 for a definition of Morita equivalent).

**Definition 36** (Étale Groupoid) An **étale groupoid** is a Lie groupoid whose source (and hence target) map is a local diffeomorphism.

**Definition 37** (Orbifold Groupoid) An **orbifold groupoid** is a Lie groupoid that is Morita equivalent to a proper étale groupoid.

*Remark 38* There are subtle differences in how "orbifold groupoid" is defined in the literature. Our definition above matches that of Pronk-Scull [22, Definition 2.7]. However, others refer to only proper étale groupoids as "orbifold groupoids"; see, for instance, [3, 10]. Further, some authors restrict their attention to effective orbifolds [17, 18]. It follows from the Slice Theorem and associated Tube Theorem [8, Theorems 2.3.3, 2.4.1] that a proper and locally free group action corresponds to an orbifold groupoid as in Definition 37. Conversely, if an action groupoid is Morita equivalent to a proper étale groupoid, then it is proper and locally free. This follows from the fact that weak equivalences, and hence Morita equivalences, preserve stabilisers and properness (see, for instance, [16, Subsection 2.7]).

Our goal is to localise a specified sub-2-category of Lie groupoids whose objects are action groupoids that satisfy a desired set of properties $\mathcal{P}$. Our selection of properties comes from those defined above. Specifically, $\mathcal{P}$ is any subset (possibly empty) of the following list of properties:

$$\mathcal{P} \subseteq \left\{ \begin{array}{c} \text{free, locally free, transitive, effective, compact, discrete,} \\ \text{proper, is an orbifold groupoid} \end{array} \right\}. \tag{3}$$

*Remark 39* As mentioned earlier, there could be redundancy when one takes $\mathcal{P}$ to include more than one property. For example, as per Remark 38, being an orbifold groupoid is the same as locally free and proper in the case of action groupoids. However, one will see that in our proofs below, one could replace "orbifold groupoid" with being Morita equivalent to any class of groupoids that one wishes; the same argument goes through. Since this argument is different than those for proper, locally free, etc., we wish to keep the redundancy.

Similar results already appear in the literature for a few specific sub-classes of Lie groupoids. For instance, in [21], Pronk localises étale Lie groupoids using the method we outlined in Subsection 4.1. In [25], Roberts localises Lie groupoids, proper Lie groupoids, étale Lie groupoids, and étale proper Lie groupoids using the method we outlined in Subsection 4.2, which is equivalent to the method of Subsection 4.1 by Proposition 30.

We generalise these scattered results that appear in the literature below by considering action groupoids satisfying the properties $\mathcal{P}$. In fact, we go further. We begin by considering action groupoids with so-called equivariant functors between them, defined as follows.

**Definition 40** (The 2-Category $\mathbf{ActGpd}_{\mathcal{P}}$) Let $\mathbf{ActGpd}_{\mathcal{P}}$ be the sub-2-category of **LieGpoid** in which

- an object is an action Lie groupoid $\mathcal{G} = G \ltimes X$ coming from an action of a Lie group $G$ on a manifold $X$ satisfying $\mathcal{P}$,
- a 1-cell is an **equivariant functor** which is induced from an equivariant map: so there exists a Lie group homomorphism $\widetilde{\varphi}\colon G \to H$ and a smooth map $\varphi_0\colon X \to Y$ such that the functor $\varphi\colon G \ltimes X \to H \ltimes Y$ is defined on arrows by
$$\varphi_1(g, x) = (\widetilde{\varphi}(g), \varphi_0(x)),$$
for all $(g, x) \in G \times X$,
- a 2-cell is a natural transformation.

Let $W_{\mathcal{P}}$ be the class of equivariant weak equivalences and $\mathsf{sW}_{\mathcal{P}}$ the class of equivariant surjective submersive weak equivalences.

We use the recipe of Roberts [27] to produce a localisation $\mathbf{AnaLieGpoid}_{\mathcal{P}}$ of $\mathbf{ActGpd}_{\mathcal{P}}$ at $W_{\mathcal{P}}$, which has the equivariant anafunctors as 1-cells. Using [27, Theorem 3.24], we will show that this bicategory is equivalent to $\mathbf{ActGpd}_{\mathcal{P}}[W_{\mathcal{P}}^{-1}]$. Finally, we show that both of these bicategories are equivalent to $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$, the full sub-bicategory of $\mathbf{LieGpoid}[W^{-1}]$ whose objects are action groupoids satisfying $\mathcal{P}$.

We begin by constructing our localisation of $\mathbf{ActGpd}_{\mathcal{P}}$ at $W_{\mathcal{P}}$.

**Definition 41** ($\mathcal{P}$-anafunctor) A $\mathcal{P}$**-anafunctor** is a generalised morphism
$$G \ltimes X \xleftarrow[\varphi]{\simeq} K \ltimes Y \xrightarrow[\psi]{} H \ltimes Z$$

in which all groupoids involved are action groupoids satisfying $\mathcal{P}$, with $\psi$ equivariant and $\varphi \in \mathsf{sW}_{\mathcal{P}}$.

We will compose $\mathcal{P}$-anafunctors using the strict pullback. So we need to verify that $\mathbf{ActGpd}_{\mathcal{P}}$ is also closed under strict pullbacks.

**Lemma 42.** *Let $\mathcal{G} = G \ltimes X$, $\mathcal{H} = H \ltimes Y$, and $\mathcal{K} = K \ltimes Z$, and let $\varphi \colon \mathcal{G} \to \mathcal{K}$ and $\psi \colon \mathcal{H} \to \mathcal{K}$ be equivariant functors in $\mathbf{ActGpd}_{\mathcal{P}}$ (with their defining group maps $\widetilde{\varphi}, \widetilde{\psi}$) such that $\varphi \in \mathsf{sW}_{\mathcal{P}}$. Then the strict pullback groupoid $\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}$ is an action groupoid of a $(G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H)$-action that satisfies properties $\mathcal{P}$, and $\mathrm{pr}_1$ and $\mathrm{pr}_2$ are equivariant with respect to the restricted projection homomorphisms from $G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H$ with $\mathrm{pr}_2 \in \mathsf{sW}_{\mathcal{P}}$.*

*Proof* By Proposition 12, $\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}$ is a Lie groupoid and $\mathrm{pr}_2 \in \mathsf{sW}$. Since $G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H$ is a closed subgroup of the Lie group $G \times H$, it is a Lie subgroup [8, Corollary 1.10.7]. It is straightforward to check that $\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}$ is isomorphic to $(G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H) \ltimes (X{}_{\varphi_0}{\times}_{\psi_0}Y)$. The restricted projection functors $\mathrm{pr}_1$ and $\mathrm{pr}_2$ from $\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}$ are equivariant with respect to the restricted projection functors on $G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H$.

To check that the properties in $\mathcal{P}$ are preserved, first observe that if $G$ and $H$ are compact/discrete, then so is $G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H$. An examination of the stabilisers of the $(G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H)$-action on $X{}_{\varphi_0}{\times}_{\psi_0}Y$ yields immediately that if the $G$- and $H$-actions are effective/free/locally free, then so is the $(G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H)$-action. If $\mathcal{G}$ and $\mathcal{H}$ are proper, then we check that the $(G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H)$-action is proper: if we have a sequence $((g_i, h_i), (x_i, y_i))_{i\in\mathbb{N}}$ such that the image under $(s, t)$ converges to $((x, y), (x', y'))$, then $((x_i, y_i), (g_i x_i, h_i y_i)) \to ((x, y), (x', y'))$, and so $x_i \to x$ and $y_i \to y$. Then the fact that $G$ acts properly on $X$ gives us an element $g$ such that $g_i \to g$ and $gx = x'$, and similarly $h \in H$ such that $h_i \to h$ such that $h_i y_i \to y'$. Thus the sequence must converge to $((g, h), (x, y))$.

Suppose $\mathcal{H}$ is transitive. Hence for fixed $((x_1, y_1), (x_2, y_2)) \in X{}_{\varphi_0}{\times}_{\psi_0}Y$, there exists $h \in H$ so that $hy_1 = y_2$. Define $(g, x_1) := \mathbf{FF}_{\varphi}{}^{-1}\Big(x_1, x_2, \big(\widetilde{\psi}(h), \psi(y_1)\big)\Big)$. Then $(g, h) \in G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H$, and the action of $G{}_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}H$ is transitive.

Suppose that $\mathcal{H}$ is an orbifold groupoid; that is, there is a proper étale groupoid $\mathcal{E}$, a groupoid $\mathcal{L}$, and two weak equivalences $\beta \colon \mathcal{L} \to \mathcal{H}$ and $\beta' \colon \mathcal{L} \to \mathcal{E}$. Since $\varphi$ is a weak equivalence, so is $\mathrm{pr}_2 \colon \mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H} \to \mathcal{H}$ by Proposition 12. Thus, again by Proposition 12, $(\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}){}_{\mathrm{pr}_2}{\times}_{\beta}\mathcal{L}$ is a Lie groupoid, and we have the Morita equivalence

$$\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H} \xleftarrow[\mathrm{pr}_1]{\simeq} (\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}){}_{\mathrm{pr}_2}{\times}_{\beta}\mathcal{L} \xrightarrow[\beta' \circ \mathrm{pr}_2]{\simeq} \mathcal{E},$$

where $\beta' \circ \mathrm{pr}_2$ is a weak equivalence by Lemma 8. Thus, the strict pullback $\mathcal{G}{}_{\varphi}{\times}_{\psi}\mathcal{H}$ is an orbifold groupoid. $\square$

*Remark 43* Throughout the proof of Lemma 42, we did not require $\mathcal{K}$ to satisfy $\mathcal{P}$, only that $\mathcal{G}$ and $\mathcal{H}$ do. Moreover, for transitive and orbifold groupoids, our proof shows that we only needed to require $\mathcal{H}$ to be transitive/an orbifold groupoid, and not $\mathcal{G}$.

Thus we know that we can define the composition of $\mathcal{P}$-anafunctors using the strict pullback and get another $\mathcal{P}$-anafunctor. We can now construct a bicategory localising $\mathbf{ActGpd}_{\mathcal{P}}$ following the method of Subsection 4.2, provided $\mathsf{sW}_{\mathcal{P}}$ is a so-called "bi-fully faithful singleton strict pretopology". We have already verified that $\mathsf{sW}_{\mathcal{P}}$ satisfies the conditions this entails: All identity arrows are in $\mathsf{sW}_{\mathcal{P}}$, which is immediate. $\mathsf{sW}_{\mathcal{P}}$ must be closed under strict pullback, which is Lemma 42. $\mathsf{sW}_{\mathcal{P}}$ must be closed under composition, which follows from Lemma 8 and the fact that surjective submersions and equivariant maps are closed under composition. Finally, elements of $\mathsf{sW}_{\mathcal{P}}$ must be representably fully faithful, which is inherited from **AnaLieGpoid**. Thus, by [27, Theorem 3.20] we have:

**Proposition 44.** *There is a bicategory* $\mathbf{AnaActGpd}_{\mathcal{P}}$ *whose objects are those of* $\mathbf{ActGpd}_{\mathcal{P}}$*, arrows are* $\mathcal{P}$*-anafunctors, and* 2*-cells are the natural transformations of Definition 29.*

To compare $\mathbf{AnaActGpd}_{\mathcal{P}}$ to $\mathbf{ActGpd}_{\mathcal{P}}[W_{\mathcal{P}}^{-1}]$, we again have to confirm that "weak equivalences" as defined by Roberts are the same as ours here. Representable fully faithfulness is the same as smooth fully faithfulness by Lemma 10. We check the $\mathsf{sW}_{\mathcal{P}}$-locally split condition using the following lemma showing that $\mathbf{ActGpd}_{\mathcal{P}}$ admits weak pullbacks, from which the required $\mathsf{sW}_{\mathcal{P}}$-locally split condition follows.

**Lemma 45.** *Let* $\mathcal{G} = G \ltimes X$ *and* $\mathcal{H} = H \ltimes Y$ *be objects in* $\mathbf{ActGpd}_{\mathcal{P}}$*, and let* $\varphi\colon \mathcal{G} \to \mathcal{K}$ *and* $\psi\colon \mathcal{H} \to \mathcal{K}$ *be functors with* $\varphi \in W$*. Then the weak pullback* $\mathcal{G}\,{}_{\varphi}\overset{\mathrm{w}}{\times}{}_{\psi}\mathcal{H}$ *is isomorphic to an action groupoid of a* $(G \times H)$*-action on its object space* $Z$ *that satisfies properties* $\mathcal{P}$*, and* $\mathrm{pr}_1$ *and* $\mathrm{pr}_3$ *are equivariant with respect to the projection homomorphisms from* $G \times H$ *with* $\mathrm{pr}_3 \in \mathsf{sW}_{\mathcal{P}}$*.*

*Proof* By Proposition 14, $\mathcal{G}\,{}_{\varphi}\overset{\mathrm{w}}{\times}{}_{\psi}\mathcal{H}$ is a Lie groupoid and $\mathrm{pr}_3 \in \mathsf{sW}$. Let $Z$ be its object space. It is straightforward to check that the group action of $G \times H$ on $Z$

$$((g,h),(x,k,y)) \mapsto (gx, \psi(h,y)k\varphi(g,x)^{-1}, hy)$$

yields an isomorphism from $(G \times H) \ltimes Z$ to $\mathcal{G}\,{}_{\varphi}\overset{\mathrm{w}}{\times}{}_{\psi}\mathcal{H}$, and that $\mathrm{pr}_1$ and $\mathrm{pr}_3$ are equivariant with respect to the projection homomorphisms $G \times H \to G$ and $G \times H \to H$ respectively.

Now we consider the selected subset of properties $\mathcal{P}$. If $G$ and $H$ are compact resp. discrete, then so is $G \times H$. An examination of the stabilisers of the $(G \times H)$-action on $Z$ immediately yields that if the $G$- and $H$- actions are effective/free/locally free, then so is the $(G \times H)$-action.

Suppose the actions of $G$ and $H$ are proper. To show that the $(G \times H)$-action is proper, consider the map $(s,t)\colon (G \times H) \times Z \to Z \times Z$ defined by

$$((g,h),(x,k,y)) \to ((x,k,y),(gx,\psi(k,y)k\phi(g,x)^{-1},hy)).$$

If we have a sequence $((g_i,h_i),(x_i,k_i,y_i))_{i\in\mathbb{N}}$ such that the image converges to $((x,k,y),(x',k',y'))$, we see that because the source map of the pair groupoid is the first projection map, we have $x_i \to x$, $k_i \to k$, and $y_i \to y$. Moreover, because the action of $G$ on $X$ is proper, there exists $g$ such that $g_i \to g$ and $gx = x'$. Similarly there must exist $h \in H$ so that we have $h_i \to h$. Lastly, if we define $k'' = \psi(h,y)k\varphi(g,x)^{-1}$ we know that $\psi(h_i,y_i)k_i\varphi(g_i,x_i)^{-1} \to k''$ and thus because $\mathcal{K}_1$ is Hausdorff and limits in Hausdorff spaces are unique, $k'' = k'$. Thus we have shown that the limit of the sequence $((g_i,h_i),(x_i,k_i,y_i))$ is $((g,h),(x,k,y))$.

Suppose that $\mathcal{H}$ is a transitive action groupoid. Fix $((x,k,y),(x',k',y')) \in Z$. Then there exists $h \in H$ with $y' = hy$. Since $\varphi$ is a weak equivalence, there exists $g \in G$ such that $\varphi(g,x) = (k')^{-1} \cdot \psi(h,y) \cdot k$. Thus $(g,h) \cdot (x,k,y) = (x',k',y')$. It follows that $(G \times H) \ltimes Z$ is a transitive action groupoid.

Lastly, if $\mathcal{H}$ is an orbifold groupoid, then a similar argument to that found in the proof of Lemma 42 shows that $\mathcal{G}{}_{\varphi}\overset{\mathrm{w}}{\times}{}_{\psi}\mathcal{H}$ is an orbifold groupoid.

Thus any chosen properties $\mathcal{P}$ are preserved. □

*Remark 46* In the proof of Lemma 45, we did not require both of the action groupoids to be transitive (resp. orbifold groupoids); we only required the action groupoid for the $H$-action to satisfy these, and $\varphi$ to be a weak equivalence.

**Corollary 47.** *A functor $\varphi\colon \mathcal{G} \ltimes X \to H \ltimes Y$ is in $W_{\mathcal{P}}$ if and only if*

*(a) $\varphi$ is equivariant and smoothly fully faithful,*
*(b) there exists a $\psi\colon K \ltimes Z \to H \ltimes Y$ in $\mathsf{sW}_{\mathcal{P}}$,*
*(c) there exists an equivariant functor $\sigma\colon K \ltimes Z \to G \ltimes X$, and*
*(d) there exists a natural transformation $\eta\colon \varphi \circ \sigma \Rightarrow \psi$.*

*Proof* Suppose $\varphi \in W_{\mathcal{P}}$. Then by definition, $\varphi$ is smoothly fully faithful. By Lemma 18, there exist $\psi\colon \mathcal{K} \to H \ltimes Y$ in $\mathsf{sW}_{\mathcal{P}}$, a functor $\sigma\colon \mathcal{K} \to G \ltimes X$, and a natural transformation $\eta\colon \varphi \circ \sigma \Rightarrow \psi$. In fact, we can choose $\mathcal{K} = (G \ltimes X){}_{\varphi}\overset{\mathrm{w}}{\times}{}_{\mathrm{id}_{H\ltimes Y}}(H \ltimes Y)$, $\psi = \mathrm{pr}_3$, and $\sigma = \mathrm{pr}_1$ where $\varphi\colon G \ltimes X \to H \ltimes Y$ is in $W_{\mathcal{P}}$. Thus Lemma 45 ensures that $\mathcal{K}$ is an action groupoid $K \ltimes Z$, $\psi \in \mathsf{sW}_{\mathcal{P}}$, and $\sigma$ is equivariant. The converse follows immediately from Lemma 18. □

It follows from Corollary 47 and [27, Theorem 3.24] that:

**Proposition 48.** *The inclusion* $\mathbf{AnaActGpd}_{\mathcal{P}} \to \mathbf{ActGpd}_{\mathcal{P}}[W_{\mathcal{P}}^{-1}]$ *is an equivalence of bicategories, where this inclusion takes a* 2*-cell to its equivalence class.*

# 6 The Equivalence of $\mathbf{AnaActGpd}_{\mathcal{P}}$ and $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$

In the previous section, we constructed a bicategory $\mathbf{AnaActGpd}_{\mathcal{P}}$ out of the equivariant action groupoids satisfying the chosen properties $\mathcal{P}$, with 1-cells given by $\mathcal{P}$-anafunctors. In this section, we will show that this bicategory $\mathbf{AnaActGpd}_{\mathcal{P}}$ is equivalent to the full sub-bicategory $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ of $\mathbf{LieGpoid}[W^{-1}]$ whose objects are action groupoids of Lie group actions satisfying $\mathcal{P}$. Thus we can start in the localised category of Lie groupoids and restrict to action groupoids with property $\mathcal{P}$. Alternatively, we can use the "smaller" bicategory $\mathbf{AnaActGpd}_{\mathcal{P}}$ constructed in the previous section, and these two categories are equivalent. Moreover, we get that these bicategories are equivalent to the category created by localising action groupoids with property $\mathcal{P}$ at all $\mathcal{P}$-weak equivalences using the original Pronk localisaton of Section 4.1 for free. This is the content of Theorem 55.

The objects of $\mathbf{AnaActGpd}_{\mathcal{P}}$ and $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ are the same, and every 1-cell of $\mathbf{AnaActGpd}_{\mathcal{P}}$ is a particular kind of generalised morphism, thus defining a 1-cell in $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$. Similarly, every 2-cell of $\mathbf{AnaActGpd}_{\mathcal{P}}$ also represents a 2-cell of $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$. So we have an inclusion:

**Definition 49** (The Pseudofunctor $I_{\mathcal{P}}$) Define $I_{\mathcal{P}}\colon \mathbf{AnaActGpd}_{\mathcal{P}} \to \mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ to be the assignment sending objects to themselves, sending a $\mathcal{P}$-anafunctor to itself as a generalised morphism, and sending a 2-cell between $\mathcal{P}$-anafunctors to its equivalence class as a 2-cell between generalised morphisms.

The goal of this section is to show that this inclusion is a pseudofunctor which induces an equivalence of bicategories. Thus we have to check that the inclusion $I_{\mathcal{P}}$ respects the compositions and unitors detailed in Appendix A, and that it is essentially surjective and fully faithful: any generalised morphism between two objects of $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ admits a 2-cell from itself to a $\mathcal{P}$-anafunctor, and that any 2-cell between $\mathcal{P}$-anafunctors can be represented by a unique 2-cell from $\mathbf{AnaActGpd}_{\mathcal{P}}$.

We begin with the generalised morphisms. Our strategy will be to show that any generalised morphism is equivalent to a $\mathcal{P}$-anafunctor induced by a bibundle. The theory of bibundles offers another method of localising **LieGpoid** with a more geometric flavour; see [9, 11, 15] for details. We do not require the full theory here, but simply borrow the necessary concepts.

**Proposition 50.** *Any generalised morphism*

$$\mathcal{G} = G \ltimes X \underset{\varphi}{\overset{\simeq}{\longleftarrow}} \mathcal{K} \underset{\psi}{\longrightarrow} H \ltimes Z = \mathcal{H}$$

*between objects in* $\mathbf{ActGpd}_{\mathcal{P}}$ *admits a 2-cell from itself to a* $\mathcal{P}$*-anafunctor* $\mathcal{G} \underset{\chi}{\overset{\simeq}{\longleftarrow}} \mathcal{L} \underset{\omega}{\longrightarrow} \mathcal{H}$.

*Proof* To have a $\mathcal{P}$-anafunctor $\mathcal{G} \underset{\chi}{\overset{\simeq}{\longleftarrow}} \mathcal{L} \underset{\omega}{\longrightarrow} \mathcal{H}$ requires that $\mathcal{L}$ be an action groupoid of an action of $G \times H$ which satisfies our selected properties $\mathcal{P}$. We will define our action groupoid $\mathcal{L}$ as a quotient of a groupoid $\widetilde{\mathcal{L}}$ which we create out of compositions in $\mathbf{LieGpoid}[W^{-1}]$ and $\mathbf{AnaLieGpoid}$; that is, weak and strict pullbacks.

By Proposition 14, $\mathcal{G}_{\mathrm{id}_{\mathcal{G}}} \overset{\mathrm{w}}{\times}_{\varphi} \mathcal{K}$ and $\mathcal{K}_{\psi} \overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}} \mathcal{H}$ are Lie groupoids and $\mathrm{pr}_1 \colon \mathcal{G}_{\mathrm{id}_{\mathcal{G}}} \overset{\mathrm{w}}{\times}_{\varphi} \mathcal{K} \to \mathcal{G}$, $\mathrm{pr}_3 \colon \mathcal{G}_{\mathrm{id}_{\mathcal{G}}} \overset{\mathrm{w}}{\times}_{\varphi} \mathcal{K} \to \mathcal{K}$, and $\mathrm{pr}_1 \colon \mathcal{K}_{\psi} \overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}} \mathcal{H} \to \mathcal{K}$ are in $\mathsf{sW}$. Therefore we can define the composition of anafunctors $\widetilde{\mathcal{L}}$, that is, the strict pullback of these weak pullbacks: $\widetilde{\mathcal{L}} := (\mathcal{G}_{\mathrm{id}_{\mathcal{G}}} \overset{\mathrm{w}}{\times}_{\varphi} \mathcal{K})_{\mathrm{pr}_3} \times_{\mathrm{pr}_1} (\mathcal{K}_{\psi} \overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}} \mathcal{H})$. By Proposition 12 we know that $\widetilde{\mathcal{L}}$ is a Lie groupoid and both of its projection functors are in $\mathsf{sW}$. By Proposition 31, $\mathcal{G} \overset{\simeq}{\longleftarrow} \widetilde{\mathcal{L}} \to \mathcal{H}$ admits a 2-cell from itself to the composition of $\mathcal{G} \overset{\simeq}{\longleftarrow} \mathcal{G}_{\mathrm{id}_{\mathcal{G}}} \overset{\mathrm{w}}{\times}_{\varphi} \mathcal{K} \to \mathcal{K}$ and $\mathcal{K} \overset{\simeq}{\longleftarrow} \mathcal{K}_{\psi} \overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}} \mathcal{H} \to \mathcal{H}$ in $\mathbf{LieGpoid}[W^{-1}]$, which in turn admits a 2-cell from itself to $\mathcal{G} \overset{\simeq}{\longleftarrow} \mathcal{K} \to \mathcal{H}$. Thus we have the following 2-commutative diagram.

$$\begin{tikzcd}
 & & & \widetilde{\mathcal{L}} \arrow[dl, "\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\mathrm{pr}_2", "\simeq"'] & & & \\
 & & \mathcal{G}_{\mathrm{id}_{\mathcal{G}}} \overset{\mathrm{w}}{\times}_{\varphi} \mathcal{K} \arrow[dl, "\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\mathrm{pr}_3", "\simeq"'] \arrow[rr, phantom, "\circlearrowright"] & & \mathcal{K}_{\psi} \overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}} \mathcal{H} \arrow[dl, "\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\mathrm{pr}_3"] & & \\
 & \mathcal{G} \arrow[dl, equal] \arrow[dr, equal] \arrow[rr, phantom, "\underset{\mathrm{PR}_2}{\Rightarrow}"] & & \mathcal{K} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] \arrow[rr, phantom, "\underset{\mathrm{PR}_2}{\Rightarrow}"] & & \mathcal{H} \arrow[dl, equal] \arrow[dr, equal] & \\
\mathcal{G} & & \mathcal{G} & & \mathcal{H} & & \mathcal{H}
\end{tikzcd}$$

We define a left action of the Lie groupoid $\mathcal{K}$ on $\widetilde{\mathcal{L}}_0$. Such an action requires an anchor map, which we take here to be $a = \mathrm{pr}_3 \circ \mathrm{pr}_1 \colon \widetilde{\mathcal{L}} \to \mathcal{K}_0$, and an action map $\mathrm{act} \colon \mathcal{K}_{1\,s}\times_a \widetilde{\mathcal{L}}_0 \to \widetilde{\mathcal{L}_0}$ defined by

$$\mathrm{act}(\kappa, ((x,(g,x),y),(y,(h,z),z))) := ((x, \varphi(\kappa)\cdot(g,x), t(\kappa)), (t(\kappa), (h,z)\cdot\psi(\kappa)^{-1}, z)).$$

The required identity and associativity properties are easily verified here; see [11, 15] for more details on Lie groupoid actions.

We will define the object space of the desired action groupoid $\mathcal{L}$ by $\mathcal{L}_0 := \mathcal{K}\backslash\widetilde{\mathcal{L}}_0$. To verify that this is a manifold, it suffices to show that the action of $\mathcal{K}$ on $\mathcal{L}_0$ is free and proper by [7, Proposition 3.6.2], the general groupoid action version of the standard group action result.

The action is free since $\varphi$ is smoothly fully faithful. We claim that the map $(\mathrm{pr}_2, \mathrm{act})\colon (\mathcal{K} \ltimes \widetilde{\mathcal{L}}_0)_1 \to \widetilde{\mathcal{L}}_0^2$ is proper. To show this, fix a sequence $((\kappa_i, \ell_i))_{i\in\mathbb{N}}$ in $(\mathcal{K} \ltimes \widetilde{\mathcal{L}}_0)_1$ such that $((s,t)(\kappa_i, \ell_i)) = ((\ell_i, \kappa_i \cdot \ell_i))$ converges in $\widetilde{\mathcal{L}}_0^2$ to $(\ell, \ell')$, where for each $i$

$$\begin{aligned} \ell_i &= ((x_i, (g_i, x_i), y_i), (y_i, (h_i, z_i), z_i)), \\ \ell &= ((x, (g, x), y), (y, (h, z), z)), \text{ and} \\ \ell' &= ((x', (g', x'), y'), (y', (h', z'), z')). \end{aligned}$$

Since the source map of $\widetilde{\mathcal{L}}_0^2$ is the first projection map, we immediately have that $(\ell_i)$ converges to $\ell$, and so in particular $x = x'$, $g_i \to g$, and $y_i \to y$. We need to obtain a limit for $\kappa_i$: we know from the coordinates of the target map that $\varphi(\kappa_i)(g_i, x_i) \to (g', x')$, and that $t(\kappa_i) \to y'$. Then the arrow $(g'g^{-1}, \varphi(y))$ has target $\varphi(y')$, and since $\varphi$ is smoothly fully faithful we can define $\kappa = \mathbf{FF}_\varphi{}^{-1}(y, y', (g'g^{-1}, \varphi(y)))$ to obtain a limit for $\kappa_i$. Thus the action of $\mathcal{K}$ on $\widetilde{\mathcal{L}}_0$ is free and proper, and so its orbit space $\mathcal{L}_0 := {}_{\mathcal{K}}\backslash^{\widetilde{\mathcal{L}}_0}$ is a manifold and the quotient map $\pi_0\colon \widetilde{\mathcal{L}}_0 \to \mathcal{L}_0$ a surjective submersion.

There is also a left action of $G \times H$ on $\widetilde{\mathcal{L}}_0$ given by

$$((\widetilde{g}, \widetilde{h}), ((x, (g, x), y), (y, (h, \psi(y)), z))) \mapsto ((\widetilde{g}x, (g\widetilde{g}^{-1}, \widetilde{g}x), y), (y, (\widetilde{h}h, \psi(y)), \widetilde{h}z)).$$

This action commutes with the action of $\mathcal{K}$, and so descends to a well-defined action of $G \times H$ on $\mathcal{L}_0$. Thus we have the action groupoid $\mathcal{L} = (G \times H) \ltimes \mathcal{L}_0$.

The map $\pi_1\colon \widetilde{\mathcal{L}}_1 \to \mathcal{L}_1$ defined by sending

$$(((\widetilde{g}, x), (g, x), k), (k, (h, \psi(s(k))), (\widetilde{h}, z)))$$

to

$$((\widetilde{g}, \widetilde{h}), [(x, (g, x), s(k)), (s(k), (h, \psi(s(k))), z)])$$

is well-defined and smooth. We can show that $\pi := (\pi_0, \pi_1)\colon \widetilde{\mathcal{L}} \to \mathcal{L}$ is a smoothly fully faithful functor using the LCL property: if we consider $\mathbf{FF}_\pi : \widetilde{\mathcal{L}}_1 \to \widetilde{\mathcal{L}}^2_{0\,\pi^2}\!\times_{(s,t)} \mathcal{L}_1$ and let $p : I \to \widetilde{\mathcal{L}}^2_{0\,\pi^2}\!\times_{(s,t)} \mathcal{L}_1$ be a curve, it will have the form $p(t) = (v_\tau, w_\tau, ((g_\tau, h_\tau), [v_\tau]))$ where $v_\tau$ and $w_\tau$ are curves in $\widetilde{\mathcal{L}}_0$ and $(g_\tau, h_\tau)$ is a curve in $G \times H$. We know that $[v_\tau]$ and $[w_\tau]$ are the source and target of the arrow $((g_\tau, h_\tau), [v_\tau])$, and so $[w_\tau] = (g_\tau, h_\tau)[v_\tau]$; the smooth fully faithfulness of $\varphi$ allows us to smoothly choose $\kappa_\tau$ such that $w_\tau = (g_\tau, h_\tau)\kappa_\tau v_\tau$. Then the curve to the arrow space defined by $(g_\tau, h_\tau)\kappa_\tau$ produces the desired lift. It is a weak equivalence by Lemma 9.

Since the individual $G$- and $H$-actions on $\widetilde{\mathcal{L}}$ commute with the $\mathcal{K}$-action, the composition $\mathrm{pr}_1 \circ \mathrm{pr}_1 : \widetilde{\mathcal{L}} \to \mathcal{G}$ descends to a well-defined equivariant functor $\chi\colon \mathcal{L} \to \mathcal{G}$; similarly, $\mathrm{pr}_3 \circ \mathrm{pr}_2\colon \widetilde{\mathcal{L}} \to \mathcal{H}$ descends to a equivariant functor $\omega\colon \mathcal{L} \to \mathcal{H}$. By Lemma 8, $\chi$ is a weak equivalence, and by Lemma 15, $\chi_0$ is a surjective submersion since $\pi_0$ and $(\mathrm{pr}_1 \circ \mathrm{pr}_1)_0\colon \widetilde{\mathcal{L}}_0 \to \mathcal{G}_0$ are. It follows from the definitions that $\chi$ and $\omega$ are equivariant with respect to the projections $G \times H \to G$ and $G \times H \to H$, resp.

The functors $\mathcal{L} \xleftarrow[\pi]{\simeq} \widetilde{\mathcal{L}} \xrightarrow[\mathrm{pr}_3 \circ \mathrm{pr}_1]{\simeq} \mathcal{K}$ with the natural transformations $(\mathrm{PR}_2)\mathrm{pr}_1 : \chi \circ \pi \Rightarrow \varphi \circ (\mathrm{pr}_3 \circ \mathrm{pr}_1)$ and $(\mathrm{PR}_2)\mathrm{pr}_2 \colon \psi \circ (\mathrm{pr}_3 \circ \mathrm{pr}_1) \Rightarrow \omega \circ \pi$ provide 2-cells between the generalised morphisms $\mathcal{G} \xleftarrow{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ and $\mathcal{G} \xleftarrow{\simeq} \mathcal{L} \longrightarrow \mathcal{H}$.

Thus all that remains is to verify that $\mathcal{L}$ inherits the properties in $\mathcal{P}$.

If $G$ and $H$ are compact/discrete, then so is $G \times H$. If the actions of $G$ and $H$ on $X$ and $Y$, resp., are effective/free/locally free, then so is the action of $G \times H$ on $\mathcal{L}$.

Suppose that the $G$- and $H$-actions are proper. Let $(\ell_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{L}_1$ where for each $i$,

$$\ell_i := ((\widetilde{g}_i, \widetilde{h}_i), [(x_i, (g_i, x_i), y_i), (y_i, (h_i, \psi(y_i))_i), z_i)]),$$

and such that $((s,t)(\ell_i))$ converges in $\mathcal{L}_0^2$ to $(\ell, \ell')$ where

$$\ell = [(x, (g, x), y), (y, (h, \psi(y)), z)] \quad \text{and} \quad \ell' = [(x', (g', x'), y'), (y', (h', \psi(y')), z')].$$

Since $\pi_0$ is a surjective submersion, it admits local sections. Thus without loss of generality, we may assume that

$$((x_i, (g_i, x_i), y_i), (y_i, (h_i, \psi(y_i)), z_i)) \to ((x, (g, x), y), (y, (h, \psi(y)), z))$$

in $\widetilde{\mathcal{L}}_0$, and so there is a sequence $(\kappa_i)$ in $\mathcal{K}_1$ such that

$$\begin{aligned}
&\kappa_i(\widetilde{g}_i, \widetilde{h}_i)((x_i, (g_i, x_i), y_i), (y_i, (h_i, \psi(y_i)), z_i)) \\
&= (\widetilde{g}_i x_i, \varphi(\kappa_i)(g_i \widetilde{g}_i^{-1}, \widetilde{g}_i x_i), t(\kappa_i)), (t(\kappa_i), (\widetilde{h}_i h_i, \psi(y_i))\psi(\kappa_i)^{-1}, \widetilde{h}_i z_i) \\
&\to ((x', (g', x'), y'), (y', (h', \psi(y')), z'))
\end{aligned}$$

in $\widetilde{\mathcal{L}}_0$.

Since the $G$- and $H$- actions are proper, $(\widetilde{g}_i)$ and $(\widetilde{h}_i)$ converge to $\widetilde{g}$ and $\widetilde{h}$, resp., and so we conclude that $\varphi(\kappa_i)$ converges to $(g'', g\widetilde{g}^{-1}x')$ for some $g'' \in G$. Now we use the smooth fully faithfulness of $\varphi$ to define $\kappa := \mathbf{FF}_\varphi{}^{-1}(y, y', g'')$. Since

$$((\widetilde{g}_i, \widetilde{h}_i), (x_i, (g_i, x_i), y_i), (y_i, (h_i, \varphi(y_i)), z_i)) \to \kappa((\widetilde{g}, \widetilde{h}), (x, (g, x), y), (y, (h, \psi(y)), z))$$

we know that in the quotient $\mathcal{L}_0$ we have

$$((\widetilde{g}_i, \widetilde{h}_i), [(x_i, (g_i, x_i), y_i), (y_i, (h_i, \psi(y)), z_i)]) \to ((\widetilde{g}, \widetilde{h}), [(x, (g, x), y), (y, (h, \psi(y)), z)]).$$

This shows that $\mathcal{L}$ is a proper Lie groupoid.

Finally, if $\mathcal{G}$ is an orbifold groupoid, then there is a Lie groupoid $\mathcal{A}$, a proper étale Lie groupoid $\mathcal{E}$, and weak equivalences $\alpha\colon \mathcal{A} \to \mathcal{G}$ and $\alpha'\colon \mathcal{A} \to \mathcal{E}$. Since $\varphi$ is a weak equivalence, $\mathrm{pr}_1\colon \mathcal{G}_{\mathrm{id}_\mathcal{G}} \overset{\mathrm{w}}{\times}_\varphi \mathcal{K} \to \mathcal{G}$ is a weak equivalence by Proposition 14. Thus $\mathcal{M} := \mathcal{A}_\alpha \overset{\mathrm{w}}{\times}_{\mathrm{pr}_1} (\mathcal{G}_{\mathrm{id}_\mathcal{G}} \overset{\mathrm{w}}{\times}_\varphi \mathcal{K})$ is a Lie groupoid, and both projection maps from $\mathcal{M}$ are surjective submersive weak equivalences, again by Proposition 14. Finally, by Proposition 12, we have the generalised morphism $\mathcal{L} \xleftarrow{\simeq} \mathcal{M}_{\mathrm{pr}_3} \times_\chi \mathcal{L} \xrightarrow[\alpha' \circ \mathrm{pr}_1 \circ \mathrm{pr}_1]{} \mathcal{E}$, where $\alpha' \circ \mathrm{pr}_1 \circ \mathrm{pr}_1$ is a weak equivalence by Lemma 8. Thus, $\mathcal{L}$ is an orbifold groupoid. □

We now want to prove a result similar to Proposition 50 for 2-cells. Recall that the 2-cells in $\mathbf{ActGpd}_\mathcal{P}$ are defined by equivalence classes of diagrams connecting generalised morphisms, whereas in $\mathbf{AnaActGpd}_\mathcal{P}$ the 2-cells are given by actual natural transformations, not equivalence classes. Therefore to show that the two localisations yield equivalent localised bicategories, we need to show that there is a unique way of representing any 2-cell between two equivariant anafunctors in $\mathbf{LieGpoid}[W^{-1}]$ as a natural transformation, yielding a 2-cell of the bicategory $\mathbf{AnaActGpd}_{\mathcal{P}}$.

The proof of this will require several lemmas, following the outline of the proof of a similar result of Pronk-Scull (see [24, Section 5]), but with some necessary modifications: equivariant surjective submersive weak equivalences are not preserved under natural transformations, so we cannot follow Pronk-Scull verbatim. The first lemma below is a modified version of [24, Lemma 5.2], and proves that any 2-cell between $\mathcal{P}$-anafunctors is equivalent to a 2-cell from $\mathbf{AnaActGpd}_{\mathcal{P}}$.

**Lemma 51.** *Let $\mathcal{G} = G \ltimes X$ and $\mathcal{H} = H \ltimes Y$ be objects of $\mathbf{ActGpd}_{\mathcal{P}}$. Suppose we have two $\mathcal{P}$-anafunctors, the top and bottom of the diagram below, with a 2-cell connecting them in $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ represented by the following diagram:*

$$\begin{tikzcd}
 & & \mathcal{K} \arrow[dll, "\varphi"', "\simeq"] \arrow[drr, "\psi"] & & \\
\mathcal{G} \arrow[rr, phantom, "\Downarrow \eta_1"] & & \mathcal{L} \arrow[u, "\alpha", "\simeq"'] \arrow[d, "\alpha'"', "\simeq"] \arrow[rr, phantom, "\Downarrow \eta_2"] & & \mathcal{H} \\
 & & \mathcal{K}'. \arrow[ull, "\varphi'", "\simeq"'] \arrow[urr, "\psi'"'] & &
\end{tikzcd} \tag{4}$$

*Then this 2-cell is represented by the following 2-cell from $\mathbf{AnaActGpd}_{\mathcal{P}}$:*

$$\begin{tikzcd}
 & & \mathcal{K} \arrow[dll, "\varphi"', "\simeq"] \arrow[drr, "\psi"] & & \\
\mathcal{G} \arrow[rr, phantom, "\circlearrowleft"] & & \mathcal{K}{}_{\varphi}{\times}_{\varphi'}\mathcal{K}' \arrow[u, two heads, "\mathrm{pr}_1", "\simeq"'] \arrow[d, two heads, "\mathrm{pr}_2"', "\simeq"] \arrow[rr, phantom, "\Downarrow \nu"] & & \mathcal{H} \\
 & & \mathcal{K}'. \arrow[ull, "\varphi'", "\simeq"'] \arrow[urr, "\psi'"'] & &
\end{tikzcd} \tag{5}$$

*Proof* By Lemma 42, $\mathcal{K}{}_{\varphi}{\times}_{\varphi'}\mathcal{K}'$ is an action groupoid of a Lie group action of $K := (G \times H)_{\widetilde{\varphi}}{\times}_{\widetilde{\psi}}(G \times H)$.

Define $\widetilde{\mathcal{L}} := (\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}')_{\mathrm{pr}_1}\overset{\mathrm{w}}{\times}_{\alpha}\mathcal{L}$, and consider the following diagram, in which Proposition 14 justifies the decorations on the arrows:

$$\begin{tikzcd}
 & & \mathcal{K} \arrow[lld, bend right, "\varphi"', "\simeq"] \arrow[rrd, bend left, "\psi"] & & \\
\mathcal{G} & \mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}' \arrow[ru, bend left=15, "\simeq", "\mathrm{pr}_1"'] \arrow[rd, bend right=15, "\mathrm{pr}_2", "\simeq"'] & \widetilde{\mathcal{L}} \arrow[l, two heads, "\simeq"', "\mathrm{pr}_1"] \arrow[r, two heads, "\simeq", "\mathrm{pr}_3"'] \arrow[u, phantom, "\overset{\mathrm{PR}_2}{\Rightarrow}"] & \mathcal{L} \arrow[lu, bend right=15, "\alpha", "\simeq"'] \arrow[ld, bend left=15, "\alpha'"', "\simeq"] & \mathcal{H} \\
 & & \mathcal{K}'. \arrow[llu, bend left, "\varphi'", "\simeq"'] \arrow[rru, bend right, "\psi'"'] & &
\end{tikzcd}$$

By Lemma 10, the natural transformation

$$(\eta_1\mathrm{pr}_3)\circ(\varphi\mathrm{PR}_2)\circ(\mathrm{ID}_{\varphi'\circ\mathrm{pr}_2}\mathrm{pr}_1)\colon \varphi'\circ\mathrm{pr}_2\circ\mathrm{pr}_1 \Rightarrow \varphi'\circ\alpha'\circ\mathrm{pr}_3$$

factors as $\varphi'\mu$ for a unique natural transformation $\mu\colon \mathrm{pr}_2\circ\mathrm{pr}_1 \Rightarrow \alpha'\circ\mathrm{pr}_3$, making the lower triangle in the above diagram 2-commute. It follows from the definition of $\mu$ that

$$(\eta_1\mathrm{pr}_3)\circ(\varphi\mathrm{PR}_2) = (\varphi'\mu)\circ(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}\mathrm{pr}_1).$$

By Lemma 17, the natural transformation

$$(\psi'\mu^{-1})\circ(\eta_2\mathrm{pr}_3)\circ(\psi\mathrm{PR}_2)\colon \psi\circ\mathrm{pr}_1\circ\mathrm{pr}_1 \Rightarrow \psi'\circ\mathrm{pr}_2\circ\mathrm{pr}_1$$

factors as $\nu\mathrm{pr}_1$ for a unique natural transformation $\nu\colon \psi\circ\mathrm{pr}_1 \Rightarrow \psi'\circ\mathrm{pr}_2$. It follows from the definition of $\nu$ that

$$(\eta_2\mathrm{pr}_3)\circ(\psi\mathrm{PR}_2) = (\psi'\mu)\circ(\nu\mathrm{pr}_1).$$

This shows that the diagram (5) is indeed an equivalence, and in the same equivalence class as the 2-cell (4). □

Thus we have shown that any 2-cell between $\mathcal{P}$-anafunctors is represented by a 2-cell from $\mathbf{AnaActGpd}_{\mathcal{P}}$. We now need to prove the uniqueness of such a representative. We begin with a technical lemma showing that any equivalence can be represented using surjective submersions.

**Lemma 52.** *Given two representatives of a 2-cell which both come from* $\mathbf{AnaActGpd}_{\mathcal{P}}$ *and have the form*

$$\begin{tikzcd}
 & \mathcal{K} \arrow[ld, "\varphi"', "\simeq"] \arrow[rd, "\psi"] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowleft"] & \mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}' \arrow[u, "\mathrm{pr}_1", "\simeq"'] \arrow[d, "\mathrm{pr}_2"', "\simeq"] \arrow[r, phantom, "\Downarrow\eta,\quad \Downarrow\mu"] & \mathcal{H} \\
 & \mathcal{K}', \arrow[lu, "\varphi'", "\simeq"'] \arrow[ru, "\psi'"'] &
\end{tikzcd}$$

*the generalised morphism*

$$\mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}' \xleftarrow[\beta]{\simeq} \mathcal{M} \xrightarrow{\simeq} \mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}'$$

*inducing the equivalence between the two representatives (see Definition 24) can be chosen so that $\beta_0$ is a surjective submersion.*

*Proof* Since the two equivalences are in the same equivalence class, there exists a generalised morphism

$$\mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}' \xleftarrow[\alpha]{\simeq} \mathcal{L} \xrightarrow[\alpha']{\simeq} \mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}'$$

and natural transformations

$$\nu\colon \mathrm{pr}_1\circ\alpha \Rightarrow \mathrm{pr}_1\circ\alpha' \quad \text{and} \quad \nu'\colon \mathrm{pr}_2\circ\alpha \Rightarrow \mathrm{pr}_2\circ\alpha'$$

such that

$$(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}\alpha')\circ(\varphi\nu) = (\varphi'\nu')\circ(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}\alpha) \quad \text{and} \quad (\mu\alpha')\circ(\psi\nu) = (\psi'\nu')\circ(\eta\alpha). \quad (6)$$

Define $\mathcal{M} := (\mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}')_{\varphi\circ\mathrm{pr}_1}\times_{\varphi\circ\mathrm{pr}_1\circ\alpha}\mathcal{L}$. By Lemma 8 and the fact that surjective submersions are closed under composition, since $\mathrm{pr}_1$ and $\varphi$ are surjective submersive weak equivalences, so is $\varphi\circ\mathrm{pr}_1$; thus by Proposition 12 $\mathcal{M}$ is a Lie groupoid.

By Lemma 10 the following natural transformation between functors from $\mathcal{M}\to\mathcal{G}$,

$$(\varphi\circ\mathrm{pr}_1)\circ\mathrm{pr}_1 = \varphi\circ(\mathrm{pr}_1\circ\alpha)\circ\mathrm{pr}_2 \underset{\varphi\nu\mathrm{pr}_2}{\Longrightarrow} \varphi\circ(\mathrm{pr}_1\circ\alpha')\circ\mathrm{pr}_2,$$

factors as $\varphi\omega$ for $\omega\colon \mathrm{pr}_1\circ\mathrm{pr}_1 \Rightarrow \mathrm{pr}_1\circ(\alpha'\circ\mathrm{pr}_2)$. And since $\varphi\circ\mathrm{pr}_1 = \varphi'\circ\mathrm{pr}_2$, the natural transformation between functors $\mathcal{M}\to\mathcal{G}$,

$$(\varphi'\circ\mathrm{pr}_2)\circ\mathrm{pr}_1 = (\varphi\circ\mathrm{pr}_1)\circ\mathrm{pr}_1 = (\varphi\circ\mathrm{pr}_1\circ\alpha)\circ\mathrm{pr}_2 = \varphi'\circ(\mathrm{pr}_2\circ\alpha)\circ\mathrm{pr}_2 \underset{\varphi'\nu'\mathrm{pr}_2}{\Longrightarrow} \varphi'\circ(\mathrm{pr}_2\circ\alpha')\circ\mathrm{pr}_2,$$

which again factors by the same lemma to give $\omega'\colon \mathrm{pr}_2\circ\mathrm{pr}_1 \Rightarrow \mathrm{pr}_2\circ(\alpha'\circ\mathrm{pr}_2)$. Thus we have the following 2-commutative diagram:

$$\begin{tikzcd}
 & \mathcal{K} & \\
 & \overset{\omega}{\Rightarrow} & \\
\mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}' \arrow[uur, "\mathrm{pr}_1", "\simeq"'] \arrow[ddr, "\mathrm{pr}_2"', "\simeq"] & \mathcal{M} \arrow[l, two heads, "\simeq"', "\mathrm{pr}_1"] \arrow[r, "\simeq", "\alpha'\circ\mathrm{pr}_2"'] & \mathcal{K}_\varphi\times_{\varphi'}\mathcal{K}' \arrow[uul, "\mathrm{pr}_1"', "\simeq"] \arrow[ddl, "\mathrm{pr}_2", "\simeq"'] \\
 & \overset{\omega'}{\Rightarrow} & \\
 & \mathcal{K}'. &
\end{tikzcd}$$

We will show that $\mathcal{M}$ together with the the 2-cells $\omega$ and $\omega'$ are also an equivalence between the original 2-cells. This requires that Equations (2) hold. The first equation is straightforward from the definitions of $\omega$ and $\omega'$ and the first equation of (6):

$$\begin{aligned}\big(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}(\alpha'\circ\mathrm{pr}_2)\big)\circ(\varphi\omega) &= \big(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}(\alpha'\circ\mathrm{pr}_2)\big)\circ(\varphi\nu\mathrm{pr}_2)\\ &= \big((\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}\alpha')\circ(\varphi\nu)\big)\mathrm{pr}_2\\ &= \big((\varphi'\nu')\circ(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}\alpha)\big)\mathrm{pr}_2\\ &= (\varphi'\nu'\mathrm{pr}_2)\circ\big(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}(\alpha\circ\mathrm{pr}_2)\big)\\ &= (\varphi'\omega')\circ\big(\mathrm{ID}_{\varphi\circ\mathrm{pr}_1}(\mathrm{pr}_1)\big)\,,\end{aligned}$$

where the last line follows from the definition of $\mathcal{M}$.

To show that the second equation of (2) holds, we will make use of the fact that $\varphi$, $\varphi'$, $\mathrm{pr}_1\circ\alpha$, and $\mathrm{pr}_2\circ\alpha$ are smoothly fully faithful, and that $\nu$ and $\nu'$ are natural transformations. If $m\in\mathcal{M}_0$, then it is of the form $m=((y,y'),z)$ for $y\in\mathcal{K}_0$, $y'\in\mathcal{K}'_0$, and $z\in\mathcal{L}_0$. So we have arrows $k:=\mathbf{FF}_{\varphi}{}^{-1}((y,\mathrm{pr}_1\circ\alpha(z)),u_{\varphi(y)})$ in $\mathcal{K}_1$ and $k':=\mathbf{FF}_{\varphi'}{}^{-1}((y',\mathrm{pr}_2\circ\alpha(z)),u_{\varphi'(y')})$ in $\mathcal{K}'_1$. We claim that in fact $\omega((y,y'),z)=\nu(z)k$ and $\omega'((y,y'),z)=\nu'(z)k'$: Lemma 10 says that the lift $\omega$ obtained above is unique, and we can check that $\widehat{\omega}\colon m=((y,y'),z)\mapsto\nu(z)k$ represents another natural transformation $\mathrm{pr}_1\circ\mathrm{pr}_1\Rightarrow\mathrm{pr}_1\circ(\alpha'\circ\mathrm{pr}_2)$ which satisfies $\varphi\widehat{\omega}=\varphi\nu\mathrm{pr}_2$, and hence $\omega(m)=\widehat{\omega}(m)=\nu(z)k$. Similarly, $\omega'((y,y'),z)=\nu'(z)k'$.

Therefore, the second equation of (6), the naturality of $\eta$, and the fact that $(k,k')$ is an arrow from $(y,y')$ to $\alpha(z)$ in $\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}'$ gives us that

$$\begin{aligned}\big(\mu(\alpha'\circ\mathrm{pr}_2)\big)\circ(\psi\omega)(m) &= \mu(\alpha'(z))\cdot\psi(\nu(z))\cdot\psi(k)\\ &= \psi'(\nu'(z))\cdot\eta(\alpha(z))\cdot\psi(k)\\ &= \psi'(\nu'(z))\cdot\psi'(k')\cdot\eta(y,y')\\ &= (\psi'\omega')\circ(\eta\mathrm{pr}_1)(m).\end{aligned}$$

It follows that $\big(\mu(\alpha'\circ\mathrm{pr}_2)\big)\circ(\psi\omega)=(\psi'\omega')\circ(\eta\mathrm{pr}_1)$.

Since $\varphi\circ\mathrm{pr}_1$ and its composition with $\alpha$ are weak equivalences, so are $\mathrm{pr}_1,\alpha'\circ\mathrm{pr}_2\colon\mathcal{M}\to\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}'$, with $(\mathrm{pr}_1)_0$ a surjective submersion. Thus the generalised morphism that we require is

$$\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}' \xleftarrow[\mathrm{pr}_1]{\simeq} \mathcal{M} \xrightarrow[\alpha'\circ\mathrm{pr}_3]{\simeq} \mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}'$$

where $\beta=\mathrm{pr}_1$. □

We now prove uniqueness of the 2-cell.

**Lemma 53.** *If the two diagrams below are in the same equivalence class,*

$$\begin{tikzcd} & \mathcal{K} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] & \\ \mathcal{G} \arrow[r, phantom, "\circlearrowright"] & \mathcal{K}_\varphi \times_{\varphi'} \mathcal{K}' \arrow[u, "\mathrm{pr}_1", "\simeq"'] \arrow[d, "\mathrm{pr}_2"', "\simeq"] \arrow[r, phantom, "\Downarrow \eta, \quad \Downarrow \eta'"] & \mathcal{H} \\ & \mathcal{K}', \arrow[ul, "\varphi'", "\simeq"'] \arrow[ur, "\psi'"'] & \end{tikzcd}$$

*then* $\eta = \eta'$.

*Proof* It suffices to prove that there exists a $\gamma \in \mathsf{sW}$ such that $\eta\gamma = \eta'\gamma$, since then Lemma 17 will imply that $\eta = \eta'$, and thus that the two 2-cells are equal in **AnaActGpd**.

Since the two equivalences are in the same equivalence class, by Lemma 52, there exists a generalised morphism

$$\mathcal{K}_\varphi \times_{\varphi'} \mathcal{K}' \xleftarrow[\gamma]{\simeq} \mathcal{L} \xrightarrow[\gamma']{\simeq} \mathcal{K}_\varphi \times_{\varphi'} \mathcal{K}'$$

and natural transformations

$$\mu \colon \mathrm{pr}_1 \circ \gamma \Rightarrow \mathrm{pr}_1 \circ \gamma' \quad \text{and} \quad \mu' \colon \mathrm{pr}_2 \circ \gamma \Rightarrow \mathrm{pr}_2 \circ \gamma'$$

inducing the equivalence relation between them, in which $\gamma$ is a surjective submersive weak equivalence. By Lemma 17, we can factor and obtain there exist $\nu, \nu' \colon \gamma \Rightarrow \gamma'$ such that

$$\mu = \mathrm{pr}_1 \nu \quad \text{and} \quad \mu' = \mathrm{pr}_2 \nu'.$$

But by the first equation of (2)

$$(\mathrm{ID}_{\varphi \circ \mathrm{pr}_1} \gamma') \circ (\varphi\mu) = (\varphi'\mu') \circ (\mathrm{ID}_{\varphi \circ \mathrm{pr}_1} \gamma),$$

the fact that $\varphi \circ \mathrm{pr}_1 = \varphi' \circ \mathrm{pr}_2$ is a surjective submersive weak equivalence, and Lemma 17 applied to $(\varphi \circ \mathrm{pr}_1)\nu = (\varphi' \circ \mathrm{pr}_2)\nu'$, we have by uniqueness that $\nu = \nu'$. Since the second equation of (2) is

$$(\eta'\gamma') \circ \big((\psi \circ \mathrm{pr}_1)\nu\big) = \big((\psi' \circ \mathrm{pr}_2)\nu\big) \circ (\eta\gamma)$$

and by a standard coherence relation of a bicategory called the "middle four exchange" (see [13, (2.1.9)]), we have

$$(\eta'\gamma') \circ \big((\psi \circ \mathrm{pr}_1)\nu\big) = \big((\psi' \circ \mathrm{pr}_2)\nu\big) \circ (\eta'\gamma);$$

we combine these equalities yielding

$$\big((\psi' \circ \mathrm{pr}_2)\nu\big) \circ (\eta'\gamma) = \big((\psi' \circ \mathrm{pr}_2)\nu\big) \circ (\eta\gamma).$$

Since natural transformations are invertible in our setting, we conclude that there is a $\gamma \in \mathsf{sW}$ such that $\eta\gamma = \eta'\gamma$ as required. □

Lastly, we will verify that the inclusion $I_{\mathcal{P}}$ is a pseudofunctor and respects the operations in the localised bicategories. To prove this, we need to consider vertical compositions, horizontal compositions, and unitors in our bicategories; details and notation can be found in Appendix A.

**Proposition 54.** *The assignment* $I_{\mathcal{P}}\colon \mathbf{AnaActGpd}_{\mathcal{P}} \to \mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ *is a pseudofunctor.*

*Proof* To begin, we must show that for each pair of action groupoids $\mathcal{G} := G \ltimes X$ and $\mathcal{H} := H \ltimes Y$, $I_{\mathcal{P}}$ induces a functor $\mathbf{AnaLieGpoid}_{\mathcal{P}}(\mathcal{G},\mathcal{H}) \to \mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}(\mathcal{G},\mathcal{H})$ between the categories of 1-cells between $\mathcal{G}$ and $\mathcal{H}$, with 2-cells between those. In particular, $I_{\mathcal{P}}$ must respect vertical composition and unit 2-cells.

Suppose the first two diagrams in Definition 62 are in fact in $\mathbf{AnaLieGpoid}_{\mathcal{P}}$, and so $\varphi$, $\varphi'$, and $\varphi''$ are surjective submersive, $\mathcal{L}_1 = \mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}'$, $\mathcal{L}_2 = \mathcal{K}'{}_{\varphi'}\times_{\varphi''}\mathcal{K}''$, the vertical maps from $\mathcal{L}_i$ are projection maps, and $\mu_1$ and $\mu_2$ are trivial. We need to show that the following two diagrams are in the same equivalence class:

$$\begin{tikzcd}
 & & \mathcal{K} \arrow[dll, two heads, "\varphi", "\simeq"'] \arrow[drr, "\psi"] & & \\
\mathcal{G} \arrow[r, phantom, "\Downarrow \varphi'\mathrm{pr}_2"] & & (\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}')_{\mathrm{pr}_2}\overset{\mathrm{w}}{\times}_{\mathrm{pr}_1}(\mathcal{K}'{}_{\varphi'}\times_{\varphi''}\mathcal{K}'') \arrow[u, two heads, "\mathrm{pr}_1\circ\mathrm{pr}_1", "\simeq"'] \arrow[d, two heads, "\mathrm{pr}_3\circ\mathrm{pr}_2"', "\simeq"] & \arrow[r, phantom, "\Downarrow \kappa"] & \mathcal{H} \\
 & & \mathcal{K}'' \arrow[ull, two heads, "\varphi''"', "\simeq"] \arrow[urr, "\psi''"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathcal{K} \arrow[dll, two heads, "\varphi", "\simeq"'] \arrow[drr, "\psi"] & & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowright"] & & \mathcal{K}_{\varphi}\times_{\varphi''}\mathcal{K}'' \arrow[u, two heads, "\mathrm{pr}_1", "\simeq"'] \arrow[d, two heads, "\mathrm{pr}_2"', "\simeq"] & \arrow[r, phantom, "\Downarrow \lambda"] & \mathcal{H} \\
 & & \mathcal{K}'', \arrow[ull, two heads, "\varphi''"', "\simeq"] \arrow[urr, "\psi''"'] & &
\end{tikzcd}$$

where $\kappa$ is the natural transformation $\nu$ of Definition 62 and where $\lambda$ is the natural transformation described in Definition 65. To accomplish this, we will apply Lemma 51 to the first diagram above, and show that the resulting natural transformation on the right (the $\nu$ of the lemma) is equal to $\lambda$. By definition, $\lambda$ is the unique natural transformation such that $\lambda\mathrm{pr}_{13} = (\eta_2\mathrm{pr}_{23}) * (\eta_1\mathrm{pr}_{12})$ where $\mathrm{pr}_{ij} = (\mathrm{pr}_i, \mathrm{pr}_j)$ is the projection of $\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}'{}_{\varphi'}\times_{\varphi''}\mathcal{K}''$ and $*$ denotes horizontal composition of natural transformations. Thus it suffices to show for a fixed $(y, y', y'') \in \mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}'{}_{\varphi'}\times_{\varphi''}\mathcal{K}''$ that $\lambda\mathrm{pr}_{13}(y, y', y'') = \nu\mathrm{pr}_{13}(y, y', y'')$, where again $\nu$ is the $\nu$ of Lemma 51. Unravelling Definition 62 and the definition of $\nu$ in Lemma 51, we obtain

$$\nu(y, y'') = \psi''(\mu^{-1}((\widetilde{y}'', y''), k, z))\eta_2(\widetilde{y}'_2, \widetilde{y}'')\psi'(\widetilde{k}')\eta_1(\widetilde{y}, \widetilde{y}')\psi(k)$$

where $\mu$ is as defined in the proof of Lemma 51, $((y, y''), k, z) \in (\mathcal{K}_{\varphi}\times_{\varphi''}\mathcal{K}'')_{\mathrm{pr}_1}\overset{\mathrm{w}}{\times}_{\mathrm{pr}_1\circ\mathrm{pr}_1}\mathcal{L}$, and

$$z := ((\widetilde{y}, \widetilde{y}'_1), \widetilde{k}', (\widetilde{y}'_2, \widetilde{y}'')) \in \mathcal{L} := (\mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}')_{\mathrm{pr}_2}\overset{\mathrm{w}}{\times}_{\mathrm{pr}_1}(\mathcal{K}'{}_{\varphi'}\times_{\varphi''}\mathcal{K}'').$$

The result is independent of choice of (admissible) $k$ and $z$, and so we choose $z = ((y, y'), u_{y'}, (y', y''))$ and $k = u_y$, after which we obtain

$$\nu(y, y'') = \eta_2(y', y'')\eta_1(y, y') = \lambda(y, y'').$$

Thus $I_{\mathcal{P}}$ preserves vertical composition.

Fix a $\mathcal{P}$-anafunctor $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$, and let $\Delta\colon \mathcal{K} \to \mathcal{K}_{\varphi}\times_{\varphi}\mathcal{K}$ be the diagonal map. Then $\mathrm{pr}_1 \circ \Delta = \mathrm{pr}_2 \circ \Delta$, and $\iota_{\mathcal{G}\leftarrow\mathcal{K}\to\mathcal{H}}\Delta$ is trivial. Via $\mathcal{K}_{\varphi}\times_{\varphi}\mathcal{K} \xleftarrow[\Delta]{\simeq} \mathcal{K} \xrightarrow{=} \mathcal{K}$ it follows that the identity 2-cell of $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ in $\mathbf{AnaLieGpoid}_{\mathcal{P}}$ is equivalent to the identity 2-cell in $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$, and we conclude that $I_{\mathcal{P}}$ induces a functor $\mathbf{AnaLieGpoid}_{\mathcal{P}}(\mathcal{G}, \mathcal{H}) \to \mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}(\mathcal{G}, \mathcal{H})$.

Since the identity generalised morphism of a Lie groupoid $\mathcal{G}$ is the same as the identity anafunctor, $I_{\mathcal{P}}$ trivially preserves identity 1-cells.

By Proposition 31 for each pair of $\mathcal{P}$-anafunctors $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ and $\mathcal{H} \xleftarrow[\chi]{\simeq} \mathcal{L} \xrightarrow[\omega]{} \mathcal{I}$ there is a 2-cell in $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ from the composition as generalised morphisms to the composition as anafunctors, represented by $(\mathrm{inc}, \mathrm{id}_{\mathcal{K}_{\psi}\times_{\chi}\mathcal{L}}, \mathrm{ID}_{\varphi\circ\mathrm{pr}_1\circ\mathrm{inc}}, \mathrm{ID}_{\mathrm{pr}_3\circ\omega\circ\mathrm{inc}})$. We now check the first of three coherence conditions (namely, [13, (4.1.3)] or (M.1) of [4, page 30]), which indicates that the various compositions of $\mathcal{P}$-anafunctors yields equivalent results. Fix three $\mathcal{P}$-anafunctors

$$\begin{tikzcd}
& \mathcal{M} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] & & \mathcal{N} \arrow[dl, "\chi"', "\simeq"] \arrow[dr, "\omega"] & & \mathcal{P} \arrow[dl, "\xi"', "\simeq"] \arrow[dr, "\zeta"] & \\
\mathcal{G} & & \mathcal{H} & & \mathcal{K} & & \mathcal{L}
\end{tikzcd}$$

Then the first coherence condition reduces to showing that the vertical composition of the 2-cells induced by the inclusions $\mathcal{M}_{\psi}\times_{\chi}\mathcal{N}_{\omega}\times_{\xi}\mathcal{P} \to \mathcal{M}_{\psi}\times_{\chi}\mathcal{N}_{\omega}\overset{\mathrm{w}}{\times}_{\xi}\mathcal{P}$ and $\mathcal{M}_{\psi}\times_{\chi}\mathcal{N}_{\omega}\overset{\mathrm{w}}{\times}_{\xi}\mathcal{P} \to \mathcal{M}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{N}_{\omega}\overset{\mathrm{w}}{\times}_{\xi}\mathcal{P}$ is equal to the vertical composition of the 2-cells induced by the inclusions $\mathcal{M}_{\psi}\times_{\chi}\mathcal{N}_{\omega}\times_{\xi}\mathcal{P} \to \mathcal{M}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{N}_{\omega}\times_{\xi}\mathcal{P}$ and $\mathcal{M}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{N}_{\omega}\times_{\xi}\mathcal{P} \to$

$\mathcal{M}_\psi \overset{\mathrm{w}}{\times}_\chi \mathcal{N}_\omega \overset{\mathrm{w}}{\times}_\xi \mathcal{P}$. These two 2-cells are represented by the diagrams, resp.,

$$
\begin{tikzcd}
 & \mathcal{M}_\psi \overset{\mathrm{w}}{\times}_\chi \mathcal{N}_\omega \overset{\mathrm{w}}{\times}_\xi \mathcal{P} \arrow[dl, two heads, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\zeta\circ\mathrm{pr}_5"] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowright"] & \mathcal{Q}_1 \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{L} \\
 & \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \arrow[ul, two heads, "\varphi\circ\mathrm{pr}_1", "\simeq"'] \arrow[ur, "\zeta\circ\mathrm{pr}_4"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathcal{M}_\psi \overset{\mathrm{w}}{\times}_\chi \mathcal{N}_\omega \overset{\mathrm{w}}{\times}_\xi \mathcal{P} \arrow[dl, two heads, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\zeta\circ\mathrm{pr}_5"] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowright"] & \mathcal{Q}_2 \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\circlearrowright"] & \mathcal{L} \\
 & \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \arrow[ul, two heads, "\varphi\circ\mathrm{pr}_1", "\simeq"'] \arrow[ur, "\zeta\circ\mathrm{pr}_3"'] &
\end{tikzcd}
$$

where

$$\mathcal{Q}_1 := (\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \overset{\mathrm{w}}{\times}_\xi \mathcal{P})_{\mathrm{id}_{\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \overset{\mathrm{w}}{\times}_\xi \mathcal{P}}} \overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{M}} \overset{\mathrm{w}}{\times} \mathrm{inc}} (\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P})$$

and

$$\mathcal{Q}_2 := (\mathcal{M}_\psi \overset{\mathrm{w}}{\times}_\chi \mathcal{N}_\omega \times_\xi \mathcal{P})_{\mathrm{id}_{\mathcal{M}_\psi \overset{\mathrm{w}}{\times}_\chi \mathcal{N}_\omega \times_\xi \mathcal{P}}} \overset{\mathrm{w}}{\times}_{\mathrm{inc} \overset{\mathrm{w}}{\times} \mathrm{id}_{\mathcal{P}}} (\mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P}),$$

(note that we have suppressed some of the notation). The equivalence is established by the quadruple $(j_1, j_2, \nu_1, \nu_2)$ with $\nu_1$ and $\nu_2$ trivial and where the generalised morphism

$$\mathcal{Q}_1 \xleftarrow[j_1]{\simeq} \mathcal{M}_\psi \times_\chi \mathcal{N}_\omega \times_\xi \mathcal{P} \xrightarrow[j_2]{\simeq} \mathcal{Q}_2$$

is defined by

$$j_1(m,n,p) = ((m,n,u_{\omega(s(n))},p), u_{\mathrm{id}_{\mathcal{M}} \overset{\mathrm{w}}{\times} \mathrm{inc}(s(m,n,p))}, (m,n,p)) \quad \text{and}$$
$$j_2(m,n,p) = ((m,u_{\chi(s(n))},n,p), u_{\mathrm{inc} \overset{\mathrm{w}}{\times} \mathrm{id}_{\mathcal{P}}(s(m,n,p))}, (m,n,p)).$$

Indeed, the natural transformations of Equations 2 all reduce to trivial ones.

Finally, we check that $I_{\mathcal{P}}$ respects the unitors from each bicategory. For right unitors, the relevant coherence condition (see [13, (4.1.4)] or (M.2) of [4, page 30]) reduces to checking that the following two diagrams represent the same 2-cell.

$$\begin{tikzcd}
 & \mathcal{K}{}_{\psi}\overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}}\mathcal{H} \arrow[dl, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[d, equal] \arrow[dr, "\mathrm{pr}_3"] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowright"] & \mathcal{K}{}_{\psi}\overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}}\mathcal{H} \arrow[d, "\mathrm{pr}_1"', "\simeq"] \arrow[r, phantom, "\Uparrow \mathrm{PR}_2"] & \mathcal{H} \\
 & \mathcal{K} \arrow[ul, "\varphi", "\simeq"'] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{K}{}_{\psi}\overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}}\mathcal{H} \arrow[dl, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\mathrm{pr}_3"] & \\
\mathcal{G} \arrow[r, phantom, "\Downarrow (\varphi\circ\mathrm{pr}_1)\mathrm{PR}_2"] & \mathcal{L} \arrow[u, "\mathrm{inc}\circ\mathrm{pr}_1", "\simeq"'] \arrow[d, "\mathrm{pr}_2\circ\mathrm{pr}_3"', "\simeq"] \arrow[r, phantom, "\Downarrow \rho"] & \mathcal{H} \\
 & \mathcal{K} \arrow[ul, "\varphi", "\simeq"'] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

where

$$\mathcal{L} := (\mathcal{K}{}_{\psi}\times_{\mathrm{id}_{\mathcal{H}}}\mathcal{H})_{\mathrm{id}_{\mathcal{K}\times\mathcal{H}}}\overset{\mathrm{w}}{\times}_{\mathrm{pr}_1}((\mathcal{K}{}_{\psi}\times_{\mathrm{id}_{\mathcal{H}}}\mathcal{H})_{\varphi\circ\mathrm{pr}_1}\times_{\varphi}\mathcal{K})$$

and

$$\rho = (\rho^{\mathrm{ana}}_{\mathcal{G},\mathcal{H}}(\mathcal{G}\leftarrow\mathcal{K}\rightarrow\mathcal{H})\mathrm{pr}_3)\circ(\mathrm{pr}_3\mathrm{PR}_2).$$

The equivalence is given by the quadruple $(j_1, j_2, \nu_1, \nu_2)$ where $j_1\colon \mathcal{K} \to \mathcal{K}{}_{\psi}\overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}}\mathcal{H}$ sends $k \in \mathcal{K}_1$ to $(k, u_{s(\psi(k))}, \psi(k))$ and $j_2\colon \mathcal{K} \to \mathcal{L}$ $k$ to $((k,\psi(k)), u_{s(k,u_{\psi(s(k))},\psi(k))}, ((k,\psi(k)),k))$, and both $\nu_1$ and $\nu_2$ are trivial. Again, the natural transformations of Equations 2 are all trivial. The computation for left unitors using the corresponding coherence condition (M.2) on page 30 of [4] is similar. □

Combining the results above yields the desired equivalence of bicategories.

**Theorem 55.** *The pseudofunctor* $I_{\mathcal{P}}\colon \mathbf{AnaActGpd}_{\mathcal{P}} \to \mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ *is an equivalence of bicategories. Consequently,* $\mathbf{AnaActGpd}_{\mathcal{P}}$*,* $\mathbf{ActGpd}_{\mathcal{P}}[W_{\mathcal{P}}^{-1}]$ *and* $\mathbf{LieGpoid}[W^{-1}]_{\mathcal{P}}$ *are all equivalent bicategories.*

*Proof* By Proposition 54, $I_{\mathcal{P}}$ is a pseudofunctor. Since $I_{\mathcal{P}}$ is surjective on objects, it suffices to show that for two action groupoids $\mathcal{G} = G \ltimes X$ and $\mathcal{H} = H \ltimes Y$ satisfying $\mathcal{P}$, the restriction of $I_{\mathcal{P}}$ to the category of $\mathcal{P}$-anafunctors from $\mathcal{G}$ to $\mathcal{H}$, which maps into the category of *all* generalised morphisms between them, is an equivalence of categories. Essential surjectivity follows from Proposition 50, and fully faithfulness follows from Lemmas 51 and 53. □

*Remark 56* One of the properties considered in $\mathcal{P}$ was being an orbifold groupoid, *i.e.* being Morita equivalent to a proper étale groupoid. In fact, the proofs for this property equally apply to groupoids which are Morita equivalent to any chosen class of groupoids: the only thing we used was the Morita equivalence, not properness or étale property. Thus one could replace our definition of orbifold groupoid with groupoids Morita equivalent to proper effective étale groupoids, as defined by Moerdijk-Mrčun in [17, page 136], or indeed groupoids Morita equivalent to those with any desired properties and the results from this section would still apply.

# 7 Application: Decomposition of Equivariant Weak Equivalences

In this section, we continue to use any collection of properties $\mathcal{P}$ chosen from those in (3), excluding effectiveness. We prove that any equivariant weak equivalence in $W_{\mathcal{P}}$ decomposes into two fundamental forms of equivariant weak equivalences: a projection and an inclusion. This decomposition was originally observed in [22] in the proper étale case, and subsequently applied in a different context in [2]. Here we show that this decomposition is compatible with any of our properties in $\mathcal{P}$.

Our goal for this section is to show that any equivariant weak equivalence in $W_{\mathcal{P}}$ can be written as the composition of maps of the following form:

- a projection $G \ltimes X \to {}_K\backslash^G \ltimes {}_K\backslash^X$ where $K$ is a closed normal subgroup of $G$ which acts freely on $X$,
- an inclusion $K \ltimes X \to [G \ltimes (G \times_K X)]$ where $K$ is a closed subgroup of $G$.

Moreover, the projection and inclusion maps are equivariant weak equivalences, and satisfy our chosen properties $\mathcal{P}$ if the original weak equivalence did.

We will first consider each type of map separately, beginning with the projection. For this result, we must exclude effective from our possible properties.

**Lemma 57.** *Given an action groupoid $G \ltimes X$ satisfying any subset of properties of $\mathcal{P}$ excluding "effective", and a closed normal subgroup $K \trianglelefteq G$ for which the restricted $K$-action on $X$ is free and proper, the quotient map $\pi\colon G \ltimes X \to {}_K\backslash^G \ltimes {}_K\backslash^X$ is an equivariant surjective submersive weak equivalence, and ${}_K\backslash^G \ltimes {}_K\backslash^X$ satisfies $\mathcal{P}$.*

*Proof* By [8, Corollary 1.10.7], $K$ is a Lie subgroup of $G$ . Since the action of $K$ on $X$ is free and proper, by [8, Theorem 1.11.4, Corollary 1.11.5], ${}_K\backslash^X$ is a manifold and the action of the Lie group ${}_K\backslash^G$ on ${}_K\backslash^X$ is well-defined and smooth. Define $\pi_0\colon X \to {}_K\backslash^X$ to be the (surjective submersive) quotient map by the $K$-action, and $\pi_1\colon (G \ltimes X)_1 \to \left({}_K\backslash^G \ltimes {}_K\backslash^X\right)_1$ to send $(g, x)$ to $(Kg, \pi_0(x))$. Then $\pi := (\pi_0, \pi_1)$

is an equivariant functor with respect to the homomorphism $\widetilde{\pi}\colon G \to {}_K\backslash G$. Denote by $[x]_K$ the image $\pi_0(x)$ for any $x \in X$.

To show that $\pi$ is a weak equivalence, we consider the map $\mathbf{FF}_\pi$ and show that it is injective. Suppose that $\mathbf{FF}_\pi(g,x) = \mathbf{FF}_\pi(g',x')$, so $x = x'$ and $g = kg$ for some $k \in K$. But since $gx = g'x' = g'x = kgx$ and the action of $K$ is free, we know that $k = e$ and so $g = g'$. Thus $\mathbf{FF}_\pi$ is injective. Surjectivity is immediate and thus $\mathbf{FF}_\pi$ bijective.

Now we verify that $\mathbf{FF}_\pi$ is a diffeomorphism. Suppose that $p = (x_\tau, x'_\tau, (Kg_\tau, [x_\tau]_K))\colon I \to X^2 {}_{\pi_0^2}\!\times_{(s,t)} ({}_K\backslash G \times {}_K\backslash X)$ is a curve. Since $G \to {}_K\backslash G$ is a principal $K$-bundle, after shrinking $I$, there is a lift $\widetilde{g}_\tau$ of $Kg_\tau$ to $G$; this satisfies $\pi_0(x'_\tau) = \pi_0\left(\widetilde{g}_\tau x_\tau\right)$. Since $X \to {}_K\backslash X$ is a principal $K$-bundle, there is a curve $k_\tau$ in $K$ such that $k_\tau \widetilde{g}_\tau x_\tau = x'_\tau$. Then $\mathbf{FF}_\pi(k_\tau \widetilde{g}_\tau, x_\tau) = p$. By Item 2 of Lemma 4, $\mathbf{FF}_\pi$ is a diffeomorphism. Since $\pi_0$ is a surjective submersion, by Lemma 9, we conclude that $\pi$ is an equivariant surjective submersive weak equivalence.

It is immediate that if $G$ is discrete, compact, or the $G$-action is transitive, then so is the ${}_K\backslash G$-action. Since for any $x \in X$, the stabiliser $\mathrm{Stab}_{{}_K\backslash G}([x]_K) = \widetilde{\pi}(\mathrm{Stab}_G(x))$, it is immediate that if the $G$-action is free/locally free, so is the ${}_K\backslash G$-action.

Suppose the $G$-action is proper. Fix a sequence $(Kg_i, [x_i]_K) \in {}_K\backslash G \times {}_K\backslash X$ so that $([x_i]_K, [g_i x_i]_K)$ converges to $([x]_K, [x']_K)$ in $\left({}_K\backslash X\right)^2$. Since $X \to {}_K\backslash X$ is a principal $K$-bundle, there exist sequences $(k_i)$ and $(k'_i)$ in $K$ such that $k_i x_i \to x$ and $k'_i g_i x_i \to x'$. Since the $G$-action is proper and the sequence $(k_i x_i, k'_i g_i x_i) \to (x, x')$ in $X^2$, the sequence $(k'_i g_i k_i^{-1}, k_i x_i) \to (g, x)$ in $G \times X$ for some $g \in G$. By continuity, $(Kg_i, [x_i]_K) \to (Kg, [x]_K)$ in ${}_K\backslash G \times {}_K\backslash X$. Thus the ${}_K\backslash G$-action is proper.

Finally, if $G \ltimes X$ is an orbifold groupoid, then there is a proper étale Lie groupoid $\mathcal{E}$, and a Morita equivalence $G \ltimes X \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{\simeq} \mathcal{E}$. Composing $\varphi$ with $\pi$ yields a Morita equivalence between ${}_K\backslash G \ltimes {}_K\backslash X$ and $\mathcal{E}$, and so ${}_K\backslash G \ltimes {}_K\backslash X$ is an orbifold groupoid. $\square$

Next we consider our second type of equivariant equivalence, the inclusion $X \to G \times_K X$. We use the standard notation $G \times_K X$ for the $G$-space ${}_K\backslash(G \times X)$, where $K$ acts on $G \times X$ anti-diagonally: $k \cdot (g, x) := (gk^{-1}, kx)$.

**Lemma 58.** *Given a closed subgroup $K \leq G$ and an action groupoid $K \ltimes X$, the inclusions $i_K\colon K \to G$ and $i_X\colon X \to G \times_K X\colon x \mapsto [1_G, x]$ induce an equivariant weak equivalence $i\colon K \ltimes X \to G \ltimes (G \times_K X)$. Moreover, if $K \ltimes X$ is free, locally free, effective, proper, or is an orbifold groupoid; then so is $G \ltimes (G \times_K X)$. In the other direction, if $G \ltimes (G \times_K X)$ is free, locally free, transitive, proper, an orbifold groupoid, or $G$ is compact or discrete; then $K \ltimes X$ has the same property.*

*Proof* Since the anti-diagonal action of $K$ on $G \times X$ is free and proper, $G \times_K X$ is a manifold. It comes equipped with a $G$-action $g \cdot [g', x] := [gg', x]$. Define $i_0 = i_X$ and $i_1 = (i_K, i_X)$. Then $i\colon K \ltimes X \to G \ltimes (G \times_K X)$ is an equivariant functor with respect to the homomorphism $i_K$.

Fix a point $[g, x] \in G \times_K X$. Then $\mathbf{ES}_i(x, (g^{-1}, [g, x])) = [g, x]$, from which it follows that $\mathbf{ES}_i$ is surjective.

Fix a curve $p = [g_\tau, x_\tau]\colon I \to G \times_K X$ and $(x', (g', [g, x])) \in X_{i_X}\times_t (G \times (G \times_K X))$ such that $[g, x] = [g_0, x_0] = p(0)$. Then $[g'g, x] = [1, x']$ and so there exists $k \in K$ such that $(g'gk^{-1}, kx) = (1, x')$. So $g'g = k \in K$ and $[(g')^{-1}, x'] = (g')^{-1}[1, x'] = (g')^{-1}[g'g, x] = [g, x]$. Since $G \times X \to G \times_K X$ is a principal $K$-bundle, after shrinking $I$, there is a lift $(\widetilde{g}_\tau, \widetilde{x}_\tau)$ of $p$ through $((g')^{-1}, x') \in G \times X$. Then $\left(\widetilde{x}_\tau, \left(\widetilde{g}_\tau^{-1}, [\widetilde{g}_\tau, \widetilde{x}_\tau]\right)\right)$ is a lift of $p$ through $(x', (g', [g, x]))$. Thus $\mathbf{ES}_i$ is a surjective submersion.

Fix $(x, x', (g', [g, x''])) \in G \times (G \times_K X)$. There exists a unique $k \in K$ such that $(1_G, x) = (gk^{-1}, kx'')$; that is, $g = k$ and $x'' = k^{-1}x$. Similarly, there exists a unique $k' \in K$ such that $(1_G, x') = (g'g(k')^{-1}, k'x'')$; that is, $g' = k'k^{-1}$ and $x' = k'k^{-1}x$. Hence $(x, x', (g', [g, x''])) = \mathbf{FF}_i(k'k^{-1}, x)$, so $\mathbf{FF}_i$ is surjective; and the uniqueness of the choices show that $\mathbf{FF}_i$ is injective, and hence bijective.

Let $p = (x_\tau, x'_\tau, (g'_\tau, [g_\tau, x_\tau]))\colon I \to G \times (G \times_K X)$ be a curve. Then the fact that $G \times X \to G \times_K X$ is a principal $K$-bundle allows us to lift to $[g_\tau, x_\tau]$ to a path $(g_\tau, x_\tau) \in G \times X$, and we see that $g'_\tau$ is a curve in $K$ and $\mathbf{FF}_i(g'_\tau, x_\tau)$ is a lift of $p$ to $K \times X$. Thus $\mathbf{FF}_i$ is a surjective submersion, and hence a diffeomorphism. We conclude that $i$ is an equivariant weak equivalence.

It is straightforward to verify that $\mathrm{Stab}_G([g, x]) = g\mathrm{Stab}_K(x)g^{-1}$. Thus the $K$-action on $X$ is free (resp. locally free) if and only if the $G$-action on $G \times_K X$ is free (resp. locally free), and the $G$-action is effective if the $K$-action is. If $G$ is compact or discrete, then so is $K$, then the corresponding action on $G \times_K X$ is as well.

Now suppose the $K$-action is proper. Let $(g_i, [g'_i, x_i])$ be a sequence in $G \times (G \times_K X)$ such that $([g'_i, x_i], [g_i g'_i, x_i])$ converges to $([g', x], [g'', x'])$ in $(G \times_K X)^2$. Since $G \times X \to G \times_K X$ is a principal $K$-bundle, there exist sequences $(k_i)$ and $(k'_i)$ in $K$ such that $(g'_i k_i^{-1}, k_i x_i) \to (g', x)$ and $(g_i g'_i (k'_i)^{-1}, k'_i x_i) \to (g'', x')$. In particular, $(k_i x_i, k'_i k_i^{-1} k_i x_i) \to (x, x')$ in $X^2$. Since the $K$-action is proper, there exists $\widehat{k} \in K$ such that $(k'_i k_i^{-1}, k_i x_i) \to (\widehat{k}, x)$. But then $g'_i k_i^{-1} k_i (k'_i)^{-1} \to g' \widehat{k}^{-1}$, and consequently $g_i \to g'' \widehat{k} (g')^{-1}$, in $G$. Thus $(g_i, [g'_i, x_i]) \to (g'' \widehat{k} (g')^{-1}, [g', x])$ in $G \times (G \times_K X)$. Thus the $G$-action on $G \times_K X$ is proper.

Suppose the $G$-action on $G \times_K X$ is transitive. Fix $x, x' \in X$. There exists $g \in G$ such that $g \cdot [1_G, x] = [1_G, x']$. In particular, there is a $k \in K$ so that $(gk^{-1}, kx) = (1_G, x')$; that is, $x' = kx$. Thus the $K$-action on $X$ is transitive.

Finally, since $i$ is a weak equivalence between $K \ltimes X$ and $G \ltimes (G \times_K X)$, if one is an orbifold groupoid, so is the other. □

We now combine the two previous lemmas into the following decomposition theorem.

**Theorem 59 (Decomposition of Equivariant Weak Equivalences).** *Let $G \ltimes X$ and $H \ltimes Y$ satisfy properties $\mathcal{P}$ (except for "effective"), and let $\varphi\colon G \ltimes X \to H \ltimes Y$ be an equivariant weak equivalence induced by a proper homomorphism $\widetilde{\varphi}\colon G \to H$. Then $\varphi$ factors as $i \circ \pi$ where $\pi$ and $i$ are equivariant weak equivalences in* $\mathbf{ActGpd}_{\mathcal{P}}$ *of the forms as in Lemmas 57 and 58, resp.*

*Proof* **Claim 1:** ${}_{\ker(\widetilde{\varphi})}\backslash^{G}$ is a Lie group isomorphic to $\Rightarrow (\widetilde{\varphi})$.
Proof of Claim 1: ${}_{\ker(\widetilde{\varphi})}\backslash^{G}$ is a Lie group [8, Corollary 1.11.5 and Proposition 1.11.8], and $\widetilde{\varphi}$ descends to a smooth bijective homomorphism $\widehat{\varphi}\colon {}_{\ker(\widetilde{\varphi})}\backslash^{G} \to \Rightarrow (\widetilde{\varphi})$. Since $\widetilde{\varphi}$ is proper, it is closed, and so $\Rightarrow (\widetilde{\varphi})$ is a closed subgroup of $H$, and hence a Lie subgroup [8, Corollary 1.10.7]. Since $\widetilde{\varphi}$ is proper, so is $\widehat{\varphi}$, and so it is a homeomorphism; *i.e.* its inverse is a continuous homomorphism. By [8, Corollary 1.10.9], continuous homomorphisms are smooth, and so $\widehat{\varphi}$ is an isomorphism of Lie groups; Claim 1 is proved.

**Claim 2:** $\varphi_0(X)$ and ${}_{\ker(\widetilde{\varphi})}\backslash^{X}$ are diffeomorphic manifolds.
Proof of Claim 2: It follows from the equivariance of $\varphi$ and the injectivity of $\mathbf{FF}_{\varphi}$ that $\ker(\widetilde{\varphi})$ acts freely on $X$. Since $\widetilde{\varphi}$ is proper, $\ker(\widetilde{\varphi})$ is a compact submanifold of $G$. Thus ${}_{\ker(\widetilde{\varphi})}\backslash^{X}$ is a manifold [8, Theorem 1.11.4]. Since $\varphi_0$ is $\ker(\widetilde{\varphi})$-invariant, it descends to a smooth surjection $\psi\colon {}_{\ker(\widetilde{\varphi})}\backslash^{X} \to \varphi_0(X)$. Now suppose $x, x' \in X$ such that $\psi([x]) = \psi([x'])$. Then $\varphi_0(x) = \varphi_0(x')$, and since $\mathbf{FF}_{\varphi}$ is a diffeomorphism, there exists a (unique) $k \in \ker(\widetilde{\varphi})$ such that $x = k \cdot x'$. It follows that $\psi$ is a smooth bijection.

Fix a curve $p = y_\tau\colon I \to \varphi_0(X)$. Shrinking $I$, since $\mathbf{ES}_{\varphi}$ is surjective submersive, there is a lift $q = (x_\tau, (h_\tau, y_\tau))\colon I \to X_{\varphi_0}\times_t (H \times Y)$ of $p$. By the smooth fully faithfulness of $\varphi$, the curve $h_\tau$ is contained in $\Rightarrow (\widetilde{\varphi})$. By Claim 1, we identify $\Rightarrow (\widetilde{\varphi})$ with ${}_{\ker\varphi}\backslash^{G}$, and since $G \to {}_{\ker(\varphi)}\backslash^{G}$ is a principal $(\ker(\varphi))$-bundle, after shrinking $I$ again, there is a lift $g_t$ of $h_t$ to $G$. But then $y_t = \psi\left([s(g_t)]_{\ker(\varphi)}\right)$, which proves that $\psi$ is a diffeomorphism. This proves Claim 2.

By Claim 2 and Lemma 57, $\pi := (\widetilde{\varphi}, \varphi_0)\colon G \ltimes X \to {}_{\ker(\varphi)}\backslash^{G} \ltimes \varphi_0(X)$ is an equivariant weak equivalence. It is straightforward to check that $(\widehat{\varphi}, \psi)$ is an isomorphism of Lie groupoids between ${}_{K}\backslash^{G} \ltimes {}_{K}\backslash^{X}$ and $\Rightarrow (\widetilde{\varphi}) \ltimes \varphi_0(X)$; we identify these. Let $i = (i_{\Rightarrow(\widetilde{\varphi})}, i_{\Rightarrow(\varphi_0)})$, where the two components are inclusions of the images into $H$ and $Y$, resp. By Claim 1, we have the following factorisation

$$G \ltimes X \xrightarrow{\pi} \Rightarrow (\widetilde{\varphi}) \ltimes \varphi_0(X) \xrightarrow{i} H \ltimes Y.$$

To obtain the desired decomposition, by Lemma 58, it suffices to show that $Y$ is $H$-equivariantly diffeomorphic to $H \times_{\Rightarrow(\widetilde{\varphi})} \varphi_0(X)$.

Define $\chi\colon H \times_{\Rightarrow(\widetilde{\varphi})} \varphi_0(X) \to Y$ to be the smooth map given by $\chi([h, \varphi_0(x)]) := h \cdot \varphi_0(x)$. Suppose $\chi([h, \varphi_0(x)]) = \chi([h', \varphi_0(x')])$. Since $\mathbf{FF}_{\varphi}$ is a diffeomorphism, there exists a unique $g \in G$ such that $\mathbf{FF}_{\varphi}(g, x) = (x, x', ((h')^{-1}h, \varphi_0(x)))$, and so $x' = g \cdot x$ and $\widetilde{\varphi}(g) = (h')^{-1}h$. Thus $[h, \varphi_0(x)] = [h', \varphi_0(x')]$, from which it follows

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that $\chi$ is injective. For a fixed $y \in Y$, since $\mathbf{ES}_\varphi$ is surjective, there exists $(x,(h,y)) \in X_\varphi{\times}_t(H \times Y)$ with $(h,y) : y \to \varphi(x)$ and thus $y = h^{-1}\varphi(x) = \chi(h^{-1}, \varphi(x))$. Thus $\chi$ is bijective.

Let $p = y_\tau : I \to Y$ be a curve. Since $\mathbf{ES}_\varphi$ is a surjective submersion, shrinking $I$, there is a lift $q = (x_\tau, (h_\tau, y_\tau))$ of $p$ to $X_{\varphi_0}{\times}_t(H \times Y)$. The curve $[h_\tau^{-1}, \varphi(x_\tau)]$ has image $p$ via $\chi$, and thus $\chi$ is a diffeomorphism. This shows that $\varphi$ decomposes into $i \circ \pi$ as desired.

It remains to show that the domain of $i$ is an action groupoid satisfying $\mathcal{P}$ (except for "effective"). But this follows from the preservation of these properties as stated in Lemmas 57 and 58. □

We now have the following immediate corollary of Proposition 50 and Theorem 59.

**Corollary 60.** *For any subset of properties of $\mathcal{P}$ which does not include "effective", given a generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ between objects $\mathcal{G} := G \ltimes X$ and $\mathcal{H} := H \ltimes Y$ of $\mathbf{ActGpd}_\mathcal{P}$, there is a 2-cell from the generalised morphism to a $\mathcal{P}$-anafunctor $\mathcal{G} \xleftarrow[\chi]{\simeq} \mathcal{L} \xrightarrow[\omega]{} \mathcal{H}$ in which $\chi$ decomposes into functors $\pi$ and $i$ as described in Lemmas 57 and 58.*

Note that if the generalised morphism in the corollary is a Morita equivalence, then $\omega$ also will decompose. We exclude effective actions from this result because they are *not* preserved under equivariant weak equivalences, as the following example shows.

*Example 61* Let $G$ be the four-element dihedral group, with elements $(e,e), (e,\tau), (\tau,e)$ and $(\tau,\tau)$. This acts on the set $X$ consisting of four points laid out in the cardinal directions, $N, S, E, W$: $(\tau, e)$ reflects so that $N$ and $S$ switch and $E, W$ are fixed, and $(e,\tau)$ reflects so that $E$ and $W$ switch and $N, S$ are fixed, and $(\tau,\tau)$ rotates by half a turn and has no fixed points. This is an effective action.

The subgroup $K = \langle(\tau,\tau)\rangle$ acts freely, so we can take the quotient by $K$. Then ${}_K\backslash^{X}$ consists of two points $[N] = [S]$ and $[E] = [W]$. Both the projected points have isotropy $\mathbb{Z}/2 = {}_K\backslash^{G}$, and this action is not effective.

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# Appendix A Bicategorical Details

Here we include the details of unitors and horizontal and vertical composition of 2-cells in the bicategories **LieGpoid**$[W^{-1}]$ and **AnaLieGpoid**.

## A.1 Details of the construction of LieGpoid$[\boldsymbol{W}^{-1}]$

We start by giving an explicit description of 2-cell compositions in **LieGpoid**$[W^{-1}]$.

**Definition 62** (Vertical Composition in **LieGpoid**$[W^{-1}]$) Given representatives of 2-cells as in the following diagrams

$$\begin{tikzcd}
& & \mathcal{K} \arrow[dll, "\varphi"', "\simeq"] \arrow[drr, "\psi"] & & \\
\mathcal{G} & \Downarrow \mu_1 & \mathcal{L}_1 \arrow[u, "\alpha", "\simeq"'] \arrow[d, "\alpha'"', "\simeq"] & \Downarrow \eta_1 & \mathcal{H} \\
& & \mathcal{K}' \arrow[ull, "\varphi'", "\simeq"'] \arrow[urr, "\psi'"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & \mathcal{K}' \arrow[dll, "\varphi'"', "\simeq"] \arrow[drr, "\psi'"] & & \\
\mathcal{G} & \Downarrow \mu_2 & \mathcal{L}_2 \arrow[u, "\beta", "\simeq"'] \arrow[d, "\beta'"', "\simeq"] & \Downarrow \eta_2 & \mathcal{H} \\
& & \mathcal{K}'', \arrow[ull, "\varphi''", "\simeq"'] \arrow[urr, "\psi''"'] & &
\end{tikzcd}$$

define their **vertical composition** to be the equivalence class of the diagram

$$\begin{tikzcd}
& & \mathcal{K} \arrow[dll, "\varphi"', "\simeq"] \arrow[drr, "\psi"] & & \\
\mathcal{G} & \Downarrow \lambda & \mathcal{L}_1 {}_{\alpha'}\overset{\mathrm{w}}{\times}_{\beta} \mathcal{L}_2 \arrow[u, "\alpha \circ \mathrm{pr}_1", "\simeq"'] \arrow[d, "\beta' \circ \mathrm{pr}_3"', "\simeq"] & \Downarrow \nu & \mathcal{H} \\
& & \mathcal{K}'' \arrow[ull, "\varphi''", "\simeq"'] \arrow[urr, "\psi''"'] & &
\end{tikzcd}$$

where $\lambda = (\mu_2 \mathrm{pr}_3) \circ (\varphi' \mathrm{PR}_2) \circ (\mu_1 \mathrm{pr}_1)$, and $\nu = (\eta_2 \mathrm{pr}_3) \circ (\psi' \mathrm{PR}_2) \circ (\eta_1 \mathrm{pr}_1)$. See [21, Subsection 2.3] or [24, Section 3] for details.

We define horizontal composition using left and right whiskering, and vertical composition.

**Definition 63** (Whiskering & Horizontal Composition in **LieGpoid**$[W^{-1}]$) The **left whiskering** of a generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{L} \xrightarrow[\psi]{} \mathcal{H}$ with a 2-cell $[\alpha, \alpha', \mu, \eta]$ represented by the diagram

$$\begin{tikzcd}
& \mathcal{M} \arrow[dl, "\chi"', "\simeq"] \arrow[dr, "\omega"] & \\
\mathcal{H} \arrow[r, phantom, "\Downarrow \mu"] & \mathcal{N} \arrow[u, "\alpha"', "\simeq"] \arrow[d, "\alpha'", "\simeq"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{K} \\
& \mathcal{M}' \arrow[ul, "\chi'", "\simeq"'] \arrow[ur, "\omega'"'] &
\end{tikzcd}$$

is the 2-cell $[\beta, \beta', \mathrm{ID}_{\varphi \circ \mathrm{pr}_1}, \eta \mathrm{pr}_3]$ represented by the diagram

$$\begin{tikzcd}
& \mathcal{L}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{M} \arrow[dl, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\omega\circ\mathrm{pr}_3"] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowleft"] & \mathcal{L}_{\psi}\overset{\mathrm{w}}{\times}_{\chi\circ\alpha}\mathcal{N} \arrow[u, "\beta"'] \arrow[d, "\beta'"] \arrow[r, phantom, "\Downarrow \eta\mathrm{pr}_3"] & \mathcal{K} \\
& \mathcal{L}_{\psi}\overset{\mathrm{w}}{\times}_{\chi'}\mathcal{M}', \arrow[ul, "\varphi\circ\mathrm{pr}_1", "\simeq"'] \arrow[ur, "\omega'\circ\mathrm{pr}_3"'] &
\end{tikzcd}$$

where $\beta(\ell, h, n) = (\ell, h, \alpha(n))$ and $\beta'(\ell, h, n) = (\ell, \mu(s(n))h, \alpha'(n))$. That $\beta$ and $\beta'$ are weak equivalences follows from the fact that $\alpha$ and $\alpha'$ are. Symmetrically, the **right whiskering** of a generalised morphism $\mathcal{H} \xleftarrow[\chi]{\simeq} \mathcal{M} \xrightarrow[\omega]{} \mathcal{K}$ with a 2-cell $[\alpha, \alpha', \mu, \eta]$ represented by the diagram

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] & \\
\mathcal{G} \arrow[r, phantom, "\Downarrow \mu"] & \mathcal{N} \arrow[u, "\alpha"', "\simeq"] \arrow[d, "\alpha'", "\simeq"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathcal{H} \\
& \mathcal{L}' \arrow[ul, "\varphi'", "\simeq"'] \arrow[ur, "\psi'"'] &
\end{tikzcd}$$

is the 2-cell $[\beta, \beta', \mu \mathrm{pr}_1, \mathrm{ID}_{\omega\circ\mathrm{pr}_3}]$ represented by the diagram

$$\begin{tikzcd}
& \mathcal{L}_{\psi}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{M} \arrow[dl, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\omega\circ\mathrm{pr}_3"] & \\
\mathcal{G} \arrow[r, phantom, "\Downarrow \mu\mathrm{pr}_1"] & \mathcal{N}_{\psi\circ\alpha}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{M} \arrow[u, "\beta"'] \arrow[d, "\beta'"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{K} \\
& \mathcal{L}'_{\psi'}\overset{\mathrm{w}}{\times}_{\chi}\mathcal{M}, \arrow[ul, "\varphi'\circ\mathrm{pr}_1", "\simeq"'] \arrow[ur, "\omega\circ\mathrm{pr}_3"'] &
\end{tikzcd}$$

where $\beta(\ell, h, n) = (\alpha(n), h, m)$ and $\beta'(\alpha'(n), h\eta(s(n))^{-1}, m)$.

Finally, the **horizontal composition** of two 2-cells represented by the diagrams below

$$\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] & & \mathcal{L} \arrow[dl, "\chi"', "\simeq"] \arrow[dr, "\omega"] & \\
\mathcal{G} & \mathcal{M} \arrow[u, "\simeq", "\alpha"'] \arrow[d, "\simeq"', "\alpha'"] \arrow[l, phantom, "\Downarrow \eta_1"] \arrow[r, phantom, "\Downarrow \eta_2"] & \mathcal{H} & \mathcal{N} \arrow[u, "\simeq", "\beta"'] \arrow[d, "\simeq"', "\beta'"] \arrow[l, phantom, "\Downarrow \mu_1"] \arrow[r, phantom, "\Downarrow \mu_2"] & \mathcal{I} \\
 & \mathcal{K}' \arrow[ul, "\varphi'", "\simeq"'] \arrow[ur, "\psi'"'] & & \mathcal{L}' \arrow[ul, "\chi'", "\simeq"'] \arrow[ur, "\omega'"'] &
\end{tikzcd}$$

is given by the left whiskering of the right 2-cell with the generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$, vertically composed with the right whiskering of the left 2-cell with the generalised morphism $\mathcal{H} \xleftarrow[\chi']{\simeq} \mathcal{L}' \xrightarrow[\omega']{} \mathcal{I}$. Alternatively, one could switch the order of the 2-cells in the vertical composition, using appropriate whiskerings. The end result is independent of the order; see [21, Subsection 2.3] or [24, Section 3] for details.

Finally, we define the unitors for this bicategory.

**Definition 64** (Unitors in $\mathbf{LieGpoid}[W^{-1}]$) For a pair of Lie groupoids $(\mathcal{G}, \mathcal{H})$, the **left unitor** $\lambda_{\mathcal{G},\mathcal{H}}$ is a natural transformation assigning to each generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ the 2-cell represented by the diagram

$$\begin{tikzcd}
 & \mathcal{G}_{\mathrm{id}_{\mathcal{G}}}\overset{\mathrm{w}}{\times}_{\varphi}\mathcal{K} \arrow[dl, "\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\psi\circ\mathrm{pr}_3"] \arrow[d, equal] & \\
\mathcal{G} & \mathcal{G}_{\mathrm{id}_{\mathcal{G}}}\overset{\mathrm{w}}{\times}_{\varphi}\mathcal{K} \arrow[d, "\mathrm{pr}_3"', "\simeq"] \arrow[l, phantom, "\Downarrow \mathrm{PR}_2"] \arrow[r, phantom, "\circlearrowright"] & \mathcal{H} \\
 & \mathcal{K}. \arrow[ul, "\varphi", "\simeq"'] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

The **right unitor** of $(\mathcal{G}, \mathcal{H})$ $\rho_{\mathcal{G},\mathcal{H}}$ is a natural transformation assigning to each generalised morphism $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ the 2-cell represented by the diagram

$$\begin{tikzcd}
 & \mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}}\mathcal{H} \arrow[dl, "\varphi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\mathrm{pr}_3"] \arrow[d, equal] & \\
\mathcal{G} & \mathcal{K}_{\psi}\overset{\mathrm{w}}{\times}_{\mathrm{id}_{\mathcal{H}}}\mathcal{H} \arrow[d, "\mathrm{pr}_1"', "\simeq"] \arrow[l, phantom, "\circlearrowright"] \arrow[r, phantom, "\Uparrow \mathrm{PR}_2"] & \mathcal{H} \\
 & \mathcal{K}. \arrow[ul, "\varphi", "\simeq"'] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

## A.2 Details of the construction of AnaLieGpoid

We now describe the compositions of 2-cells for the category **AnaLieGpoid**, starting with vertical composition.

**Definition 65** (Vertical Composition in **AnaLieGpoid**) Let $\mathcal{G} \xleftarrow[\varphi_i]{\simeq} \mathcal{K}_i \xrightarrow[\psi_i]{} \mathcal{H}$ be anafunctors ($i = 1, 2, 3$), and let $\eta$ be a transformation between the first and second, and $\mu$ a transformation between the second and third. Define the **vertical composition** $\mu \circ \eta$ to be unique natural transformation $\nu$ so that $(\mu \mathrm{pr}_{23}) \circ (\eta \mathrm{pr}_{12}) = \nu \mathrm{pr}_{13}$, where $\mathrm{pr}_{12}$, $\mathrm{pr}_{23}$, and $\mathrm{pr}_{13}$ are the projections as indicated in the diagram below:

$$\begin{tikzcd}
 & & \mathcal{K}_1 {}_{\varphi_1}\!\times_{\varphi_3} \mathcal{K}_3 \arrow[dll, bend right] \arrow[drr, bend left] & & \\
 & & \mathcal{K}_1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{K}_2 {}_{\varphi_2}\!\times_{\varphi_3} \mathcal{K}_3 \arrow[u, "\mathrm{pr}_{13}"', "\simeq"] \arrow[dl, "\mathrm{pr}_{12}"', "\simeq"] \arrow[dr, "\mathrm{pr}_{23}", "\simeq"'] \arrow[dd, phantom, "\circlearrowright"] \arrow[dll, phantom, "\circlearrowright"] \arrow[drr, phantom, "\circlearrowright"] & & \\
\mathcal{K}_1 \arrow[ddrr, bend right, "\psi_1"'] \arrow[dr, phantom, "\eta \Rightarrow"] & \mathcal{K}_1 {}_{\varphi_1}\!\times_{\varphi_2} \mathcal{K}_2 \arrow[l, twoheadrightarrow] \arrow[dr] & & \mathcal{K}_2 {}_{\varphi_2}\!\times_{\varphi_3} \mathcal{K}_3 \arrow[r, twoheadrightarrow] \arrow[dl] & \mathcal{K}_3 \arrow[ddll, bend left, "\psi_3"] \arrow[dl, phantom, "\mu \Rightarrow"] \\
 & & \mathcal{K}_2 \arrow[d, "\psi_2"'] & & \\
 & & \mathcal{H}. & &
\end{tikzcd}$$

Such a $\nu$ exists by Lemma 17 since $\mathrm{pr}_{13} \in \mathsf{sW}$.

We can now verify that the identity 2-cell of an anafunctor $\mathcal{G} \xleftarrow{\simeq} \mathcal{K} \to \mathcal{H}$ is given by the natural transformation

$$\iota_{\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}} \colon (\mathcal{K}_{\varphi}\!\times_{\varphi}\mathcal{K})_0 \to \mathcal{H}_1 \colon (y_1, y_2) \mapsto \psi(\mathbf{FF}_{\varphi}{}^{-1}((y_1, y_2), u_{\varphi(y_1)})).$$

as claimed. Consider another anafunctor $\mathcal{G} \xleftarrow[\varphi']{\simeq} \mathcal{K}' \xrightarrow[\psi']{} \mathcal{H}$ and 2-cell $\eta$ from the first to the second, $(\eta \circ \iota_{\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}})(y, y') = \eta(y_0, y') \cdot \iota_{\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}}(y, y_0)$ for any $y_0$ such that $(y, y_0) \in \mathcal{K}_{\varphi}\!\times_{\varphi}\mathcal{K}$ and $(y_0, y') \in \mathcal{K}_{\varphi}\!\times_{\varphi'}\mathcal{K}'$. We wish to show $(\eta \circ \iota_{\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}})(y, y') = \eta(y, y')$. If $k := \mathbf{FF}_{\varphi}{}^{-1}((y, y_0), u_{\varphi(y)})$, then $\iota_{\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}}(y, y_0) = \psi(k)$ by definition. But $(k, u_{y'})$ is an arrow in $\mathcal{K}_{\varphi}\!\times_{\varphi'}\mathcal{K}'$ from $(y, y')$ to $(y_0, y')$. By naturality of $\eta$, we get the desired equality. A similar argument shows that $(\iota_{\mathcal{G} \leftarrow \mathcal{K} \rightarrow \mathcal{H}} \circ \mu)(y'', y) = \mu(y'', y)$ for any 2-cell $\mu$ from an anafunctor $\mathcal{G} \xleftarrow[\varphi'']{\simeq} \mathcal{K}'' \xrightarrow[\psi'']{} \mathcal{H}$ to $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$.

We again define horizontal composition as the vertical composition of left and right whiskering, defined below. However, we decompose this further by defining left whiskering as the vertical composition of two simpler left whiskerings.

**Definition 66** (Whiskering & Horizontal Composition in **AnaLieGpoid**) Suppose we have a general 2-cell of the form

$$\begin{tikzcd}
& \mathcal{L} \arrow[dl, two heads, "\chi"', "\simeq"] \arrow[dr, "\omega"] & \\
\mathcal{H} \arrow[r, phantom, "\circlearrowright"] & \mathcal{L}_{\chi}\times_{\chi'}\mathcal{L}' \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{I} \\
& \mathcal{L}'. \arrow[ul, two heads, "\chi'", "\simeq"'] \arrow[ur, "\omega'"'] &
\end{tikzcd} \tag{A1}$$

The **left whiskering** of this 2-cell with an anafunctor of the form $\mathcal{K} \xleftarrow{=} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ is the natural transformation $\eta\mathrm{PR}_2$ in the following diagram:

$$\begin{tikzcd}
& \mathcal{K}_{\psi}\times_{\chi}\mathcal{L} \arrow[dl, two heads, "\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\omega\circ\mathrm{pr}_2"] & \\
\mathcal{K} \arrow[r, phantom, "\circlearrowright"] & \mathcal{K}_{\psi}\times_{\chi\circ\mathrm{pr}_1}(\mathcal{L}_{\chi}\times_{\chi'}\mathcal{L}') \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\Downarrow\eta\mathrm{PR}_2"] & \mathcal{I} \\
& \mathcal{K}_{\psi}\times_{\chi'}\mathcal{L}'. \arrow[ul, two heads, "\mathrm{pr}_1", "\simeq"'] \arrow[ur, "\omega'\circ\mathrm{pr}_2"'] &
\end{tikzcd}$$

The **left whiskering** of the 2-cell (A1) with an anafunctor of the form $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow{=} \mathcal{K}$ is the unique natural transformation $\eta'$ for which $\eta'(\mathrm{pr}_2 \circ (\mathrm{id}_{\mathcal{L}\times\mathcal{L}'}, \iota)) = \eta$; here, $\mathrm{pr}_2 \in \mathsf{sW}$ is the second projection functor on

$$\mathcal{M} := (\mathcal{L}_{\chi}\times_{\chi'}\mathcal{L}')_{\varphi\circ\chi\circ\mathrm{pr}_1}\times_{\varphi\circ\chi'\circ\mathrm{pr}_1}(\mathcal{L}_{\varphi\circ\chi}\times_{\varphi\circ\chi'}\mathcal{L}')$$

and $\iota\colon \mathcal{L}_{\chi}\times_{\chi'}\mathcal{L}' \to \mathcal{L}_{\varphi\circ\chi}\times_{\varphi\circ\chi'}\mathcal{L}'$ is the canonical inclusion.

$$
\begin{tikzcd}
 & & \mathcal{L} \arrow[drr, "\omega"] & & \\
\mathcal{L}_{\chi}\times_{\chi'}\mathcal{L}' \arrow[r, "(\mathrm{id}_{\mathcal{L}\times\mathcal{L}'},\iota)"'] & \mathcal{M} \arrow[ur, bend left, two heads, "\mathrm{pr}_1\circ\mathrm{pr}_1"] \arrow[ur, phantom, "\simeq \circlearrowleft"'] \arrow[r, two heads, "\mathrm{pr}_2", "\simeq"'] \arrow[dr, bend right, two heads, "\mathrm{pr}_2\circ\mathrm{pr}_1"'] \arrow[dr, phantom, "\circlearrowleft \simeq"] & \mathcal{L}_{\varphi\circ\chi}\times_{\varphi\circ\chi'}\mathcal{L}' \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] & \Downarrow\eta' & \mathcal{I} \\
 & & \mathcal{L}' \arrow[urr, "\omega'"'] & &
\end{tikzcd}
$$

Then if we have a general anafunctor $\mathcal{G} \xleftarrow{\simeq} \mathcal{K} \to \mathcal{H}$, we note that it is the composition of $\mathcal{G} \xleftarrow{\simeq} \mathcal{K} \xrightarrow{=} \mathcal{K}$ and $\mathcal{K} \xleftarrow{=} \mathcal{K} \to \mathcal{H}$, and so we define the **left whiskering** with the 2-cell (A1) by the vertical composition of the two left whiskerings above.

Finally, the **right whiskering** of an anafunctor $\mathcal{I} \xleftarrow[\xi]{\simeq} \mathcal{M} \xrightarrow[\zeta]{} \mathcal{J}$ with the 2-cell (A1) is the natural transformation

$\mu\colon (\mathcal{L}_{\omega}\times_{\xi}\mathcal{M})_0\,{}_{\chi\circ\mathrm{pr}_1}\!\times_{\chi'\circ\mathrm{pr}_1}(\mathcal{L}'{}_{\omega'}\times_{\xi}\mathcal{M})_0 \to \mathcal{J}_1 \colon ((w,v_1),(w',v_2)) \mapsto \mathbf{FF}_{\xi}{}^{-1}(v_1,v_2,\eta(w,w'))$
in the diagram

$$
\begin{tikzcd}
 & \mathcal{L}_{\omega}\times_{\xi}\mathcal{M} \arrow[dl, "\chi\circ\mathrm{pr}_1"', "\simeq"] \arrow[dr, "\zeta\circ\mathrm{pr}_2"] & \\
\mathcal{H} \arrow[r, phantom, "\circlearrowleft"] & (\mathcal{L}_{\omega}\times_{\xi}\mathcal{M})_{\chi\circ\mathrm{pr}_1}\times_{\chi'\circ\mathrm{pr}_1}(\mathcal{L}'{}_{\omega'}\times_{\xi}\mathcal{M}) \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathcal{J} \\
 & \mathcal{L}'{}_{\omega'}\times_{\xi}\mathcal{M}. \arrow[ul, "\chi'\circ\mathrm{pr}_1", "\simeq"'] \arrow[ur, "\zeta\circ\mathrm{pr}_2"'] &
\end{tikzcd}
$$

The **horizontal composition** of the two 2-cells

$$
\begin{tikzcd}
 & \mathcal{K} \arrow[dl, "\varphi"', "\simeq"] \arrow[dr, "\psi"] & & \mathcal{L} \arrow[dl, "\chi"', "\simeq"] \arrow[dr, "\omega"] & \\
\mathcal{G} \arrow[r, phantom, "\circlearrowleft"] & \mathcal{K}_{\varphi}\times_{\varphi'}\mathcal{K}' \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{H} \arrow[r, phantom, "\circlearrowleft"] & \mathcal{L}_{\chi}\times_{\chi'}\mathcal{L}' \arrow[u, two heads, "\simeq"] \arrow[d, two heads, "\simeq"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathcal{J} \\
 & \mathcal{K}' \arrow[ul, "\varphi'", "\simeq"'] \arrow[ur, "\psi'"'] & & \mathcal{L}' \arrow[ul, "\chi'", "\simeq"'] \arrow[ur, "\omega'"'] &
\end{tikzcd}
$$

is given by the left whiskering of the right 2-cell with the anafunctor $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$, vertically composed with the right whiskering of the left 2-cell with the anafunctor $\mathcal{H} \xleftarrow[\chi']{\simeq} \mathcal{L}' \xrightarrow[\omega']{} \mathcal{J}$. Alternatively, one could switch the order of the 2-cells in the vertical composition, using appropriate whiskerings. In fact, that the decomposition of a horizontal composition into the various vertical compositions of the various whiskerings above always yields the same result is proved in [27, Lemmas 3.14, 3.15].

We end this appendix with the unitors for **AnaLieGpoid**.

**Definition 67** (Unitors in **AnaLieGpoid**) For a pair of Lie groupoids $(\mathcal{G}, \mathcal{H})$, the **left unitor** $\lambda^{\text{ana}}_{\mathcal{G},\mathcal{H}}$ is the natural transformation assigning to each anafunctor $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ the natural transformation

$$\left((\mathcal{G}_{\text{id}_{\mathcal{G}}}\times_{\varphi}\mathcal{K})_{\text{pr}_1}\times_{\varphi}\mathcal{K}\right)_0 \to \mathcal{H}_1 \colon ((x, y_1), y_2) \mapsto \mathbf{FF}_{\varphi}{}^{-1}(y_1, y_2, u_x).$$

The **right unitor** of $(\mathcal{G}, \mathcal{H})$, denoted $\rho^{\text{ana}}_{\mathcal{G},\mathcal{H}}$, is the natural transformation assigning to each anafunctor $\mathcal{G} \xleftarrow[\varphi]{\simeq} \mathcal{K} \xrightarrow[\psi]{} \mathcal{H}$ the natural transformation

$$\left((\mathcal{K}_{\psi}\times_{\text{id}_{\mathcal{H}}}\mathcal{H})_{\varphi\circ\text{pr}_1}\times_{\varphi}\mathcal{K}\right)_0 \to \mathcal{H}_1 \colon ((y_1, z), y_2) \mapsto \mathbf{FF}_{\varphi}{}^{-1}(y_1, y_2, u_{\varphi(y_1)}).$$

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