BICATEGORIES OF FRACTIONS REVISITED: TOWARDS SMALL HOMS AND CANONICAL 2-CELLS

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Abstract. This paper introduces a set of conditions on a class of arrows in a bicategory which is weaker than the one given in [4] but still allows a bicalculus of fractions. These conditions allow us to invert a smaller collection of arrows so that in some cases we may obtain a bicategory of fractions with small hom-categories. We adapt the construction of the bicategory of fractions to work with the weaker conditions. We further discuss conditions under which there are canonical representatives for 2-cells, and how pasting of 2-cells can be simplified in the presence of certain pseudo pullbacks and introduce how this can be applied to orbifolds.

1. Introduction

This paper introduces a set of conditions on a class of arrows in a bicategory which is weaker than the one given in [4] but still allows us to form the localization as a bicalculus of fractions. One potential issue with localizations which are constructed as categories, or bicategories, of fractions is that the hom-sets, or hom-categories, may not be small, as there is no guarantee in general that the fractions with a given domain and codomain form a set. To ensure that we do get a set, we need the class of arrows $\mathfrak{W}$ to be inverted be *locally small*, i.e., for any given object C there is only a *set* of arrows in $\mathfrak{W}$ with codomain C.

We may try to find a locally small subclass of the arrows to be inverted which generates the larger class in the sense that it induces an equivalent category (or bicategory) of fractions. This subclass may not satisfy all of the conditions for forming a (bi)category of fractions, so we consider whether any of the conditions can be weakened. When an arrow can be factored as a composite of arrows that are to be inverted, this arrow will receive an inverse in any localization that adds inverses for the arrows in the factorization. This observation leads us to consider the second condition of [4], the requirement that the class of arrows to be inverted is closed under composition, as an axiom that could potentially be weakened. We cannot completely omit it: some version of this axiom is needed to be able to define horizontal composition in the bicategory of fractions. However, we can replace it by the following condition:

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[**WB2**] For each pair of composable arrows $B \xrightarrow{v} C \xrightarrow{w} D$ in $\mathfrak{W}$, there is an arrow $A \xrightarrow{u} B$ such that $A \xrightarrow{wvu} D$ is in $\mathfrak{W}$.

When a class of arrows satisfies this condition together with the other conditions for a bicalculus of fractions given in [4], it generates (through composition) a larger class of arrows that satisfies all the bicalculus of fractions conditions. In this paper we will carefully consider all the conditions for the bicalculus of fractions and give more optimal versions of these conditions, and then provide an adjusted construction of the bicategory of fractions. This construction is still given with arrows that are single spans rather than zig-zags. This also provides us with a slightly weaker set of conditions for the classical construction of the category of fractions as given by Gabriel and Zisman in [2], spelled out in Corollary 4.10.

Our motivating example for this is the bicategory of orbispaces, given as the bicategory of fractions of proper étale groupoids of suitable topological spaces with respect to the class of essential equivalences as in described in [3, 1]. A priori, the hom categories in this category are not small unless one requires all spaces to be second countable topological manifolds. We can work with a larger class of spaces when using the following observation. The class of essential equivalences has a subclass of essential covering maps that is locally small.

Another issue when working with a (bi)category of fractions is that one always works with equivalence classes in the homs. For categories, arrows are given by equivalence classes; for bicategories the same is true for 2-cells. This makes the hom-categories in the bicategory of fractions a priori very large and somewhat mysterious and hard to work with. Horizontal composition of 2-cells for instance is rather cumbersome to describe and calculate. Our second goal in this paper is to address this issue by providing conditions under which there are canonical representatives for 2-cells and under which the horizontal composition operation is significantly simplified. In our motivating example of orbispaces, essential equivalences have several nice cancellation properties that allow for a simplification of the 2-cell structure and allow us to use canonical representatives for 2-cells when this is convenient.

We prove two types of results about the 2-cell structure: about the choice of representatives for 2-cells, and about conditions that allow us to simplify the pasting of 2-cells. Each representative diagram for a 2-cell in the bicategory of fractions, as in diagram (1) in Section 3 is given by two 2-cells in the original bicategory. The 'left-hand' 2-cell α is invertible, and we think of this as the cell that allows the 'right-hand' 2-cell β to be defined. We focus on the role of the left-hand 2-cell. Tommasini indirectly addresses the question of when a 2-cell can be represented by a diagram with a given left-hand 2-cell in [9]. In general this is not always possible, and moreover, two diagrams with the same left-hand 2-cell but different right-hand 2-cells may still represent the same 2-cell in the bicategory of fractions, so the universal homomorphism mapping a bicategory to its bicategory of fractions is in general neither 2-faithful nor 2-full. However, under conditions we develop, the situation simplifies and for each pair of spans we may choose any left-hand 2-cell and show that each 2-cell in the bicategory of fractions can then be uniquely represented by a diagram involving the given left-hand 2-cell.

Additionally, for the case when the bicategory has certain pseudo pullbacks, we develop results to simplify the horizontal composition of 2-cells in the bicategory

of fractions. Overall, our goal is to make the role of 2-cells in the bicategory of fractions more transparent. In our motivating example of orbifolds these conditions are satisfied; this will be explored further in [6].

This paper is structured as follows. In Section 2, we introduce the new, weakened, conditions on a class $\mathfrak{W}$ to give rise to a bicalculus of fractions, and develop some theory on liftings of 2-cells related to the fourth condition on $\mathfrak{W}$, and on relating squares required by the third contition. In Section 3 we give the new bicategory of fractions construction $\mathcal{B}(\mathfrak{W}^{-1})$, a generalization of the one given in [4], with horizontal composition of arrows and 2-cells adjusted to account for the weaker assumption. In Section 4 we investigate the connection between our new construction and the original construction of [4], and show that if $\mathfrak{W}$ satisfies the weaker conditions of Section 2, then the class of arrows obtained by taking the closure of $\mathfrak{W}$ under composition satisfies the original conditions from [4] and gives a bi-equivalent bicategory of fractions. Additionally, we introduce the notion of covering subclasses of arrows, designed to allow us to pass to an even smaller subclass of arrows to obtain a locally small subclass of a given class of arrows. In Section 5 we introduce conditions that allow us to simplify the form of the 2-cells in the bicategory of fractions and obtain canonical representatives for the equivalence classes, and in Section 6 we investigate the case when the original bicategory has certain pseudo pullbacks and show how this can be used to simplify horizontal composition of 2-cells in the bicategory of fractions. In Section 7 we indicate how this work applies to orbispaces, to be further explored in [6]. The last section is an appendix giving associativity of 2-cells and proving associativity coherence.

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2. Weaker Conditions for a Bicalculus of Fractions

In the first part of this section we introduce the new conditions on a class of arrows in a bicategory that will give rise to a bicalculus of fractions. These are a weakening of the conditions **BF1**–**BF5** given in [4]. In the second part of this section we develop general results about the structure of the 2-cells in a bicategory with a class of arrows satisfying our new conditions.

2.1. **The New Conditions.** We list our new conditions on a class of arrows. In Section 3 we will show that these are sufficient for the existence of the bicategory of fractions, although the specific construction of this bicategory needs to be changed.

[WB1] All identities are in $\mathfrak{W}$.

[WB2] For each pair of composable arrows $B \xrightarrow{v} C \xrightarrow{w} D$ in $\mathfrak{W}$, there is an arrow $A \xrightarrow{u} B$ such that $A \xrightarrow{wvu} D$ is in $\mathfrak{W}$.

[WB3] For every pair $w\colon A \to B$, $f\colon C \to B$ with $w \in \mathfrak{W}$, there exist maps h, v, where $v \in \mathfrak{W}$, and an invertible 2-cell α as in the following diagram.

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ {\scriptstyle v}\downarrow & \overset{\alpha}{\Longrightarrow} & \downarrow{\scriptstyle w} \\ C & \xrightarrow[f]{} & B \end{array}$$

[WB4] For any 2-cell

$$\alpha\colon w \circ f \Rightarrow w \circ g$$

with $w \in \mathfrak{W}$, there exists $u \in \mathfrak{W}$ and a 2-cell

$$\beta\colon f \circ u \Rightarrow g \circ u$$

such that $\alpha \circ u = w \circ \beta$. Furthermore, the collection of such pairs (u, β) has the following property: when (u_1, β_1) and (u_2, β_2) are two such pairs, there exist arrows s, t, such that $u_1 \circ s$ and $u_2 \circ t$ are in $\mathfrak{W}$, and there is an invertible 2-cell $\varepsilon\colon u_1 \circ s \Rightarrow u_2 \circ t$ such that the following diagram commutes:

$$\begin{array}{ccc} f \circ u_1 \circ s & \xrightarrow{\beta_1 \circ s} & g \circ u_1 \circ s \\ {\scriptstyle f\circ\varepsilon}\downarrow & & \downarrow{\scriptstyle g\circ\varepsilon} \\ f \circ u_2 \circ t & \xrightarrow[\beta_2 \circ t]{} & g \circ u_2 \circ t. \end{array}$$

[WB5] When $w \in \mathfrak{W}$ and there is an invertible 2-cell $\alpha\colon v \Rightarrow w$, then $v \in \mathfrak{W}$.

Remarks 2.1. (1) The original condition **BF1** stated that all equivalences were in the class $\mathfrak{W}$. It is well-known that it is sufficient to replace this with the given **[WB1]**; see for instance, [9].

(2) Condition **[WB2]** is a significantly weaker version of the original condition **BF2**, which required that $\mathfrak{W}$ be closed under composition.

(3) Conditions **[WB3]** and **[WB5]** are the same as the old conditions **BF3** and **BF5** respectively.

(4) When α and β are 2-cells as in condition **[WB4]**, we will refer to β as a *lifting of α with respect to w*. In [4], condition **BF4** additionally required that if α is invertible, it has a lifting β that is invertible. We will show in Proposition 2.3 that this assumption is not needed, as it can be derived from the other assumptions.

2.2. **Properties of Liftings of 2-Cells.** In this section we prove that our condition **[WB4]**, together with the conditions **[WB1]**–**[WB3]** and **[WB5]**, imply the original condition **BF4**. To do this, we develop some properties of the 2-cell liftings that **[WB4]** requires, and show that they can be chosen to respect composition.

We assume throughout this section that $\mathfrak{W}$ is a class of arrows satisfying conditions **[WB1]**-**[WB5]**. We begin by showing that for fixed $w \in \mathfrak{W}$, the collection of the liftings of cells given by **[WB4]** inherits the vertical composition structure in the sense that the vertical composition of two liftings gives a lifting for the vertical composition of the original cells.

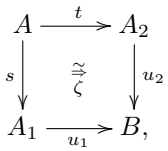
Lemma 2.2. *Let $\mathfrak{W}$ satisfy* **[WB1]**–**[WB5]**. *Suppose that we have arrows*

$$B \underset{h}{\overset{f}{\underset{g}{\rightrightarrows}}} C \xrightarrow{w} D$$

with $w \in \mathfrak{W}$, and let $\alpha_1\colon wf \Rightarrow wg$ and $\alpha_2\colon wg \Rightarrow wh$ be 2-cells. Then there exists an arrow $u\colon A \to B$ in $\mathfrak{W}$ with 2-cells $\beta_1\colon fu \Rightarrow gu$ and $\beta_2\colon gu \Rightarrow hu$ such that $w\beta_1 = \alpha_1 u$ and $w\beta_2 = \alpha_2 u$. It follows that $w(\beta_2 \cdot \beta_1) = (\alpha_2 \cdot \alpha_1)u$.

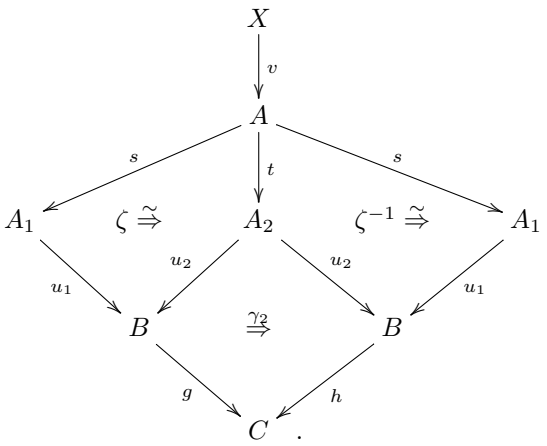
Proof. We begin by choosing two arbitrary arrows and cells as in condition **[WB4]**: let $u_1 \colon A_1 \to B$ and $u_2 \colon A_2 \to B$ be two arrows in $\mathfrak{W}$ with 2-cells $\gamma_1 \colon fu_1 \Rightarrow gu_1$ and $\gamma_2 \colon gu_2 \Rightarrow hu_2$ such that $w\gamma_1 = \alpha_1 u_1$ and $w\gamma_2 = \alpha_2 u_2$.

Since u_1 and u_2 are in $\mathfrak{W}$, condition **[WB3]** gives us a square



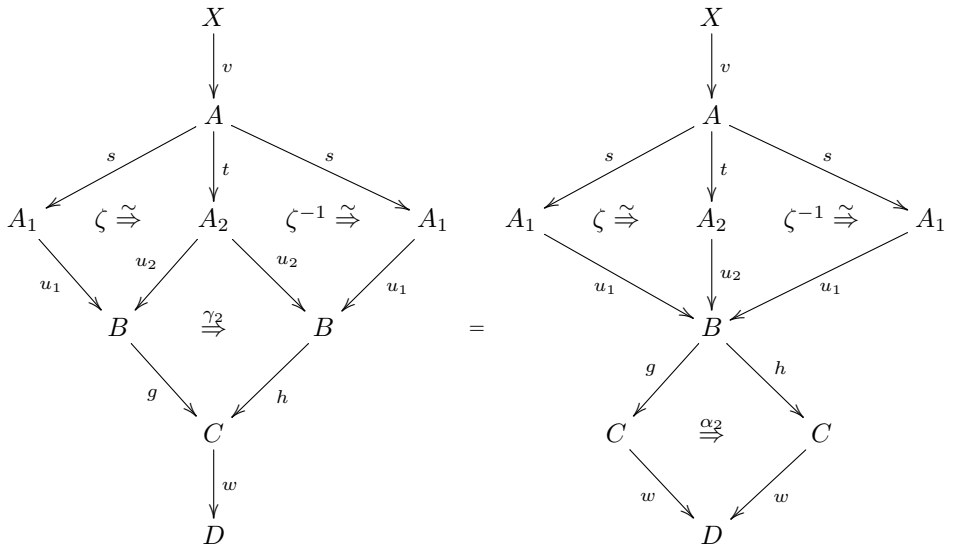
with $s \in \mathfrak{W}$. By Condition **[WB2]**, there is an arrow $v \colon X \to A$ such that the composition $u_1 sv$ is in $\mathfrak{W}$.

We claim that the following arrow and 2-cells satisfy the conditions of this lemma: $u = u_1 sv$, $\beta_1 = \gamma_1 sv$ and $\beta_2 = ((h\zeta^{-1}) \cdot (\gamma_2 t) \cdot (g\zeta)) \circ v$, as in the diagram,



To prove this claim, first note that since γ_1 was chosen to satisfy **[WB4]**, $w\beta_1 = w\gamma_1 sv = \alpha_1 u_1 sv = \alpha_1 u$. Now using the fact that γ_2 was also chosen so that

$w\gamma_2 = \alpha_2 u_2$, we calculate $w\beta_2$ in the following diagrams:



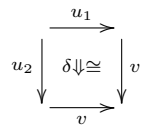
and this is clearly equal to $\alpha_2 u_1 s v = \alpha_2 u$, as required. □

We now use this lemma to prove that whenever the 2-cell $\alpha\colon wf \Rightarrow wg$ is invertible, there is at least one choice of a pair (u, β) for **[WB4]** such that β is also invertible.

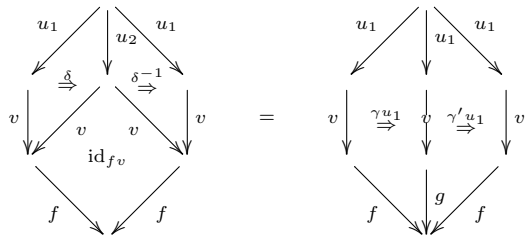
Proposition 2.3. *Let $\mathfrak{W}$ satisfy the conditions* **[WB1]**–**[WB5]**. *If $w \in \mathfrak{W}$ and $\alpha\colon wf \Rightarrow wg$ is an invertible 2-cell, then there is an arrow $u \in \mathfrak{W}$ with an invertible 2-cell $\beta\colon fu \Rightarrow gu$ such that $w\beta = \alpha u$.*

Proof. We begin by applying Lemma 2.2 to the case where $h = f$, $\alpha_1 = \alpha$ and $\alpha_2 = \alpha^{-1}$. This gives us an arrow $v \in \mathfrak{W}$ and 2-cells $\gamma\colon fv \Rightarrow gv$ and $\gamma'\colon gv \Rightarrow fv$ such that $w\gamma = \alpha v$ and $w\gamma' = \alpha^{-1} v$. So $w(\gamma' \cdot \gamma) = (\alpha^{-1} \cdot \alpha)v = \mathrm{id}_{wf} v$. This does not guarantee that γ and γ' are inverses, but we will show that there is a further lifting v' such that $vv' \in \mathfrak{W}$ and γv and $\gamma v'$ are inverses.

We create v' in two stages. First we will find u_1 such that $(\gamma' u_1)(\gamma u_1) = \mathrm{id}_{fu_1}$, and then find w_1 such that $(\gamma u_1 w_1)(\gamma' u_1 w_1) = \mathrm{id}_{fu_1 w_1}$. To find u_1, we observe that both $w(\gamma'\gamma) = id_{wf} v$ and $w \circ \mathrm{id}_{fv} = \mathrm{id}_{wf} v$. Thus, $(v, \gamma' \cdot \gamma)$ and (v, id_{fv}) are both pairs of liftings of id_{wf} with respect to w as in **[WB4]**. The second half of **[WB4]** gives a relationship between any two such pairs, so applying that here gives two maps, u_1 and u_2, and an invertible 2-cell,



with $vu_i \in \mathfrak{W}$ and such that



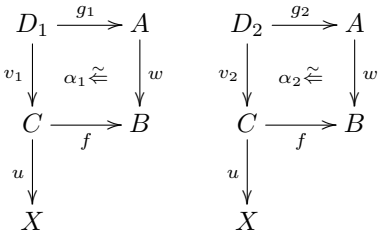
The left-hand side of this equation is equal to the identity 2-cell, id_{fvu_1}, so $\gamma' u_1 \cdot \gamma u_1 = \mathrm{id}_{fvu_1}$.

Now we create w_1 via the same argument applied to the 2-cells $\gamma u_1 \cdot \gamma' u_1$ and id_{gvu_1}. We know that $w(\gamma u_1 \cdot \gamma' u_1) = (\alpha \cdot \alpha^{-1})vu_1 = \mathrm{id}_{wg}vu_1 = \mathrm{id}_{wgvu_1} = w\mathrm{id}_{gvu_1}$. So both $(vu_1, \gamma u_1 \cdot \gamma' u_1)$ and $(vu_1, \mathrm{id}_{gvu_1})$ are liftings of id_{wg} with respect to w, and applying the second half of **[WB4]** as above gives us w_1, w_2 and an invertible 2-cell ϵ such that $vu_1w_i \in \mathfrak{W}$ and $\gamma u_1 w_1 \cdot \gamma' u_1 w_1 = \mathrm{id}_{gvu_1w_1}$. We conclude that $\gamma' u_1 w_1 = (\gamma u_1 w_1)^{-1}$. Therefore setting $v' = u_1 w_1$, $u = vv' = vu_1w_1$ and $\beta = \gamma u_1 w_1$ satisfies the requirements of the proposition. □

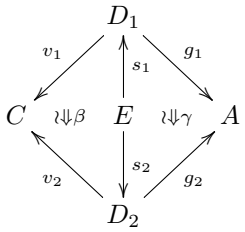
Remark 2.4. Combining the proofs for Proposition 2.3 and Lemma 2.2 shows that if α in Proposition 2.3 is invertible, for any arrow $u \in \mathfrak{W}$ with 2-cell $\beta\colon fu \Rightarrow gu$ such that $w\beta = \alpha u$, there is an arrow s such that $\beta \circ s$ is invertible.

2.3. **Squares as in Condition [WB3].** In this section we address a question related to condition **[WB3]**: if there are two squares as in **[WB3]** for the same cospan, how are these squares related to each other? This question was answered in the proof of Lemma A.1.1 in [4] for cospans where both arrows are in $\mathfrak{W}$. Here, we prove a more general result, for cospans with just one arrow in $\mathfrak{W}$ and assuming only the weaker condition **[WB2]**. This result will play a crucial role in the constructions of whiskering of 2-cells with arrows in the bicategory of fractions. It will also be used in the study of the equivalence relation on the 2-cells diagrams.

Proposition 2.5. *For $w\colon A \to B$ in $\mathfrak{W}$ and $f\colon C \to B$ any arrow in $\mathcal{B}$, and any two squares,*

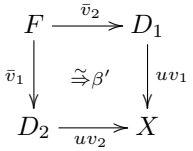


where u, uv_1 and uv_2 are all in $\mathfrak{W}$, then there are arrows s_1 and s_2 and invertible 2-cells β and γ as in



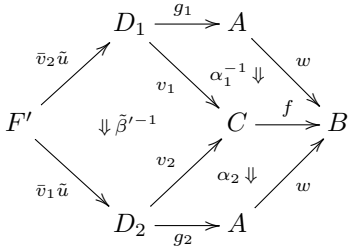
such that $uv_1s_1 \in \mathfrak{W}$, and the composites $(\alpha_2 s_2)\cdot(f\beta)$ and $(w\gamma)\cdot(\alpha_1 s_1)$ are equal.

Proof. Since uv_1 is in $\mathfrak{W}$, condition **[WB3]** gives us a square



with $\bar{v}_1 \in \mathfrak{W}$. Applying Proposition 2.3 to the 2-cell $\beta' : uv_2\bar{v}_1 \Rightarrow uv_1\bar{v}_2$, we get an arrow $\tilde{u}\colon F' \to F$ in $\mathfrak{W}$ and an invertible 2-cell $\tilde{\beta}' : v_2\bar{v}_1\tilde{u} \Rightarrow v_1\bar{v}_2\tilde{u}$.

Then we have the following invertible 2-cell from $wg_1\bar{v}_2\tilde{u}$ to $wg_2\bar{v}_1\tilde{u}$.



By applying Proposition 2.3 with respect to w, there is an arrow $\tilde{w}\colon F'' \to F'$ in $\mathfrak{W}$ with an invertible 2-cell $\gamma'\colon g_1\bar{v}_2\tilde{u}\tilde{w} \Rightarrow g_2\bar{v}_1\tilde{u}\tilde{w}$ such that $w\gamma'$ is equal to the pasting of this last diagram composed with $\tilde{w}$. Finally, by repeatedly applying condition **[WB2]** to the string of composible $\mathfrak{W}$ arrows $uv_2, \bar{v}_1, \tilde{u}, \tilde{w}$, there is an arrow $t\colon E \to F''$ such that $uv_2\bar{v}_1\tilde{u}\tilde{w}t \in \mathfrak{W}$. By condition **[WB5]** it follows that $uv_1\bar{v}_2\tilde{u}\tilde{w}t \in \mathfrak{W}$ as well. The reader may verify that $s_1 = \bar{v}_2\tilde{u}\tilde{w}t$, $s_2 = \bar{v}_1\tilde{u}\tilde{w}t$, $\beta = \tilde{\beta}'\tilde{w}t$ and $\gamma = \gamma' t$ satisfy the conditions of this proposition. □

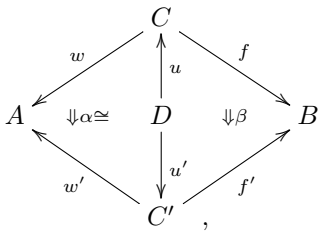
Remark 2.6. An extension of the result of Proposition 2.5 can be found in Appendix A, Proposition A.1.

3. The New Bicategory of Fractions Construction

We will now show that conditions introduced in Section 2.1 are sufficient to construct a bicategory of fractions $\mathcal{B}(\mathfrak{W}^{-1})$. Given a bicategory $\mathcal{B}$ and a class of arrows $\mathfrak{W}$ which satisfies the conditions **[WB1]**–**[WB5]**, we first describe the new bicategory $\mathcal{B}(\mathfrak{W}^{-1})$, and then show that it has the universal property of the bicategory of fractions. The objects, arrows and 2-cells of $\mathcal{B}(\mathfrak{W}^{-1})$ are defined just as in [4], but we will need to adjust the definition of composition and pasting. We begin by reminding the reader of the definition as given in [4].

- *Objects* are the objects of $\mathcal{B}$.
- *Arrows* are spans of the form $\xleftarrow{w} \quad \xrightarrow{f}$ with $w \in \mathfrak{W}$ and f an arbitrary arrow in $\mathcal{B}$.
- *2-cells* are equivalence classes of diagrams of the form

(1)



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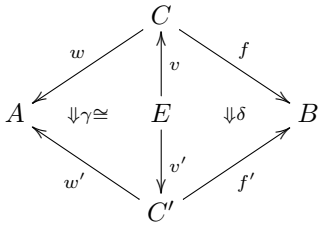
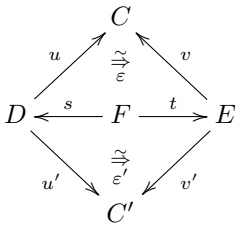
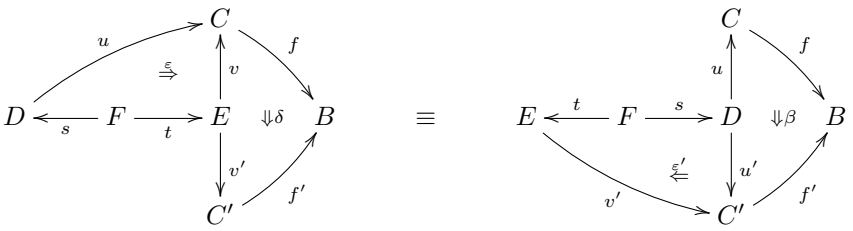
where wu and $w'u'$ are in $\mathfrak{W}$. Such a diagram (1) is *equivalent* to another such diagram



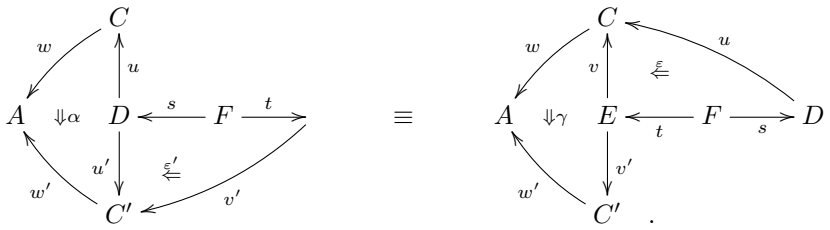
(with wv in $\mathfrak{W}$) if and only if there exists a diagram of the form



with $wus \in \mathfrak{W}$, such that



and



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Remark 3.1. In the description above, we consistently only require half of our arrow compositions to be in $\mathfrak{W}$. For example, we require only that $wv \in \mathfrak{W}$, and not the corresponding $w'v'$; similarly we only require $wus \in \mathfrak{W}$. However, since the 2-cells are invertible and $\mathfrak{W}$ satisfies **[WB5]**, the other half follows automatically.

The original condition **BF2** was used in [4] in the construction of composition of arrows and horizontal and vertical composition of 2-cells in the bicategory of fractions. In constructing these compositions under our weaker conditions, we need to adjust for the fact that $\mathfrak{W}$ is no longer closed under composition. Instead, we have the condition **[WB2]** that allows us to pre-compose with an additional arrow to get a composition in $\mathfrak{W}$. The description of the compositions in [4] relies heavily on the choices of squares as in condition **[WB3]** and liftings as in condition **[WB4]** (although, in fact, the construction only depends on the choices of the squares when they are used to compose the spans, as Tommasini [9] has shown that different choices made in the composition of 2-cells give equivalent representatives). In describing the compositions in the new bicategory of fractions, we use a collection of choices for arrows for composites as in **[WB2]** to augment the choices of squares and liftings to make sure that the necessary arrows are in $\mathfrak{W}$. We list and label these choices here before beginning the constructions so we can refer back to them.

Notation 3.2. The following choices of arrows and 2-cells will be used in the construction of the bicategory of fractions $\mathcal{B}(\mathfrak{W}^{-1})$. The first three choices really determine the construction. The last four are just short-cuts for frequently used combinations of the first three.

[C1] For each pair of composable arrows $\xrightarrow{v}\xrightarrow{u}$ in $\mathfrak{W}$ use **[WB2]** to choose an arrow $w_{u,v}$ such that $uvw_{u,v} \in \mathfrak{W}$.

[C2] For every pair $\xrightarrow{f}\xleftarrow{u}$ with $u \in \mathfrak{W}$ use **[WB3]** to choose a square

$$\begin{array}{ccc} R & \xrightarrow{f'} & T \\ {\scriptstyle u'}\downarrow & \alpha & \downarrow{\scriptstyle u} \\ S & \xrightarrow[f]{} & B \end{array}$$

When we want to stress the dependence of α on f and u, we denote this cell by $\alpha_{f,u}$.

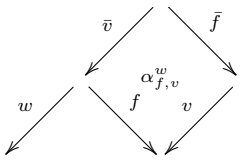
[C3] Given $\alpha\colon w \circ f \Rightarrow w \circ g$, a 2-cell with $w \in \mathfrak{W}$, choose a 1-cell $\tilde{w} \in \mathfrak{W}$ and a 2-cell

$$\tilde{\alpha}\colon f \circ u \Rightarrow g \circ u$$

such that $\alpha \circ \tilde{w} = w \circ \tilde{\alpha}$. Using Proposition 2.3, we choose $\tilde{\alpha}$ to be invertible whenever α is.

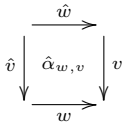
[C4] For each zig-zag, $\xleftarrow{w}\xrightarrow{f}\xleftarrow{v}$ with v and w in $\mathfrak{W}$, **[C2]** determines a square and invertible 2-cell $\alpha_{f,v}$. Compose this with the choice $w_{uv'}$ from **[C1]** to get $w_{uv'}v'w \in \mathfrak{W}$. Defining $\bar{v} = w_{uv'}v', \bar{f} = w_{uv'}f'$ and

$\alpha^w_{f,v} = w_{uv'}\alpha_{f,v}$ gives the chosen diagram



with $w\bar{v} \in \mathfrak{W}$. Note that $\bar{v}$ is not guaranteed to be in $\mathfrak{W}$.

[C5] For each cospan $\xrightarrow{w}\xleftarrow{v}$ with both arrows $w, v \in \mathfrak{W}$, apply **[C2]** to obtain a square with an invertible 2-cell $\alpha_{w,v}$. Then compose with $w_{wv'}$ from **[C1]** to get $w_{wv'}v' \in \mathfrak{W}$. Define $\hat{v} = w_{wv'}v', \hat{w} = w_{wv'}w'$ and $\hat{\alpha}_{w,v} = w_{wv'}\alpha_{w,v}$ to obtain the chosen square



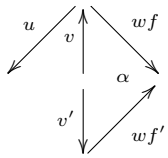
where $w\hat{v} \in \mathfrak{W}$.

[C6] For each invertible 2-cell $\alpha\colon w \circ s_1 \Rightarrow w \circ s_2$ with $w, ws_1, ws_2 \in \mathfrak{W}$, apply **[C3]** to obtain $\tilde{w} \in \mathfrak{W}$ and $\tilde{\alpha} : s_1\tilde{w} \Rightarrow s_2\tilde{w}$. Then ws_1 and $\tilde{w}$ are in $\mathfrak{W}$, so apply **[C1]** to obtain an arrow u such that $ws_1\tilde{w}u \in \mathfrak{W}$. Since $\tilde{\alpha}$ in **[C3]** is invertible, we conclude that $ws_2\tilde{w}u$ is also in $\mathfrak{W}$. Setting $\bar{\tilde{w}} = \tilde{w}u$, we get the chosen lifting

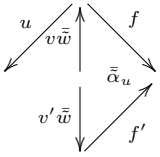
$$\bar{\tilde{\alpha}}\colon s_1\bar{\tilde{w}} \Rightarrow s_2\bar{\tilde{w}}$$

such that $ws_1\bar{\tilde{w}} \in \mathfrak{W}$.

[C7] For each configuration,



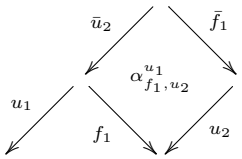
with uv and w in $\mathfrak{W}$, **[C2]** determines w_{uv} with $uvw_{uv} \in \mathfrak{W}$. Pre-composing with $\bar{\tilde{w}} = w_{uv}$ determined by **[C1]** gives the chosen 2-cell $\bar{\tilde{\alpha}}_u$ with $uv\bar{\tilde{w}} \in \mathfrak{W}$.



With these choices determined, we will now define the bicategory of fractions.

Composition of 1-Cells We define the composition of spans $A \xleftarrow{u_1} S \xrightarrow{f_1} B$ and $B \xleftarrow{u_2} T \xrightarrow{f_2} C$ in $\mathcal{B}(\mathfrak{W}^{-1})$ using the chosen square in **[C4]** of Notation

3.2,

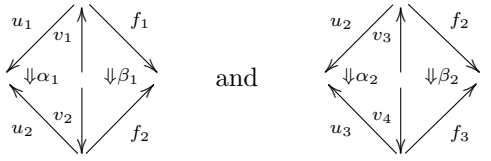


so that $u_1\bar{u}_2 \in \mathfrak{W}$. Then the composition of spans is given by

$$A \xleftarrow{u_1\bar{u}_2} \xrightarrow{f_2\bar{f}_1} C.$$

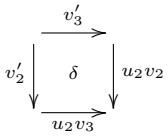
Remarks 3.3.
(1) Proposition 2.5 implies that any other choice of a square to define the composition would result in an isomorphic arrow in $\mathcal{B}(\mathfrak{W}^{-1})$. Proposition A.1 below further shows that the isomorphism is unique when certain properties with respect to the defining squares are required.
(2) Horizontal composition of 1-cells is clearly not associative in general. In Appendix A, Proposition A.3 we introduce the family of associativity 2-cells and in Appendix B, Proposition B.4, we show that this family satisfies the associativity coherence conditions. The definition of the associativity cells is a direct generalization of the ones given in [4], but the proof of coherence is a bit more involved. The appendices highlight the technical results that lead to coherence in separate propositions.

Vertical Composition of 2-Cells We define the vertical compositon of 2-cell diagrams,



.

First, since u_2v_3 and u_2v_2 are both in $\mathfrak{W}$, let

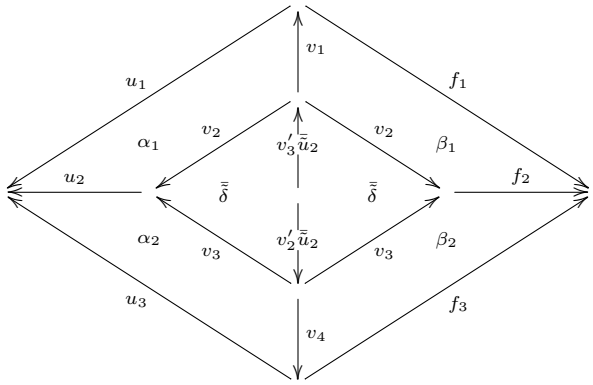


be the chosen square in **[C5]** of Notation 3.2: $\delta = \hat{\alpha}_{u_2v_3,u_2v_2}$ and $u_2v_3v_2' \in \mathfrak{W}$. Since δ is invertible, $u_2v_2v_3' \in \mathfrak{W}$ also.

Next, apply **[C6]** to $\delta : u_2v_2v_3' \Rightarrow u_2v_3v_2'$ and obtain an arrow $\bar{\bar{u}}_2 \in \mathfrak{W}$ and an invertible 2-cell $\bar{\bar{\delta}} : v_2v_3'\bar{\bar{u}}_2 \Rightarrow v_3v_2'\bar{\bar{u}}_2$. Note that $u_2v_2v_3'\bar{\bar{u}}_2 \in \mathfrak{W}$, as indicated in **[C6]**.

This gives us the following representative for the vertical composition,

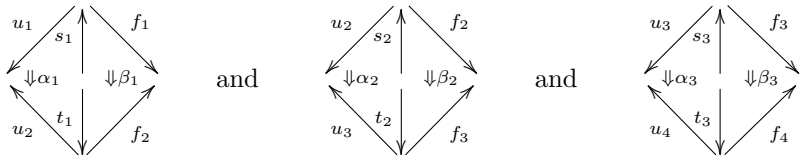
(2)



Observe that $u_2v_2v_3'\bar{\tilde{u}}_2 \in \mathfrak{W}$ by construction, and $u_1v_1v_3'\bar{\tilde{u}}_2$ and $u_3v_4v_2'\bar{\tilde{u}}_2$ are in $\mathfrak{W}$ since they are isomorphic to $u_2v_2v_3'\bar{\tilde{u}}_2$. So this diagram represents a 2-cell from $\xleftarrow{u_1}\xrightarrow{f_1}$ to $\xleftarrow{u_3}\xrightarrow{f_3}$.

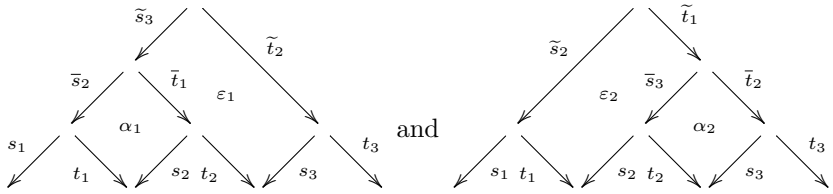
Lemma 3.4. *Vertical composition of 2-cells is strictly associative.*

Proof. Consider three vertically composable 2-cell diagrams,

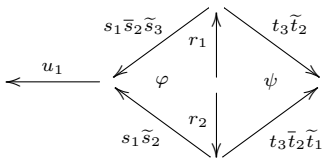


.

The two possible vertical compositions correspond to choices of squares δ_i and ε_i with $i = 1, 2$ as in



with $u_2s_2\bar{t}_1\widetilde{s}_3 \in \mathfrak{W}$ and $u_2s_2\bar{s}_3\widetilde{t}_1 \in \mathfrak{W}$. The construction of the associativity 2-cell diagrams in Appendix A can be applied to these two spans. Let the resulting 2-cell diagram be

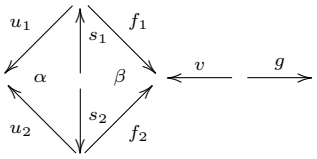


Then the reader may verify that the cells φ and ψ satisfy the equations to show that the two vertical compositions of the 2-cell diagrams are equivalent. (Use the

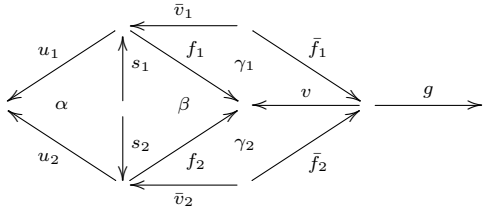
properties of the cells used to construct the associativity 2-cells as described in the appendix.) □

Horizontal Composition of 2-Cells The construction for horizontal composition in [4] is given in terms of whiskering on the left and the right. We will address the two cases in the following two subsections.

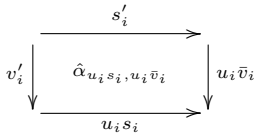
3.1. **Left Whiskering.** Suppose we have



with $u_i s_i \in \mathfrak{W}$ and α invertible, so that the left side represents a 2-cell. We begin by constructing the composites of the arrows involved. This gives us the following diagram,

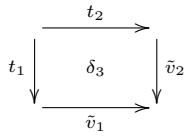


where $\gamma_1 = \alpha^{u_1}_{f_1,v}$ and $\gamma_2 = \alpha^{u_2}_{f_2,v}$ are the chosen squares of **[C4]** of Notation 3.2. The next step is to construct squares that complete the cospans $\xrightarrow{s_1}\!\!\leftarrow\!\!\!\!-^{\bar{v}_1}$ and $\xrightarrow{s_2}\!\!\leftarrow\!\!\!\!-^{\bar{v}_2}$. Neither s_i nor $\bar{v}_i$ (where $i = 1, 2$) are necessarily in $\mathfrak{W}$, but the $u_i s_i$ are by assumption, and the $u_i \bar{v}_i$ are by **[C4]**. Now take the squares chosen in **[C5]** for $i = 1, 2$,



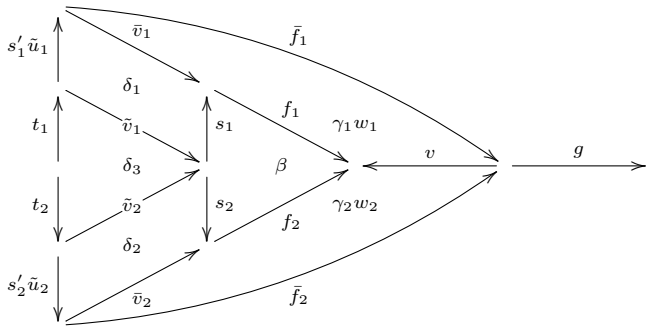
where the composites $u_i s_i v'_i$ are in $\mathfrak{W}$ and the 2-cells $\hat{\alpha}_{u_i s_i, u_i \bar{v}_i}$ are invertible. Now we have $\hat{\alpha} : u_i s_i v'_i \Rightarrow u_i \bar{v}_i s'_i$ where $u_i \in \mathfrak{W}$, and hence **[C6]** determines arrows $\tilde{u}_i$ and 2-cells $\delta_i : s_i v'_i \tilde{u}_i \Rightarrow \bar{v}_i s'_i \tilde{u}_i$. If we write $v'_i \tilde{u}_i = \tilde{v}_i$ then we have $u_i s_i \tilde{v}_i \in \mathfrak{W}$ for $i = 1, 2$.

Finally, we want to construct a square to complete the cospan $\xrightarrow{\tilde{v}_1}\!\!\leftarrow\!\!\!\!-^{\tilde{v}_2}$. Neither of the $\tilde{v}_i$ is necessarily in $\mathfrak{W}$, but the $u_i s_i \tilde{v}_i$ are. Also, since $\alpha \colon u_1 s_1 \Rightarrow u_2 s_2$ is invertible, it follows that $u_1 s_1 \tilde{v}_2 \in \mathfrak{W}$. Using a sequence of chosen squares and lifts as above, we construct a square

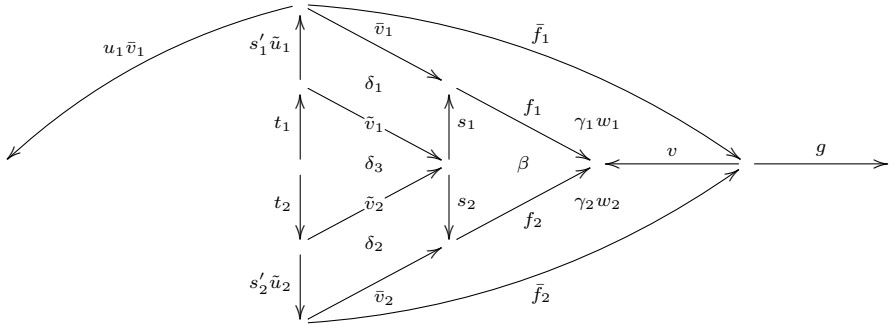


with δ_3 invertible and $u_1 s_1 \tilde{v}_1 t_1 \in \mathfrak{W}$.

To find the right-hand 2-cell in the diagram representing the left whiskering, we want to apply a choice of lifting as in condition **[WB4]** to the following diagram,



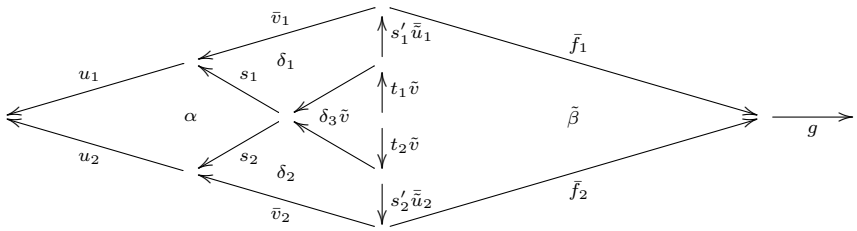
and lift with respect to v. However, we need to do this in such a way that we obtain a valid 2-cell diagram. So we will take the lifting of **[C7]** for the diagram



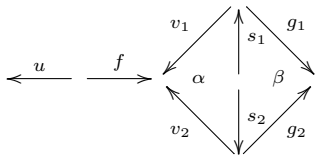
This gives us an arrow $\tilde{v} \in \mathfrak{W}$ and a 2-cell $\tilde{\beta}\colon \bar{f}_1 s_1' \tilde{u}_1 t_1 \tilde{v} \Rightarrow \bar{f}_2 s_2 \tilde{u}_2 t_2 \tilde{v}$ such that $v\tilde{\beta}$ is equal to the pasting of the previous diagram composed with $\tilde{v}$, and $u_1\bar{v}_1 s_1' \tilde{u}_1 t_1 \tilde{v} \in \mathfrak{W}$.

The resulting representative for the horizontal composition can be described by

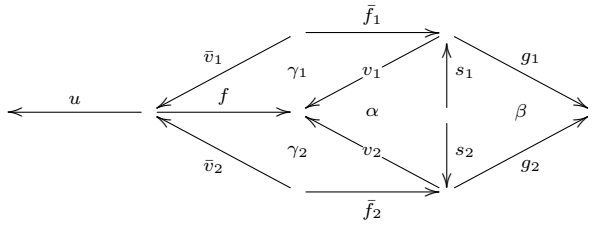
(3)



3.2. **Right Whiskering.** Consider a diagram



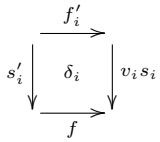
with $v_1 s_1$ and $v_2 s_2$ in $\mathfrak{W}$, and α invertible, so the right side represents a 2-cell. Again, we begin by constructing the horizontal compositions of the arrows involved using the squares of **[C2]** in Notation 3.2 to obtain the following diagram,



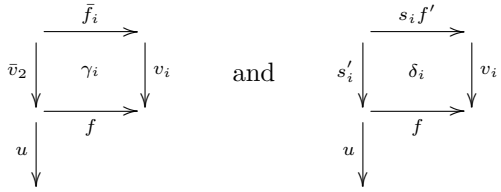
where $\gamma_i = \alpha^u_{f,v_i}$ and $u\bar{v}_i \in \mathfrak{W}$ for $i = 1, 2$.

Since $v_i s_i \in \mathfrak{W}$ for $i = 1, 2$, we have chosen squares from **[C2]** giving

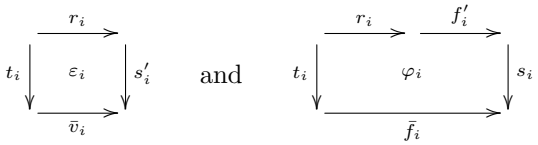
(4)



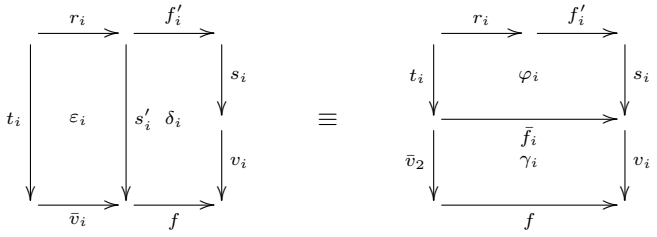
with $s'_i \in \mathfrak{W}$. Now apply Proposition 2.5 to the pairs of squares



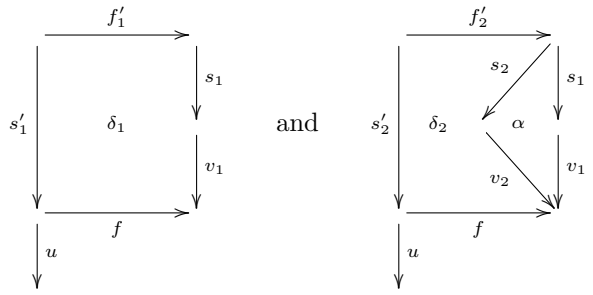
We obtain arrows and invertible 2-cells,



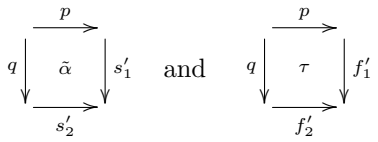
such that $u\bar{v}_i t_i \in \mathfrak{W}$ for $i = 1, 2$ and the composites of the following two pasting diagrams are equal:



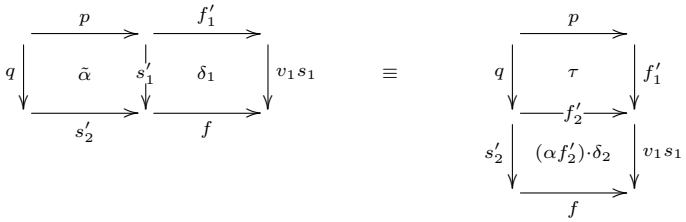
Now apply Proposition 2.5 to the following two squares, where $v_1 s_1, s'_1, u, us'_1, us_2 \in \mathfrak{W}$:



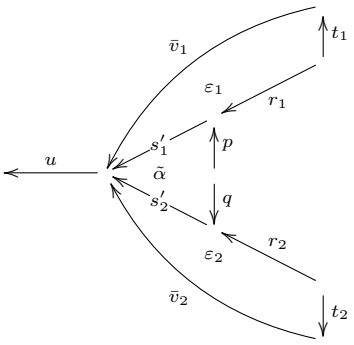
This gives us arrows and invertible 2-cells



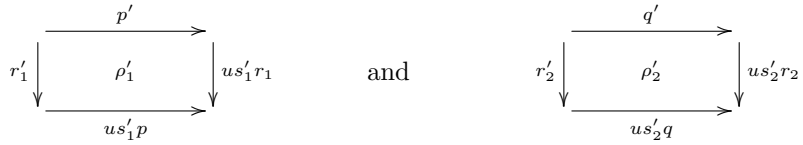
such that $s'_1 p \in \mathfrak{W}$ and the following two pasting diagrams give the same composite:



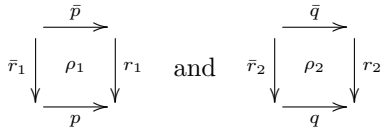
Thus far we have constructed the following part of the left-hand cell of the whiskered 2-cell diagram,



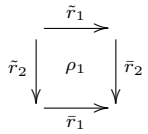
We fill in the gap in the middle by chosen liftings of chosen squares according to conditions **[WB3]** and **[WB4]**. First note that the $u\bar{v}_i t_i$ are in $\mathfrak{W}$ for $i = 1, 2$, and hence since ϵ_i is invertible, $us'_i r_i \in \mathfrak{W}$. So we have squares from **[C2]**,



and we lift with respect to us'_1 and us'_2 respectively (as in **[C3]**) and add additional arrows w_1 and w_2 to obtain arrows $\bar{p} = p'\tilde{u}_1 w_1$, $\bar{r}_1 = r'_1\tilde{u}_1 w_1 \in \mathfrak{W}$ and $\bar{q} = q'\tilde{u}_2$, $\bar{r}_2 = r'_2\tilde{u}_2 w_2 \in \mathfrak{W}$ with invertible 2-cells

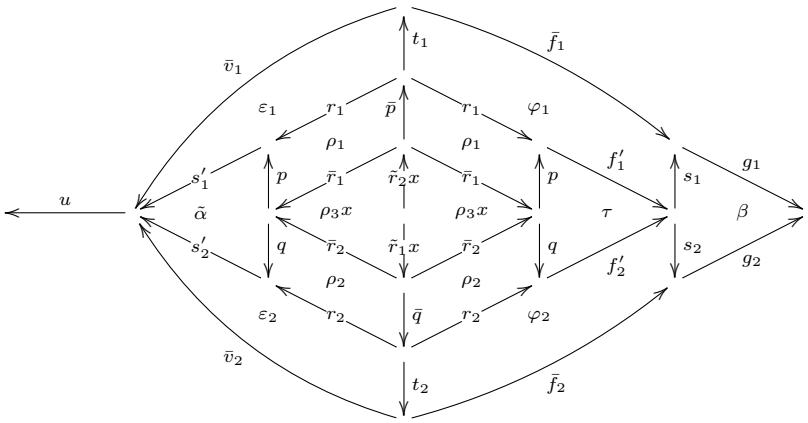


Finally, we take a chosen square according to **[C2]**,



and let x be a chosen arrow such that $u\bar{v}_1 t_1 \bar{p}\tilde{r}_2 x \in \mathfrak{W}$. Then the result of the whiskering becomes:

(5)



Remarks 3.5. (1) When the class $\mathfrak{W}$ of arrows to be inverted satisfies the traditional **BF1**–**BF5** conditions from [4], this construction reduces to the construction given in that paper when one takes the identity arrow whenever a choice of an arrow based on condition **[WB2]** is needed. The definition of horizontal whiskering here is not exactly the same as the one given in [4], but the 2-cell diagrams obtained are equivalent. This is shown in [9], where it is proved that various choices to fill the 2-cell diagrams for whiskering all result in equivalent 2-cell diagrams.

(2) The fact that the vertical composition and the horizontal whiskering operations described here are well-defined on equivalence classes of 2-cell diagrams can be checked by a long and rather tedious calculation applying the same type of operations on the arrows and cells that witness the equivalence relation.

With these definitions, we get the following:

Theorem 3.6. *For any bicategory $\mathcal{B}$ with a class $\mathfrak{W}$ of arrows that satisfies conditions* **[WB1]**–**[WB5]**, *there is a bicategory of fractions $\mathcal{B}(\mathfrak{W}^{-1})$ with a homomorphism*

$$J_{\mathfrak{W}} \colon \mathcal{B} \to \mathcal{B}(\mathfrak{W}^{-1})$$

which sends arrows in $\mathfrak{W}$ to internal equivalences. Moreover, this bicategory satisfies the following universal property: for any bicategory $\mathcal{D}$, composition with $J_{\mathfrak{W}}$ induces an equivalence of categories

$$\mathrm{Hom}(\mathcal{B}(\mathfrak{W}^{-1}), \mathcal{D}) \simeq \mathrm{Hom}_{\mathfrak{W}}(\mathcal{B}, \mathcal{D}),$$

where $\mathrm{Hom}(\mathcal{B}(\mathfrak{W}^{-1}), \mathcal{D})$ *denotes the category of homomorphisms and pseudo, resp. lax, resp. oplax, transformations and* $\mathrm{Hom}_{\mathfrak{W}}(\mathcal{B}, \mathcal{D})$ *denotes the category of homomorphisms and pseudo, resp. lax, resp. oplax transformations that send arrows in $\mathfrak{W}$ to internal equivalences.*

Remark 3.7. For further details on what it means for a transformation to send arrows to internal equivalences, see [4].

Proof. We have given definitions for all of the compositions. There are no coherence requirements on the choices of squares or liftings, so this gives a valid construction of a bicategory with all necessary properties. The resulting bicategory also has the same universal properties as the original bicategory of fractions, since the proof of [4, Theorem 21] does not depend on any specific properties of the choices made. □

A different way to derive this result will be given in Theorem 4.8.

4. Equivalences of Bicategories of Fractions

The first goal of this paper was to provide conditions under which we can take smaller classes of arrows to invert, while still obtaining an equivalent bicategory of fractions. In this section we develop a condition to allow us to restrict to a smaller subclass of arrows, namely when a subclass 'covers' the original class of arrows.

We show that if we start with a class of arrows satisfying **[WB1]**–**[WB5]**, and we have a covering subclass which satisfies **[WB1]** and **[WB5]**, then in fact the subclass will satisfy all the conditions**[WB1]**–**[WB5]** and the bicategory of fractions for the subclass is equivalent to the one for the original class of arrows. We will then apply this result to a class $\mathfrak{W}$ of arrows satisfying **[WB1]**–**[WB5]**, and consider its closure under composition and invertible 2-cells, $\widehat{\mathfrak{W}}$. We show that $\widehat{\mathfrak{W}}$ satisfies the conditions **BF1**–**BF5** of [4], and that $\mathfrak{W}$ covers $\widehat{\mathfrak{W}}$. This gives an equivalence of bicategories

$$\mathcal{B}(\mathfrak{W}^{-1}) \simeq \mathcal{B}(\widehat{\mathfrak{W}}^{-1}),$$

showing that the newly constructed bicategories of fractions of Section 3 are indeed equivalent to the ones introduced in [4].

4.1. Coverings of Arrows.

The following is our main condition on a subclass of a class of arrows.

Definition 4.1. Let $\mathfrak{W} \subseteq \mathfrak{V}$ be two classes of arrows in a bicategory $\mathcal{B}$. Then $\mathfrak{W}$ *covers* $\mathfrak{V}$ if for each arrow $v \in \mathfrak{V}$, there is an arrow u such that $vu \in \mathfrak{W}$.

We begin by verifying that some of the new bicategories of fractions conditions will descend from a class to a covering subclass.

Proposition 4.2. *Let $\mathcal{B}$ be a bicategory with a class of arrows $\mathfrak{V}$ satisfying all the conditions* [**WB1**]–[**WB5**]*, and a subclass $\mathfrak{W} \subseteq \mathfrak{V}$ which covers $\mathfrak{V}$ and satisfies conditions* [**WB1**] *and* [**WB5**]*. Then $\mathfrak{W}$ also satisfies conditions* [**WB2**]–[**WB4**].

Proof. [**WB2**] Let $A \xrightarrow{w_1} B$ and $B \xrightarrow{w_2} C$ be a pair of composable arrows in $\mathfrak{W}$. Since $\mathfrak{W} \subseteq \mathfrak{V}$ and $\mathfrak{V}$ satisfies condition [**WB2**], there is an arrow u_1 such that $w_2 w_1 u_1 \in \mathfrak{V}$. Since $\mathfrak{W}$ covers $\mathfrak{V}$, there is an arrow u_2 such that $w_2 w_1 u_1 u_2 \in \mathfrak{W}$. So $\mathfrak{W}$ satisfies condition [**WB2**].

[**WB3**] Consider a cospan of arrows $A \xrightarrow{f} C \xleftarrow{w} B$ with $w \in \mathfrak{W}$. Since $\mathfrak{V}$ satisfies [**WB3**], there is a square with an invertible 2-cell α,

$$\begin{array}{ccc} D & \xrightarrow{g} & B \\ {\scriptstyle v}\downarrow & \alpha & \downarrow{\scriptstyle w} \\ A & \xrightarrow[f]{} & C \end{array}$$

with $v \in \mathfrak{V}$. Since $\mathfrak{W}$ covers $\mathfrak{V}$, there is an arrow $\left(E \xrightarrow{u} D\right)$ such that $vu \in \mathfrak{W}$. Then the square

$$\begin{array}{ccc} E & \xrightarrow{gu} & B \\ {\scriptstyle vu}\downarrow & \alpha u & \downarrow{\scriptstyle w} \\ A & \xrightarrow[f]{} & C \end{array}$$

shows that $\mathfrak{W}$ satisfies condition [**WB3**].

[**WB4**] Let $\alpha \colon wf \Rightarrow wg$ be a 2-cell with $w \in \mathfrak{W}$. Since $w \in \mathfrak{V}$ and $\mathfrak{V}$ satisfies [**WB4**], there is an arrow $v \in \mathfrak{V}$ with a 2-cell $\beta \colon fv \Rightarrow gv$ such that $\alpha v = w\beta$. And since $\mathfrak{W}$ covers $\mathfrak{V}$, there is an arrow u such that $vu \in \mathfrak{W}$. Now take $w' = vu \in \mathfrak{W}$ and $\beta' = \beta u$. Then $w\beta' = \alpha w'$.

To check that $\mathfrak{W}$ also satisfies the second part of [**WB4**], let (w_1', β_1) and (w_2', β_2) be pairs such that $w_1', w_2' \in \mathfrak{W}$, and $\beta_1 \colon w_1' f \Rightarrow w_1' g$, $\beta_2 \colon w_2' f \Rightarrow w_2' g$ such that $\alpha w_1' = w\beta_1$ and $\alpha w_2' = w\beta_2$. Since $w, w_1', w_2' \in \mathfrak{V}$ and we assume that $\mathfrak{V}$ satisfies [**WB4**], there are arrows s, t such that $w_1' s, w_2' t \in \mathfrak{V}$, and invertible 2-cell $\varepsilon : w_1' s \Rightarrow w_2' t$ such that

$$\begin{array}{ccc} f w_1' s & \xrightarrow{\beta_1 s} & g w_1' s \\ {\scriptstyle f\varepsilon}\downarrow & & \downarrow{\scriptstyle g\varepsilon} \\ f w_2' t & \xrightarrow[\beta_2 t]{} & g w_2' t \end{array}$$

commutes. Since $w_1' s \in \mathfrak{V}$, there is an arrow u such that $w_1' s u \in \mathfrak{W}$. Then $w_2' t u \in \mathfrak{W}$ as well, since $\varepsilon u : w_1' s u \Rightarrow w_2' t u$ is an invertible 2-cell and $\mathfrak{W}$ is closed under invertible 2-cells by condition [**WB5**]. So define $s' = su$, $t' = tu$, and

$\varepsilon' = \varepsilon u \colon w_1' s' \overset{\sim}{\Rightarrow} w_2' t'$ to obtain a commutative diagram

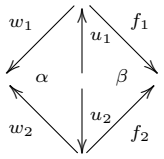
$$\begin{array}{ccc} f w_1' s' & \xrightarrow{\beta_1 s'} & g w_1' s' \\ {\scriptstyle f\varepsilon}\downarrow & & \downarrow{\scriptstyle g\varepsilon'} \\ f w_2' t' & \xrightarrow[\beta_2 t']{} & g w_2' t' \end{array}$$

as required. □

Theorem 4.3. *Let $\mathcal{B}$ be a bicategory with a class of arrows $\mathfrak{V}$ satisfying the conditions* **[WB1]**-**[WB5]** *and a class $\mathfrak{W} \subseteq \mathfrak{V}$ which covers $\mathfrak{W}$ and satisfies* **[WB1]** *and* **[WB5]**. *Then the induced bicategories of fractions are bi-equivalent:*

$$\mathcal{B}(\mathfrak{W}^{-1}) \simeq \mathcal{B}(\mathfrak{V}^{-1}).$$

Proof. By the universal property of $\mathcal{B}(\mathfrak{V}^{-1})$ there is a canonical pseudofunctor $J \colon \mathcal{B}(\mathfrak{W}^{-1}) \to \mathcal{B}(\mathfrak{V}^{-1})$, which is the identity on objects and sends the span (w, f) in $\mathcal{B}(\mathfrak{W}^{-1})$ to the span (w, f) in $\mathcal{B}(\mathfrak{V}^{-1})$. By making the appropriate choices of representatives, we may also assume that J maps the 2-cell represented by the diagram



in $\mathcal{B}(\mathfrak{W}^{-1})$ to the 2-cell represented by this same diagram in $\mathcal{B}(\mathfrak{V}^{-1})$.

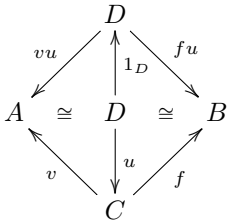
It is obvious that J is an isomorphism on objects. To show that it is essentially surjective on arrows, let

$$A \xleftarrow{v} C \xrightarrow{f} B$$

be an arrow in $\mathcal{B}(\mathfrak{V}^{-1})$. Since $\mathfrak{W}$ covers $\mathfrak{V}$, there is an arrow $\left(D \xrightarrow{u} C\right)$ such that $vu \in \mathfrak{W}$. So the span

$$A \xleftarrow{vu} D \xrightarrow{fu} B$$

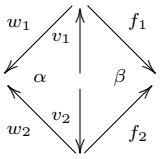
is in the image of J. Furthermore, there is an invertible 2-cell



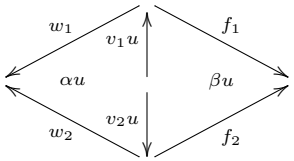
showing that J is essentially surjective on arrows.

It remains to show that J is fully faithful on 2-cells. To show that it is full on 2-cells, consider the 2-cell represented by the diagram,

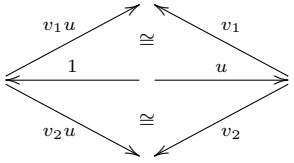
(6)



with $w_1, w_2 \in \mathfrak{W}$ and $w_1v_1, w_2v_2 \in \mathfrak{V}$. Since $\mathfrak{W}$ covers $\mathfrak{V}$, there is an arrow u such that $w_1vu \in \mathfrak{W}$. Since $\alpha u\colon w_1v_1u \Rightarrow w_2v_2u$ is invertible, it follows that $w_2v_2u \in \mathfrak{W}$ as well. Hence, the 2-cell represented by



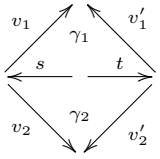
is in the image of J. This diagram represents the same 2-cell as (6), since the following gives an equivalence between them:



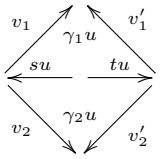
To verify that J is faithful on 2-cells, consider two 2-cells between the same spans of arrows

(7)

and suppose that these diagrams represent the same 2-cell in $\mathcal{B}(\mathfrak{V}^{-1})$. This means that there is an equivalence given by arrows s and t with 2-cells γ_1 and γ_2 as in



such that the appropriate diagrams of 2-cells commute and $w_1v_1s \in \mathfrak{V}$. Since $\mathfrak{W}$ covers $\mathfrak{V}$, there is an arrow u such that $w_1v_1su \in \mathfrak{W}$. So the diagram



represents an equivalence of the diagrams in (7) in $\mathcal{B}(\mathfrak{W}^{-1})$. We conclude that J is fully faithful on 2-cells, and hence is a biequivalence of bicategories. □

Remark 4.4. This theorem implies that the choices made in constructing the bicategory of fractions in Section 3 do not matter, since $\mathfrak{W}$ is a cover of itself, and Theorem 4.3 provides an equivalence of bicategories created with different choices.

Remark 4.5. Our notion of a covering of a class of arrows is a dual notion to that of the right saturation of a class of arrows defined in [10]. The right saturation enlarges the class of arrows to be inverted, rather than restricting to a smaller subclass.

The right saturation of a class $\mathcal{W}$ of arrows consists of those arrows $f\colon C \to D$ for which there exist arrows $g\colon B \to C$ and $h\colon A \to B$ such that gh and fg are both in $\mathcal{W}$. If $\mathcal{W}$ satisfies the conditions **BF1**-**BF5**, then so does its saturation, and the saturation gives rise to an equivalent bicategory of fractions. It is not difficult to use [**WB3**] to show that if $\mathcal{W} \subseteq \mathcal{V}$ covers $\mathcal{V}$, then $\mathcal{V}$ is a subset of the saturation of $\mathcal{W}$. This does not immediately imply the equivalence of the induced bicategories of fraction, because $\mathcal{W}$ may not satisfy **BF2**. However, Theorem 4.3 implies that the equivalences of bicategories of fractions in [10] apply when we replace **BF2** with [**WB2**].

Remark 4.6. In the case where one is only interested is obtaining a smaller version of $\mathcal{B}(\mathfrak{W}^{-1})(X,Y)$ for certain objects X in the bicategory $\mathcal{B}$, there is a local version of Theorem 4.3. Given an object X in $\mathcal{B}$ and a class of arrows $\mathfrak{V}$ in $\mathcal{B}$, we say that a subclass $\mathfrak{W} \subseteq \mathfrak{V}$ *covers* $\mathfrak{V}$ *at* X when the class $\mathfrak{W}/X$ of arrows in $\mathfrak{W}$ with codomain X covers the class $\mathfrak{V}/X$ of arrows in $\mathfrak{V}$ with codomain X. We write $\mathcal{B}_{\mathfrak{W}}(X,Y)$ for the category for spans from X to Y with reverse arrows in $\mathfrak{W}$ and 2-cells as defined in bicategory of fractions for $\mathfrak{W}$. Now, if $\mathfrak{V}$ satisfies conditions [**WB1**]–[**WB5**] and $\mathfrak{W} \subseteq \mathfrak{V}$ satisfy condition [**WB5**], there is an equivalence of categories

$$\mathcal{B}_{\mathfrak{W}}(X,Y) \xrightarrow{\sim} \mathcal{B}(\mathfrak{V}^{-1})(X,Y),$$

for any object Y in $\mathcal{B}$.

4.2. **Closure Under Composition.** Given a class of arrows $\mathfrak{W}$ in a bicategory $\mathcal{B}$, let $\widehat{\mathfrak{W}}$ denote the class obtained from $\mathfrak{W}$ by closure under composition and invertible 2-cells. So $\widehat{\mathfrak{W}}$ is the smallest class of arrows in $\mathcal{B}$ such that

- $\mathfrak{W} \subseteq \widehat{\mathfrak{W}}$;
- If $f_1, f_2 \in \widehat{\mathfrak{W}}$, and $f_2 \circ f_1$ is defined, then $f_2 \circ f_1 \in \widehat{\mathfrak{W}}$;
- If $f \in \widehat{\mathfrak{W}}$ and $\alpha\colon f \xRightarrow{\sim} g$ is an invertible 2-cell in $\mathcal{B}$, then $g \in \widehat{\mathfrak{W}}$.

Then each arrow $w \in \widehat{\mathfrak{W}}$ will have an invertible 2-cell $\alpha : w \xRightarrow{\sim} w_n \circ \cdots \circ w_1$ with codomain a finite composite of arrows $w_1, \ldots, w_n \in \mathfrak{W}$.

Lemma 4.7. *If $\mathfrak{W}$ satisfies the conditions* **[WB1]**–**[WB5]**, *then $\widehat{\mathfrak{W}}$ defines a wide subcategory which satisfies the conditions from* [4] *for constructing a bicategory of fractions.*

Proof. Since $\mathfrak{W}$ contains all identities, so does $\widehat{\mathfrak{W}}$, so $\widehat{\mathfrak{W}}$ satisfies condition **BF1** from [4]. And $\widehat{\mathfrak{W}}$ has been created to be closed under composition, verifying **BF2**. Conditions **BF3**–**BF5** are equivalent to conditions **[WB3]**–**[WB5]** (and **BF3** and **BF5** are identical to their weaker versions); see Remark 2.1. So it suffices to check conditions **[WB3]**–**[WB5]**.

Since every arrow in $\widehat{\mathfrak{W}}$ is isomorphic to a composition of finitely many arrows in $\mathfrak{W}$, repeated application of **[WB3]** for $\mathfrak{W}$ gives us **[WB3]** for $\widehat{\mathfrak{W}}$.

To verify condition **[WB4]**, suppose that $\alpha\colon wf \Rightarrow wg$ and $\gamma\colon w_n \cdots w_1 \xRightarrow{\sim} w$ with $w_1, \ldots, w_n \in \mathfrak{W}$. Repeatedly applying **[WB4]** for $\mathfrak{W}$ gives us arrows w_{n-k} and 2-cells $\beta_{n-k}\colon w_{n-k-1}\cdots w_1 f w'_n \cdots w'_{n-k} \Rightarrow w_{n-k-1}\cdots w_1 g w'_n \cdots w'_{n-k}$ for $k = 0, \ldots, n-1$ such that $w_{n-k}\cdots w_{n-1}w_n\beta_{n-k} = ((\gamma^{-1}f)\cdot\alpha\cdot(\gamma f))w_n 1 w'_{n-1}\cdots w'_{n-k}$. So β_1 with $w'_n w'_{n-1}\cdots w'_1$ is the required lifting.

To check the compatibility condition in **[WB4]**, consider $\alpha\colon wf \Rightarrow wg$ with liftings $\alpha'\colon fw' \Rightarrow gw'$ and $\alpha''\colon fw'' \Rightarrow gw''$. Since $w', w'' \in \widehat{\mathfrak{W}}$, there are arrows $w'_1, \ldots, w'_k, w''_1, \ldots, w''_\ell$ in $\mathfrak{W}$ with invertible 2-cells, $\delta\colon w'_k \cdots w'_1 \Rightarrow w'$ and $\gamma\colon w''_\ell \cdots w''_1 \Rightarrow w''$. By repeatedly applying condition **[WB2]** for $\mathfrak{W}$ there are arrows u', u'' such that $w'u' \in \mathfrak{W}$ and $w''u'' \in \mathfrak{W}$. Hence we can apply **[WB4]** for $\mathfrak{W}$ to the liftings $\alpha' u'\colon fw'u' \Rightarrow gw'u'$ and $\alpha'' u''\colon fw''u'' \Rightarrow gw''u''$ and obtain arrows s, t and a 2-cell $\varepsilon\colon w'u's \Rightarrow w''u''t$ showing compatibility of these liftings. This then gives us also the required arrows $u's$ and $u''t$ with the cell ε to establish compatibility for the original liftings.

Finally, $\widehat{\mathfrak{W}}$ satisfies condition **BF5** by construction. □

Theorem 4.8. *If $\mathfrak{W}$ satisfies the conditions* **[WB1]**–**[WB5]**, *then $\mathcal{B}(\mathfrak{W}^{-1}) \simeq \mathcal{B}(\widehat{\mathfrak{W}}^{-1})$, where $\mathcal{B}(\widehat{\mathfrak{W}}^{-1})$ is the bicategory of fractions from* [4] *and $\mathcal{B}(\mathfrak{W}^{-1})$ is the bicategory of fractions defined in Section 3.*

Proof. We have shown that whenever a class of arrows $\mathfrak{W}$ satisfies the stronger conditions **BF1-BF5**, the resulting bicategory of fractions is equivalent to the traditional one from [4]; see Remarks 3.5(1) and 4.4. So $\mathcal{B}(\widehat{\mathfrak{W}}^{-1})$ may be taken to be the classical bicategory of fractions and Theorem 4.3 now gives us the equivalence of the resulting bicategories of fractions. □

Corollary 4.9. *When $\mathfrak{W}$ satisfies the conditions* **[WB1]**–**[WB5]**, *the inclusion pseudo functor $J_{\mathfrak{W}}\colon \mathcal{B} \to \mathcal{B}(\mathfrak{W}^{-1})$ satisfies the universal property for the bicategory of fractions.*

Proof. A pseudo functor $\mathcal{B} \to \mathcal{C}$ sends the arrows in $\mathfrak{W}$ to equivalences if and only if it sends the arrows in $\widehat{\mathfrak{W}}$ to equivalences. □

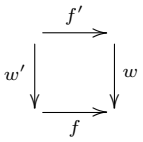
This result also applies to results for categories of fractions in the 1-category case given in [2].

Corollary 4.10. *A class of arrows W in a category $\mathcal{C}$ allows for the construction of a category of right fractions $\mathcal{C}[W^{-1}]$ if it satisfies the following conditions:*

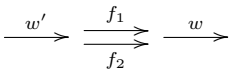
(1) *W contains all identities;*

(2) *For any pair of composable arrows* $B \xrightarrow{v} C \xrightarrow{w} D$ *in* W *there is an arrow* $A \xrightarrow{u} B$ *such that* $A \xrightarrow{wvu} D$ *is in* W*;*

(3) *For any arrow* $w \in W$ *and any arrow* f *which shares its codomain with* w*, there is an arrow* $w' \in W$ *and an arrow* f' *such that the following square is defined and commutes:*



(4) *Given* $w \in W$ *and parallel arrows* f_1, f_2 *such that* $wf_1 = wf_2$*, then there is an arrow* $w' \in W$ *such that* $f_1 w' = f_2 w'$*,*

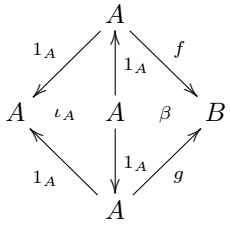


Examples 4.11. (1) When one wants to add the inverse for an arrow w in a monoid, the class W in the traditional Gabriel-Zisman construction of [2] would be required to contain all powers of w. In our case W only needs to contain a cofinal set of powers of w.

(2) Consider the category of atlases and atlas maps for manifolds. In order to obtain the category containing all smooth maps between manifolds using the original conditions, one needs takes the category of fractions with respect to the atlas refinements. With the new theory we may restrict ourselves to refinements in which no charts are repeated, or any other family of refinements that satisfies condition **[WB2]**.

5. Simplifying 2-Cell Representatives

The universal homomorphism $J_{\mathfrak{W}} \colon \mathcal{B} \to \mathcal{B}(\mathfrak{W}^{-1})$ is defined by the identity on objects, and takes an arrow $f \colon A \to B$ to the generalized arrow $A \xleftarrow{1_A} A \xrightarrow{f} B$ and a 2-cell $\alpha \colon f \Rightarrow g$ to a 2-cell diagram of the form below.



As Tommasini observed in Remark 3.5 of [9], this universal homomorphism is neither 2-full nor 2-faithful in general. The map $J_{\mathfrak{W}}$ fails to be 2-full because not every 2-cell between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$ needs to have a representative of this particular form. The map $J_{\mathfrak{W}}$ fails to be 2-faithful because two 2-cell diagrams of this form, say with distinct right cells β and γ, could represent the same 2-cell in the bicategory of fractions when there is an arrow $t \in \mathfrak{W}$ such that $\beta t = \gamma t$. This leads us to consider the more general issue of the equivalence relation on the 2-cell diagrams.

In this section we discuss some variations of **[WB4]** and consider when a 2-cell in the bicategory of fractions can be represented by a 2-cell diagram with a given

left-hand side. In the following section, we will look at choosing these left-hand sides to have nice additional properties that will simplify some of the composition constructions. In some cases representatives with a given left-hand side will even be unique. We will prove in [6] that some of these properties hold for the case of essential equivalences between orbifold étale groupoids. In fact they apply more generally to any fully faithful maps between étale topological groupoids.

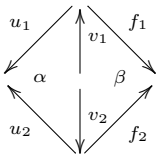
The following properties will be required in various parts of this section.

Definition 5.1. A class $\mathfrak{W}$ of arrows in a bicategory $\mathcal{B}$ satisfies the condition

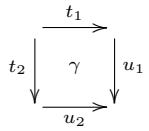
- **[UWB4]:** (unitary **[WB4]**) if for any 2-cell $\alpha\colon wf \Rightarrow wg$ with $w \in \mathfrak{W}$ there is a 2-cell $\tilde{\alpha}\colon f \Rightarrow g$ such that $w\tilde{\alpha} = \alpha$.
- **[UUWB4]:** if for any 2-cell $\alpha\colon wf \Rightarrow wg$ with $w \in \mathfrak{W}$ there is a *unique* 2-cell $\tilde{\alpha}\colon f \Rightarrow g$ such that $w\tilde{\alpha} = \alpha$.
- **[co-UWB4]:** if for any 2-cell $\alpha\colon fw \Rightarrow gw$ with $w \in \mathfrak{W}$ there is a 2-cell $\alpha'\colon f \Rightarrow g$ such that $\alpha' w = \alpha$.
- **[co-UUWB4]:** if for any 2-cell $\alpha\colon fw \Rightarrow gw$ with $w \in \mathfrak{W}$ there is a *unique* 2-cell $\alpha'\colon f \Rightarrow g$ such that $\alpha' w = \alpha$.

Lemma 5.2. *Let $\mathfrak{W}$ be a class of arrows in $\mathcal{B}$ satisfying the conditions* **[WB1]**–**[WB5]** *and* **[co-UWB4]**. *Given any 2-cell diagram*

(8)

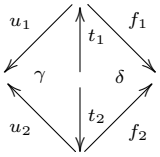


in $\mathcal{B}(\mathfrak{W}^{-1})$ and any square



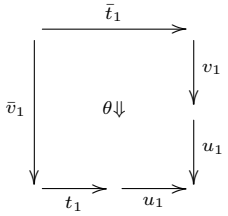
in $\mathcal{B}$ with $u_i t_i \in \mathfrak{W}$ for $i = 1, 2$, there is a 2-cell δ such that the diagram

(9)

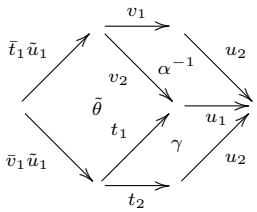


represents the same 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ as (8).

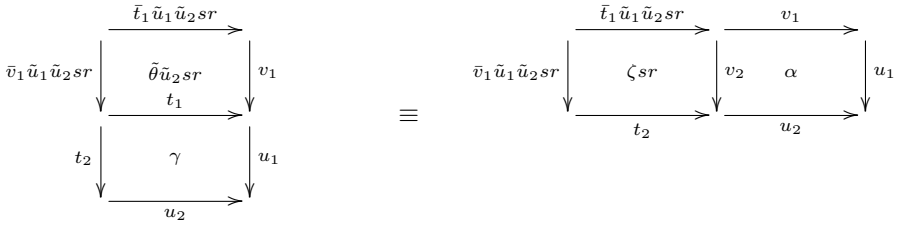
Proof. By **[WB3]** there is a square



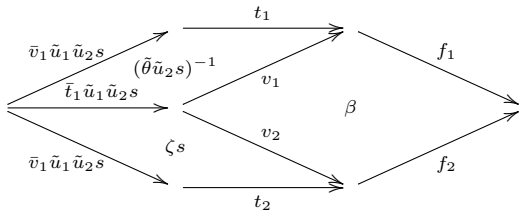
with $\bar{v}_1 \in \mathfrak{W}$ and θ invertible. By **[WB4]** there is an arrow $\tilde{u}_1 \in \mathfrak{W}$ and an invertible 2-cell $\tilde{\theta}\colon (v_1\bar{t}_1)\tilde{u}_1 \Rightarrow (t_1\bar{v}_1)\tilde{u}_1$. Now consider the pasting of the diagram



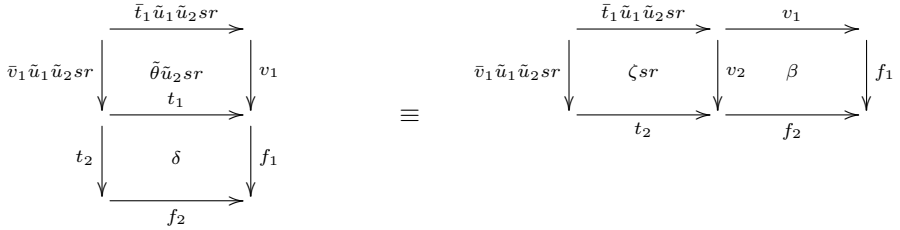
By **[WB4]** there is an arrow $\tilde{u}_2 \in \mathfrak{W}$ with an invertible 2-cell $\zeta\colon (v_2(\bar{t}_1\tilde{u}_1))\tilde{u}_2 \Rightarrow (t_2(\bar{v}_1\tilde{u}_1))\tilde{u}_2$. Finally, we need to ensure that certain compositions of arrows are in $\mathfrak{W}$. First consider the composition of arrows $\bar{v}_1\tilde{u}_1\tilde{u}_2$. Each of the three arrows in this composition is in $\mathfrak{W}$, so by **[WB2]** there is an arrow s such that $\bar{v}_1\tilde{u}_1\tilde{u}_2 s \in \mathfrak{W}$. Furthermore, $u_2t_2 \in \mathfrak{W}$ as well, so there is an arrow r such that $(u_2t_2)(\bar{v}_1\tilde{u}_1\tilde{u}_2 s)r \in \mathfrak{W}$. Then we have the following equality of pastings of 2-cells:



We want to construct a cell δ such that β and δ fit into a similar equality of 2-cell pastings. So consider the following pasting diagram,



By condition **[co-UWB4]** there is a 2-cell $\delta\colon f_1t_1 \Rightarrow f_2t_2$ such that $\delta\bar{v}_1\tilde{u}_1\tilde{u}_2 s$ is equal to the pasting of this diagram. Then we get that



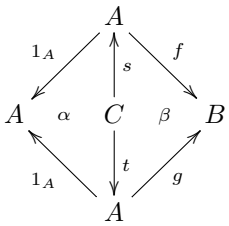
and hence we conclude that with δ thus defined, (9) is equivalent to (8). □

One might hope that adding condition **[co-UUWB4]** to the conditions of Lemma 5.2 would imply uniqueness of the 2-cell δ, or in general, that the **[WB1]**–**[WB5]** conditions together with **[co-UUWB4]** imply that a given 2-cell has at most one diagram with a given left hand side. Unfortunately, this is not the case, because we

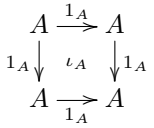
do not require that the arrows s and t are in $\mathfrak{W}$ in the definition of the equivalence relation on 2-cell diagrams, and so **[co-UUWB4]** does not apply where we need it. However, they will be in $\mathfrak{W}$ if the left hand arrows in the spans are identity arrows. Hence, we obtain the uniqueness result for cells between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$. So we have the following result.

Proposition 5.3. *Suppose that $\mathfrak{W}$ be a class of arrows in a bicategory $\mathcal{B}$ satisfying conditions* **[WB1]**–**[WB5]** *and* **[co-UUWB4]**. *Then the universal homomorphism $U_{\mathfrak{W}}\colon \mathcal{B} \to \mathcal{B}(\mathfrak{W}^{-1})$ is 2-full and 2-faithful.*

Proof. To show that the homomorphism is 2-full, consider an arbitrary 2-cell between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$. This will have a representative of the form



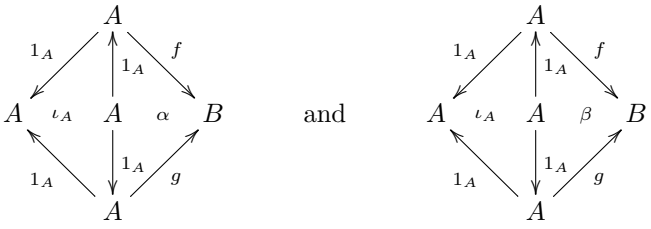
Note that there is a square



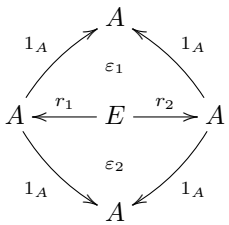
and Lemma 5.2 says that we can represent the 2-cell between $J_{\mathfrak{W}}(f)$ and $J_{\mathfrak{W}}(g)$ using this square on the left side. Thus, the 2-cell is the image of a 2-cell in $\mathcal{B}$.

To show that the map $J_{\mathfrak{W}}$ is 2-faithful, suppose that we have two 2-cells $J_{\mathfrak{W}}(\alpha)$ and $J_{\mathfrak{W}}(\beta)$, represented by

(10)



which represent the same 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$. Then there must be maps $r_1, r_2\colon E \rightrightarrows A$ with 2-cells ε_1, ε_2 as in



satisfying the equations to make the two diagrams in (10) equivalent. Since the left hand squares are just identities, this implies that $\varepsilon_1 = \varepsilon_2\colon r_1 \Rightarrow r_2$. The other

equation then implies that $\alpha \circ \varepsilon_1 = \beta \circ \varepsilon_1$. Since ε_1 is invertible, this implies that $\alpha r_1 = \beta r_1$.

Now note that $1_A r_1 \in \mathfrak{W}$, so $r_1 \in \mathfrak{W}$. Hence, **[co-UUWB4]** implies that there is a unique $\gamma\colon f \Rightarrow g$ such that $\gamma r_1 = \alpha r_1$. Hence, $\alpha = \beta$. □

We also have the following uniqueness result.

Theorem 5.4. *Let $\mathfrak{W}$ be a class of arrows in a bicategory $\mathcal{B}$ satisfying conditions* **[WB1]**–**[WB5]** *and* **[co-UUWB4]** *and the 3-for-2 property. Then each 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ has a unique representative with a given left-hand 2-cell.*

Proof. In the proof of Proposition 5.3, uniqueness followed because $r_1 \in \mathfrak{W}$ allowed us to apply the **[co-UUWB4]** condition. In this case, the 2-for-3 condition says that if u_1 and $u_1 r_1$ are both in $\mathfrak{W}$, then r_1 is also in $\mathfrak{W}$. So the same argument applies.

□

6. Bicategories with Pseudo Pullbacks

We now apply the ideas of Section 5 to representing generalized 2-cells using pseudo-pullbacks. If a bicategory has all pseudo pullbacks of the form

$$\begin{array}{ccc} P & \xrightarrow{\bar{w}} & \\ {\scriptstyle \bar{f}}\downarrow & \rho & \downarrow{\scriptstyle f} \\ & \xrightarrow[w]{} & \end{array}$$

where $w \in \mathfrak{W}$, and the class $\mathfrak{W}$ is stable under these pseudo pullbacks in the sense that $w \in \mathfrak{W}$ implies that $\bar{w} \in \mathfrak{W}$, it is possible to use the pseudo pullbacks as chosen squares as in **[C2]** of Notation 3.2 in the construction of $\mathcal{B}(\mathfrak{W}^{-1})$. This makes the construction of this bicategory more canonical; see [10] for instance.

We are interested in a different use of the pseudo pullbacks: as the left-hand sides of the generalized 2-cell diagrams. This will allow us to simplify the horizontal composition operations. It will require some additional assumptions on $\mathcal{B}$, so we will develop conditions under which each 2-cell has a representative diagram where α is a pseudo pullback. The first condition is the following.

Definition 6.1. We say that $\mathfrak{W}$ is *pullback closed* if for any pseudo pullback

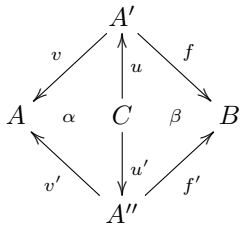
$$\begin{array}{ccc} P & \xrightarrow{\bar{u}} & B \\ {\scriptstyle \bar{v}}\downarrow & \rho & \downarrow{\scriptstyle v} \\ A & \xrightarrow[u]{} & C \end{array}$$

with arrows $u, v \in \mathfrak{W}$, the composite $u\bar{v}$ is again in $\mathfrak{W}$.

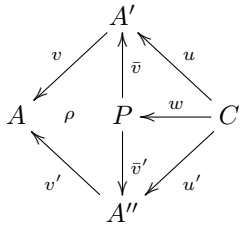
Since ρ is invertible, **[WB5]** will imply that $v\bar{u} \in \mathfrak{W}$ as well.

Proposition 6.2. *If $\mathcal{B}$ has all pullbacks for cospans with at least one leg in $\mathfrak{W}$, and $\mathfrak{W}$ satisfies conditions* **[WB1]**–**[WB5]** *together with* **[co-UWB4]** *and is pullback closed, then each 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ has a representative with the left hand 2-cell a pseudo pullback.*

Proof. For any 2-cell diagram,



we have an induced universal arrow



such that $\rho w = \alpha$. Since $\mathfrak{W}$ is pullback closed, $\bar{v}v$ and $\bar{v}'v'$ are in $\mathfrak{W}$, and so Lemma 5.2 shows that we can represent the 2-cell using the pseudo pullback.

□

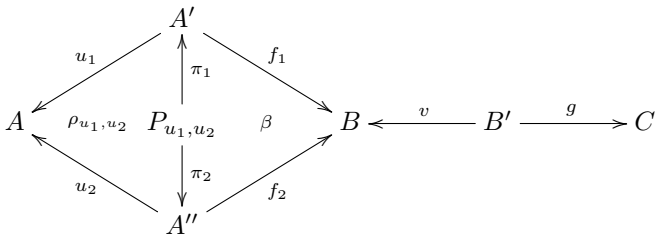
Moreover, the argument from Thereom 5.4 gives the following.

Proposition 6.3. *If $\mathfrak{W}$ satisfies conditions* **[WB1]**–**[WB5]** *and* **[co-UUWB4]**, *and is pullback closed and satisfies the 3-for-2 condition, the representation of a 2-cell is unique.*

Vertical composition of 2-cells is not simplified by taking representatives with pseudo pullbacks. In fact it is slightly complicated, since we need to calculate the vertical composition of the 2-cell diagrams and then construct an equivalent 2-cell diagram that has the pseudo pullback on the left-hand side, using the lifting as in the proof of Lemma 5.2. However, the horizontal whiskering operations can be significantly simplified by using pseudo pullbacks, as we show in the following two subsections.

6.1. **Left Whiskering With Pullbacks.** Throughout this subsection, we will assume that $\mathfrak{W}$ satisfies all conditions of Proposition 6.2: conditions **[WB1]**–**[WB5]**, pullback closed, and **[co-UWB4]**. We consider whiskering of the form

(11)



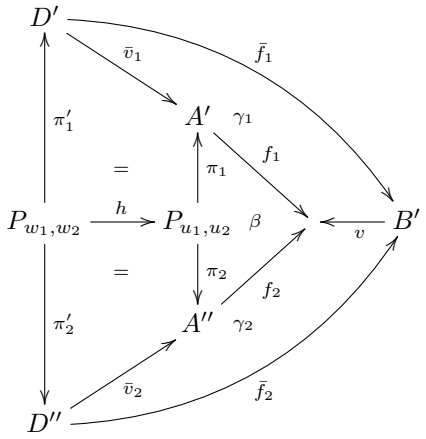
We construct the composition of the 1-cells using chosen squares γ_1 and γ_2 as in Section 3.1,

$$\begin{array}{ccc} D' & \xrightarrow{\bar{f}_1} & B' \\ {\scriptstyle \bar{v}_1}\downarrow & \gamma_1 & \downarrow{\scriptstyle v} \\ A' & \xrightarrow[f_1]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} D'' & \xrightarrow{\bar{f}_2} & B' \\ {\scriptstyle \bar{v}_2}\downarrow & \gamma_2 & \downarrow{\scriptstyle v} \\ A'' & \xrightarrow[f_2]{} & B \end{array}$$

such that $w_1 := u_1\bar{v}_1$ and $w_2 := u_2\bar{v}_2$ are in $\mathfrak{W}$. Let

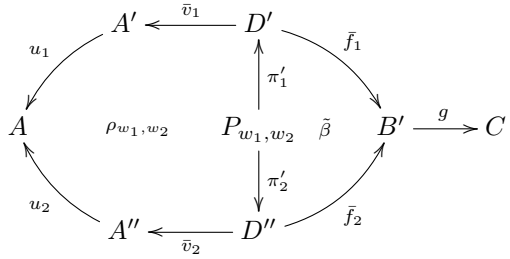
$$\begin{array}{ccc} P_{w_1,w_2} & \xrightarrow{\pi_1'} & D' \\ {\scriptstyle \pi_2'}\downarrow & \rho_{w_1,w_2} & \downarrow{\scriptstyle w_1=u_1\bar{v}_1} \\ D'' & \xrightarrow[w_2=u_2\bar{v}_2]{} & A \end{array}$$

be the pseudo pullback. Then there is a unique arrow $h\colon P_{w_1,w_2} \to P_{u_1,u_2}$ such that $\pi_1 h = \bar{v}_1\pi_1'$, $\pi_2 h = \bar{v}_2\pi_2'$ and $\rho_{u_1,u_2}h = \rho_{w_1,w_2}$. Finally, let $\tilde{\beta}$ be the lifting of the diagram,



with respect to v. Then the result of whiskering as in (11) is given by

(12)

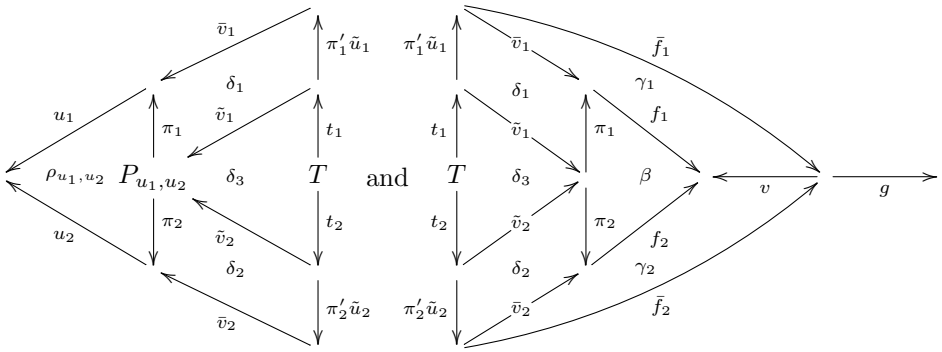


Lemma 6.4. *Diagram* (12) *is equivalent to the diagram* (3) *obtained for this type of whiskering in Section 3.1.*

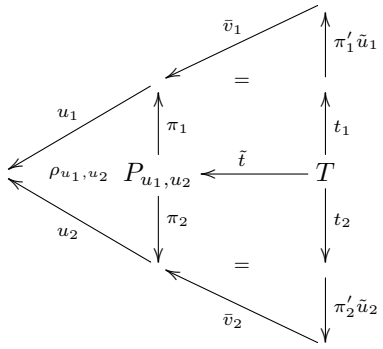
Proof. It was shown in [9] that any choice of the squares and liftings in the composition construction of Section 3.1 give equivalent 2-cell diagrams as long as we use the composition squares from **[C2]** of Notation 3.2 for the composition of the 1-cells and the squares have the right properties. The only place where the chosen

squares are essential is in the composition of the 1-cells, so with the exception of the cells γ_1 and γ_2 we can replace all cells used in the whiskering algorithm from Section 3.1 with cells and squares we have just constructed above. So we will redo the construction from Section 3.1 and use the universal properties of the pseudo pullbacks to adjust the squares to obtain a 2-cell diagram that is clearly equivalent to (12).

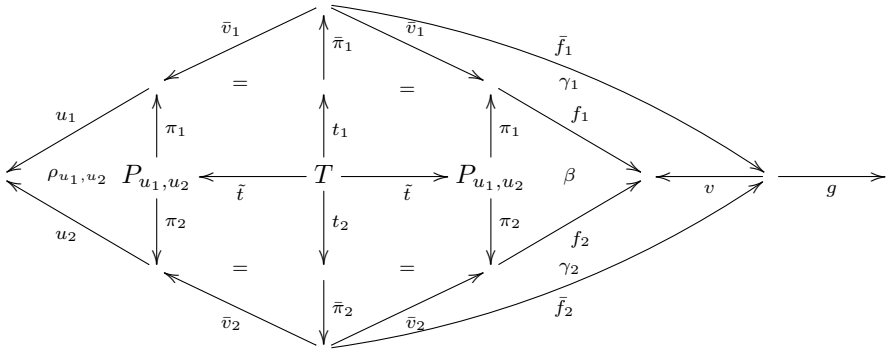
Recall that in Section 3.1 we used chosen squares δ_1, δ_2 and δ_3 to obtain diagrams (13)



By the universal property of the pseudo pullback there is an arrow $\tilde{t}\colon T \to P_{u_1,u_2}$ such that the following diagram pastes to the same 2-cell as the first diagram in (13),



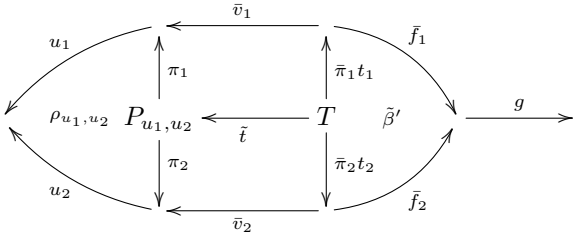
We now replace the chosen squares δ_1, δ_2 by the new commuting squares in this diagram and let $\delta_3 = \mathrm{id}_{\tilde{t}}$. We obtain the following diagram,



This is almost a 2-cell diagram: we just need to take a lifting $\tilde{\beta}'$ of the right-hand side with respect to v as in **[co-UWB4]**.

To show that the resulting 2-cell,

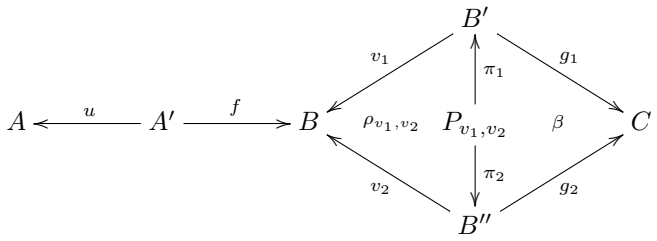
(14)



is equivalent to (12), note that there is a unique arrow $t' \colon T \to P_{w_1,w_2}$ such that $\rho_{w_1,w_2} t' = \rho_{u_1,u_2} \tilde{t}$. Now $\tilde{\beta} t'$ is another lifting of the right-hand side in (11), so the diagrams with $\tilde{\beta}'$ and $\tilde{\beta} t'$ on the left-hand side are equivalent. Hence, (12) and (14) are equivalent. □

6.2. **Right Whiskering With Pullbacks.** Throughout this section, we will assume all conditions of Proposition 6.2: conditions **[WB1]**–**[WB5]**, pullback closed, and **[co-UWB4]**. We additionally assume that $\mathfrak{W}$ satisfies the dual **[UWB4]** condition. We now consider the composition

(15)



where P_{v_1,v_2} is the pseudo pullback of v_1 and v_2. First we construct the composition of the 1-cells using chosen squares **[C2]**

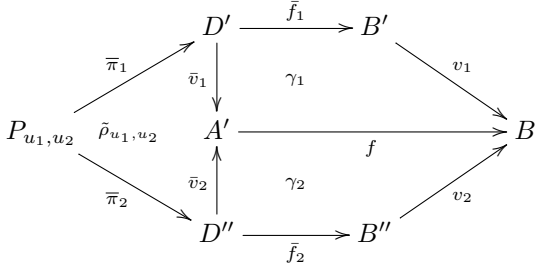
$$\begin{array}{ccc} D' & \xrightarrow{\bar{f}_1} & B' \\ {\scriptstyle \bar{v}_1}\downarrow & \gamma_1 & \downarrow{\scriptstyle v_1} \\ A' & \xrightarrow[f]{} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} D'' & \xrightarrow{\bar{f}_2} & B'' \\ {\scriptstyle \bar{v}_2}\downarrow & \gamma_2 & \downarrow{\scriptstyle v_2} \\ A' & \xrightarrow[f]{} & B \end{array}$$

such that $u_1 := u\bar{v}_1$ and $u_2 := u\bar{v}_2$ are in $\mathfrak{W}$ as in Section 3.2. Let

$$\begin{array}{ccc} P_{u_1,u_2} & \xrightarrow{\overline{\pi}_1} & D' \\ {\scriptstyle \overline{\pi}_2}\downarrow & \rho_{u_1,u_2} & \downarrow{\scriptstyle u_1} \\ D'' & \xrightarrow[u_2]{} & A \end{array}$$

be the pseudo pullback of u_1 and u_2. Note that $\rho_{u_1,u_2}\colon u\bar{v_1}\overline{\pi}_1 \Rightarrow u\bar{v}_2\overline{\pi}_2$. By **[UWB4]**, there is a lifting $\tilde{\rho}_{u_1,u_2}\colon \bar{v}_1\overline{\pi}_1 \Rightarrow \bar{v}_2\overline{\pi}_2$. This cell can be pasted with γ_1 and γ_2 to form

(16)

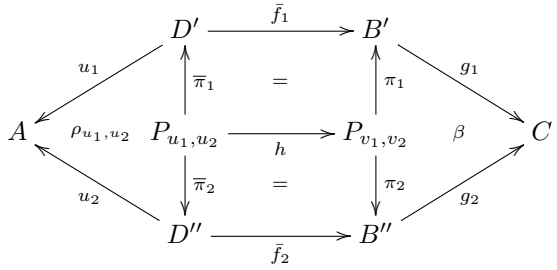


By the universal property of the pseudo pullback P_{v_1,v_2}, there is a unique arrow

(17) $$h\colon P_{u_1,u_2} \to P_{v_1,v_2} \text{ such that } \pi_1 h = \bar{f}_1\pi'_1 \text{ and } \pi_2 h = \bar{f}_2\pi'_2$$

and furthermore, $\rho_{v_1,v_2}h$ is equal to the pasting of (16). We claim that the following 2-cell diagram represents the result of whiskering (15):

(18)

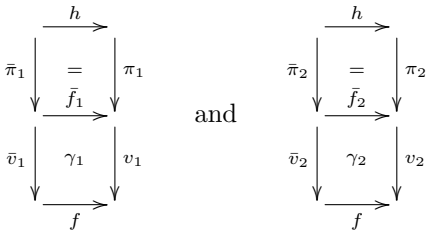


Lemma 6.5. *Diagram (18) is equivalent to the diagram (5) obtained for this type of whiskering in Section 3.2.*

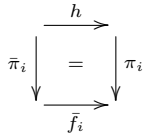
Proof. Again, we use the results from [9] [Section 4] that the equivalence classes of the resulting 2-cell diagrams in the whiskering constructions and vertical composition construction do not depend on the choice of the squares and liftings used as long as we use the chosen composition of 1-cells and the appropriate arrows are in

$\mathfrak{W}$. We will now go through the algorithm of Section 3.2 and substitute the cells above. We will show that the result is precisely (18).

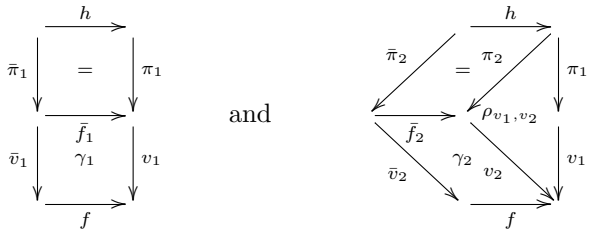
In (4), we take for δ_1 and δ_2 respectively,



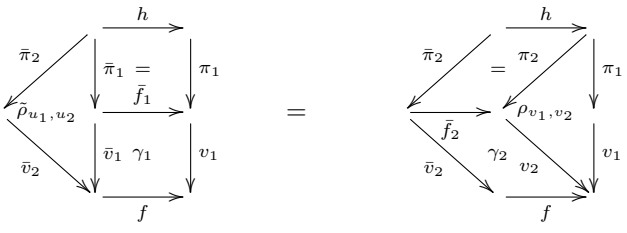
This allows us to take r_1 and r_2 to be identity arrows and $t_i = \bar{\pi}_i$ for $i = 1, 2$. Furthermore, φ_i is given by



and $\varepsilon_i = \mathrm{id}_{\bar{v}_i \bar{\pi}_i}$, for $i = 1, 2$. The next step is then to compare the pastings,



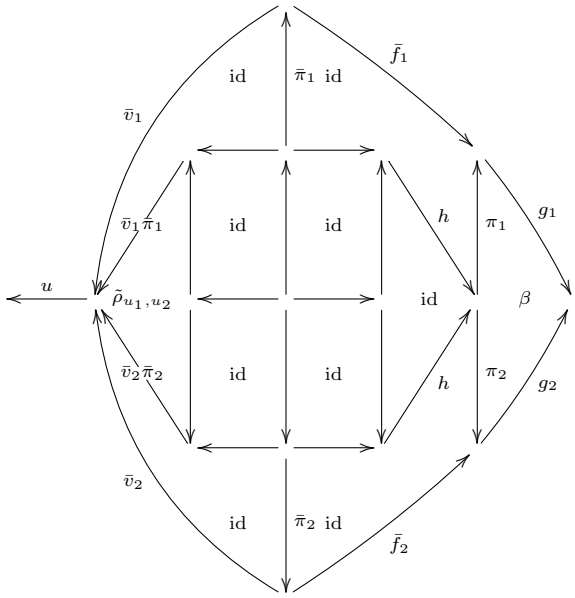
Here we may choose p and q to be identity arrows, $\tau = \mathrm{id}_h$ and $\tilde{\alpha} = \tilde{\rho}_{u_1,u_2}$, since



by (16) and (17).

Omitting the identity coherence cells, the resulting 2-cell diagram is

(19)



where all unlabeled arrows are identity arrows. Composing the cells in both the left-hand side and the right-hand side of this diagram gives us the 2-cell diagram in (18) as required. □

7. Future Directions: An Application to Orbifolds

In this section, we briefly sketch how the results in this paper apply to the bicategory of orbigroupoids. Details will be given in [6]; here we only give an overview.

One way to define orbifolds is by using the 2-category of orbigroupoids: étale groupoids internal to a category of suitable topological spaces, such as topological manifolds or some more general category of spaces. Then we consider the class of essential equivalences, maps that are categorical equivalences internal to the topological category chosen: they satisfy a suitably topologized version of being essentially surjective and fully faithful. For more details, see [1, 3]. We define orbifolds as the bicategory of fractions of orbigroupoids with respect to the class of essential equivalences. The class of essential equivalences satisfies the 3-for-2 condition [5], are pullback closed as in Definition 6.1, and satisfy the **BF** conditions from [4] and the new conditions **[UUWB4]** and **[Co-UUWB4]**. Thus, we can apply the results of this paper to get the following:

Theorem 7.1. (1) *The universal map from the 2-category of orbigroupoids to its bicategory of fractions with respect to the class $\mathfrak{W}$ of essential equivalences,*

$$J_{\mathfrak{W}} \colon \mathbf{OrbiGroupoids} \longrightarrow \mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$$

is 2-fully faithful.

(2) *Each 2-cell in* $\mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$ *has a unique representation by a 2-cell diagram with any given left hand side.*

(3) *The 2-cells in* $\mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$ *can be uniquely represented by diagrams with pseudo pullbacks as left hand 2-cells and horizontal composition can be calculated as in Section 6.*

Furthermore, there is a subclass $\mathfrak{C} \subset \mathfrak{W}$ of *essential covering maps*, defined by,

Definition 7.2. Let $\mathcal{G}$ be an étale groupoid. An *essential covering map* $\varphi^{\mathcal{U}} \colon \mathcal{G}^*(\mathcal{U}) \to \mathcal{G}$ is determined by a (non-repeating) collection of open subsets $\mathcal{U} \subseteq \mathcal{P}(\mathcal{G}_0)$ which meets every orbit of $\mathcal{G}$ (although it may not cover $\mathcal{G}_0$). Then $\mathcal{G}^*(\mathcal{U})$ is the groupoid defined by $\mathcal{G}^*(\mathcal{U})_0 = \coprod_{U \in \mathcal{U}} U$, with $\varphi_0^{\mathcal{U}} \colon \mathcal{G}(\mathcal{U})_0 \to \mathcal{G}_0$ defined by the inclusion maps. Furthermore, $\mathcal{G}(\mathcal{U})_1$ and the remaining maps are determined by the pullback diagram

$$\begin{array}{ccc} \mathcal{G}(\mathcal{U})_1 & \xrightarrow{\varphi_1^{\mathcal{U}}} & \mathcal{G}_1 \\ {\scriptstyle (s,t)}\downarrow & & \downarrow{\scriptstyle (s,t)} \\ \mathcal{G}(\mathcal{U})_0 \times \mathcal{G}(\mathcal{U})_0 & \xrightarrow[\varphi_0^{\mathcal{U}} \times \varphi_0^{\mathcal{U}}]{} & \mathcal{G}_0 \times \mathcal{G}_0 \end{array}$$

The class $\mathfrak{C}$ of essential covering maps is locally small and satisfies conditions **[WB1]**–**[WB5]**, and **[UUWB4]** and **[Co-UUWB4]**. So we get a bicategory $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ with small hom-categories, where $J_{\mathfrak{C}} \colon \mathbf{OrbiGroupoids} \longrightarrow \mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ is 2-fully faithful. Furthermore, the essential covering maps cover the essential equivalences in the sense described in Definition 4.1. Hence, there is an equivalence of bicategories, $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1}) \simeq \mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$.

Now $\mathfrak{C}$ is not pullback-closed. However, because of this equivalence of bicategories we can use the 2-cell diagrams from $\mathbf{OrbiGroupoids}(\mathfrak{W}^{-1})$ as 2-cells between arrows in $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$, and hence represent these by 2-cell diagrams involving pseudo pullbacks; these are not necessarily in the shape required of 2-cell diagrams in $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ because certain composites will not be in $\mathfrak{C}$, but they can be used as an alternate way to represent the 2-cells in this bicategory. This allows us to use the simplified composition described in Section 6. So we conclude:

Theorem 7.3. (1) *The bicategory of fractions of orbigroupoids with respect to essential covering maps,* $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ *has small hom-categories.*

(2) *The universal mapping,* $J_{\mathfrak{W}} \colon \mathbf{OrbiGroupoids} \longrightarrow \mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ *is 2-fully faithful.*

(3) *Each 2-cell in* $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ *has a unique representation by a 2-cell diagram with any given left hand side.*

(4) *The 2-cells in* $\mathbf{OrbiGroupoids}(\mathfrak{C}^{-1})$ *can be uniquely represented by diagrams with pseudo pullbacks as left hand 2-cells, and horizontal composition can be calculated as in Section 6.*

For further details, proofs, and applications, see [6].

References

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[4] D.A. Pronk, Etendues and stacks as bicategories of fractions, *Compositio Math.*, **102** (1996), pp. 243–303.
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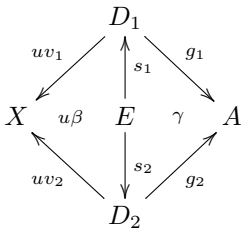
Appendix Appendix A Associativity Part I: Associativity 2-cells

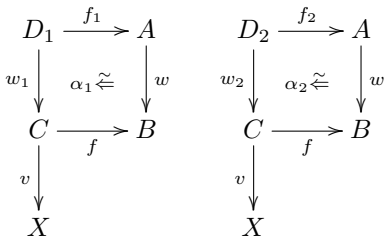
The goal of these appendices is to describe the associativity 2-cells and show that they satisfy the coherence pentagon condition. In Appendix A we will construct the associativity 2-cells, based on an extension of Proposition 2.5. In Appendix B we will sketch the coherence conditions.

Consider the 2-cells β and γ in Proposition 2.5. They give rise to a generalized 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$,



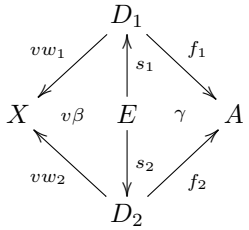
We show that this is the unique cell with this property: if β' and γ' also satisfy the conditions of Proposition 2.5, then the 2-cell diagram defined by β' and γ' is equivalent to this one.

Proposition A.1. *For $v, w\colon A \to B$ in $\mathfrak{W}$ and $f\colon C \to B$ any arrow in $\mathcal{B}$, and any two squares,*



with $vw_1, vw_2 \in \mathfrak{W}$, there is a unique 2-cell

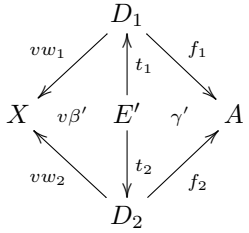
(20)



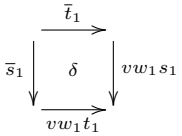
in $\mathcal{B}(\mathfrak{W}^{-1})$ such that the composites $(\alpha_2 s_2)\cdot(f\beta)$ and $(w\gamma)\cdot(\alpha_1 s_1)$ are equal.

Proof. Let

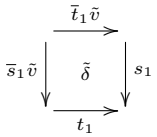
(21)



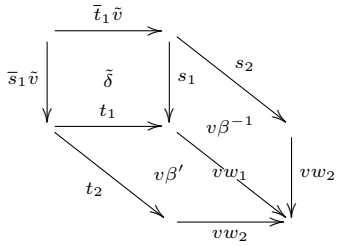
be another 2-cell diagram with the property that the composites $(\alpha_2 t_2)\cdot(f\beta')$ and $(w\gamma')\cdot(\alpha_1 t_1)$ are equal. Let



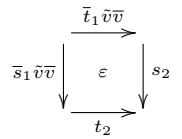
be a square as in condition **[WB3]** and let $\tilde{v}$ with



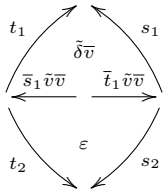
be a lifting as in **[WB4]** for δ with respect to vw_1. We use this cell in the following pasting,



Use condition **[WB4]** to obtain an arrow $\overline{v}$ and a cell

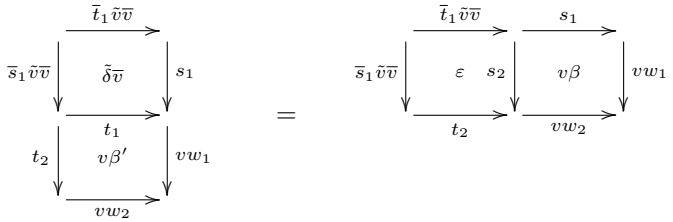


which form a lifting for this pasting with respect to vw_2. We would like to use the diagram



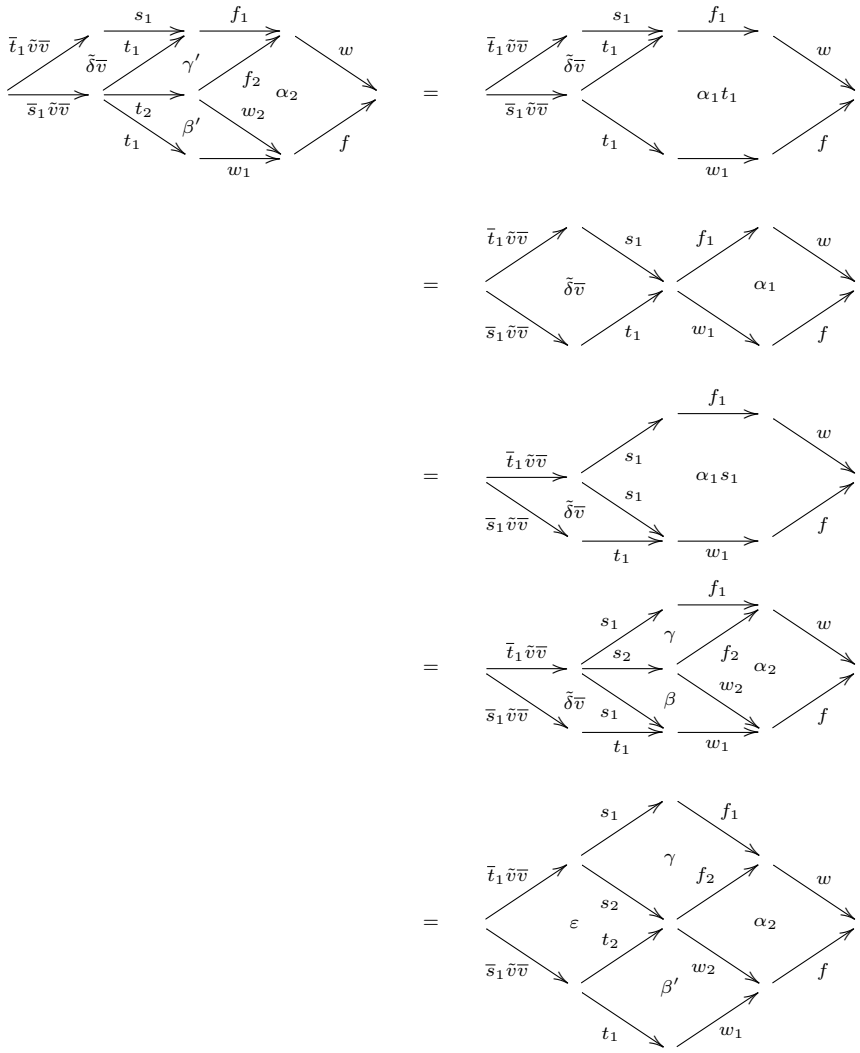
to show that the two 2-cell diagrams are equivalent. We will see that we still need to make a couple of small adjustments.

We have already that the following pastings are equal:

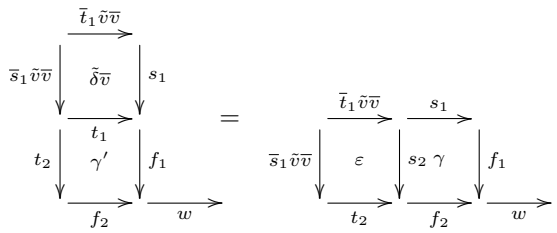


To obtain the corresponding result with γ, γ' instead of β, β', we need to compose with the arrow w in order to apply our hypothesis. We will also compose the pasting diagram we are interested in with the cells β' and α_2. This leads to the following

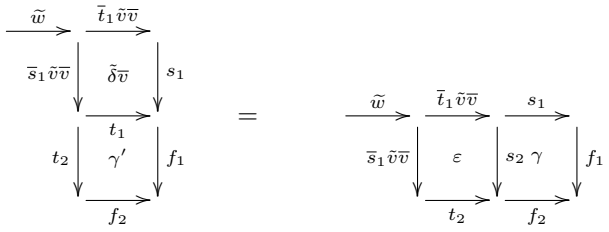
calculation,



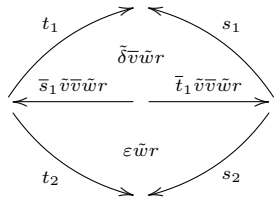
Since β' and α_2 are invertible 2-cells, we conclude that



By condition **[WB4]** there is an arrow $\widetilde{w} \in \mathfrak{W}$ such that

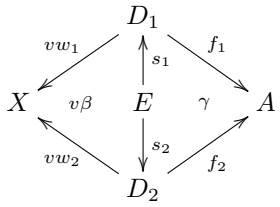


Finally, let r be an arrow such that the composition $vw_1s_1\overline{t}_1\tilde{v}\overline{v}\tilde{w}r \in \mathfrak{W}$. Then the cells



satisfy the equations to establish the fact that (20) and (21) are equivalent 2-cell diagrams, as claimed. □

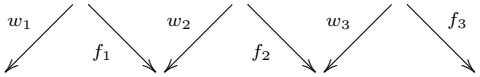
Notation A.2. We will say that the 2-cell



above *connects* the squares α_1 and α_2.

Proposition A.3. *For any path of composable spans:*

(22)



there is an associativity 2-cell

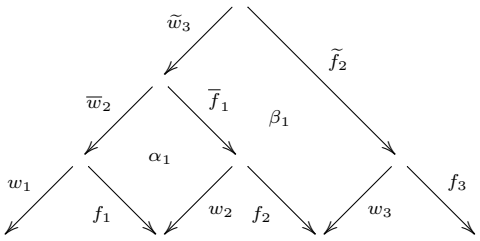
$$\alpha_{(w_3,f_3),(w_2,f_2),(w_1,f_1)} \colon (w_3, f_3) \circ ((w_2, f_2 \circ (w_1, f_1)) \Rightarrow ((w_3, f_3) \circ (w_2, f_2)) \circ (w_1, f_1)$$

between the composites as constructed in Section 3.

Proof. If we first compose the left hand pair and use the choices as described in the construction of $\mathcal{B}(\mathfrak{W}^{-1})$, we obtain $(w_3, f_3) \circ ((w_2, f_2 \circ (w_1, f_1))$ as the following

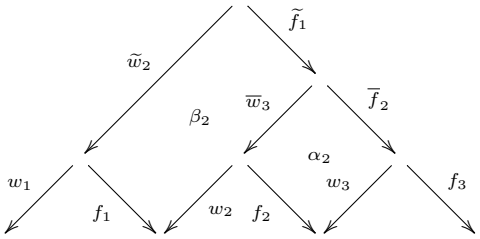
span,

(23)



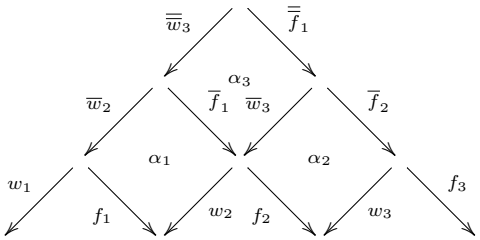
Note that $w_1\overline{w}_2\widetilde{w}_3 \in \mathfrak{W}$. If we first compose the right hand pair we get $((w_3, f_3) \circ (w_2, f_2)) \circ (w_1, f_1)$ as the span,

(24)



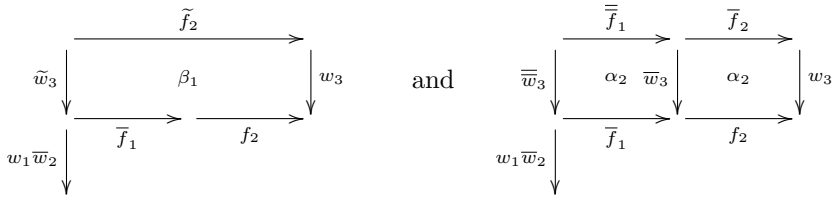
where $w_1\widetilde{w}_2 \in \mathfrak{W}$ and $w_2\overline{w}_3 \in \mathfrak{W}$. The associativity 2-cell will be a vertical composite of two 2-cells going through the intermediate:

(25)

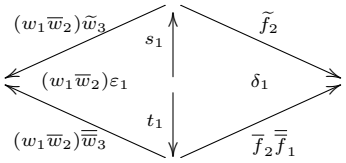


where $w_1\overline{w}_2\overline{\overline{w}}_3 \in \mathfrak{W}$ and $w_2\overline{w}_3 \in \mathfrak{W}$. We construct the associativity 2-cell as a vertical composition of two 2-cells: (23) $\Rightarrow$ (25) and (25) $\Rightarrow$ (24).

(23) $\Rightarrow$ (25): the diagrams in (23) and (25) only differ in the following chosen squares:

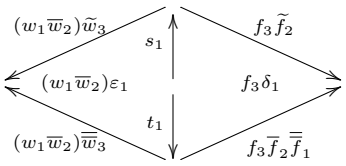


By Proposition A.1 there is a unique 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ connecting these two squares. Let

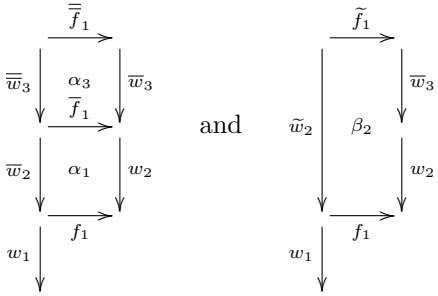


be a representing diagram for this 2-cell. Composing it with f_3 gives,

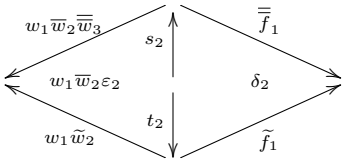
(26)



(25) $\Rightarrow$ (24): the diagrams in (25) and (24) only differ by the following two squares:

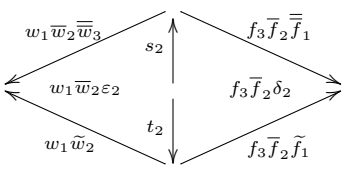


By Proposition A.1 there is a unique 2-cell in $\mathcal{B}(\mathfrak{W}^{-1})$ connecting these two squares. Let



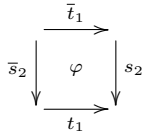
be a representing diagram for this 2-cell. Composing with $f_3\overline{f}_2$ gives,

(27)

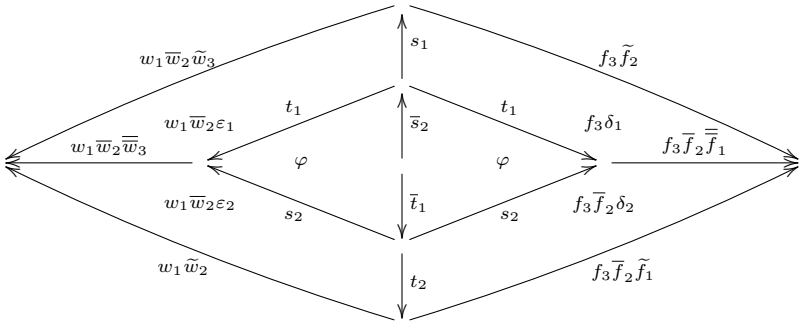


The associativity 2-cell for the composable path given in (22) is the vertical composition of (26) and (27). To calculate this composition, we use a choices as in

[C5] and **[C6]** of Notation 3.2 to obtain a square



with $w_1\overline{w}_2\overline{\overline{w}}_3 s_2\overline{t}_1 \in \mathfrak{W}$. Then the associativity 2-cell $\alpha_{(w_3,f_3),(w_2,f_2),(w_1,f_1)}$ is represented by



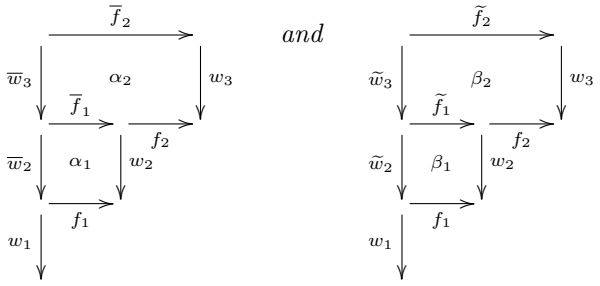
□

Appendix Appendix B Associativity Part II: Coherence

To prove the required coherence result, we will view the diagram (25) as a kind of common subdivision of (23) and (24), and break up the coherence into transitions given by Proposition A.1, and transitions with two layers of cells. There are two versions of this two layer case. They seem dual to each other, but their proofs are not, as the arrows in $\mathfrak{W}$ play very different roles. The two cases are covered in Propositions B.1 and B.2 below.

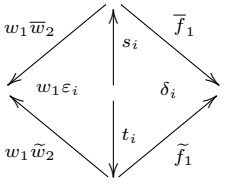
Proposition B.1. *Suppose we have two diagrams in $\mathcal{B}$,*

(28)



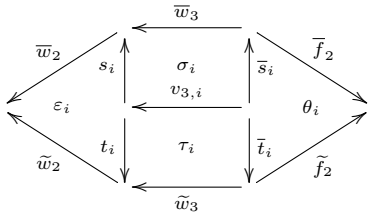
and

with two 2-cell diagrams

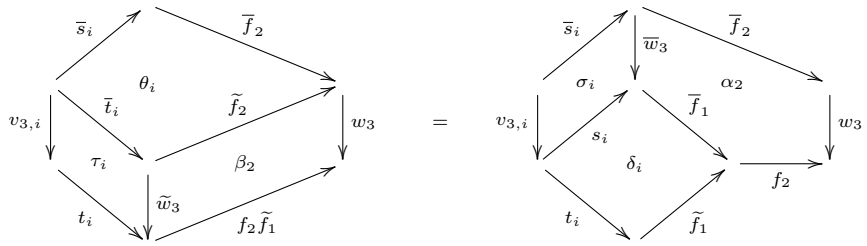


for $i = 1, 2$,

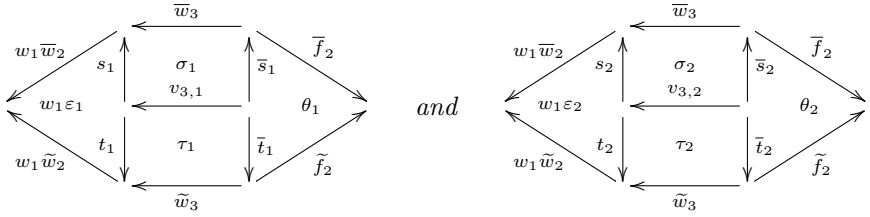
both connecting α_1 and β_1. Suppose that there are 2-cells σ_i, τ_i and θ_i for $i = 1, 2$ as in



such that $w_1\overline{w}_2\overline{w}_3\overline{s}_i \in \mathfrak{W}$ and



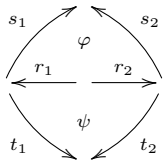
for $i = 1, 2$. Then the 2-cell diagrams,
(29)



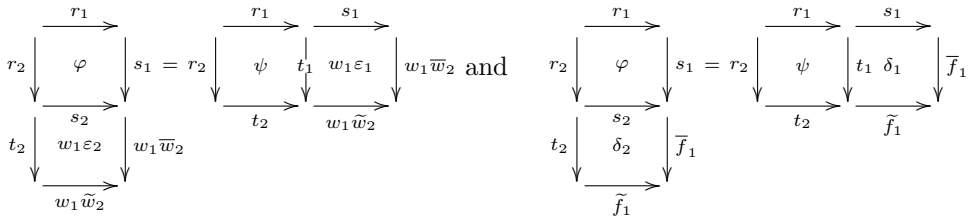
are equivalent.

Proof. By Proposition A.1 we know that

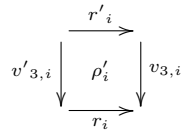
are equivalent 2-cell diagrams as they both connect the same pair of squares. So there are 2-cells



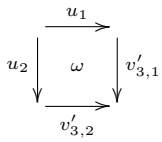
such that



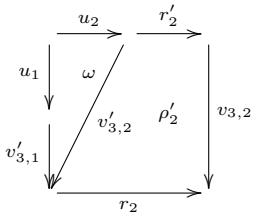
Now consider the cospan $\xrightarrow{v_{3,i}} \xleftarrow{r_i}$. Since both $w_1\overline{w}_2 s_i v_{3,i}$ and $w_1\overline{w}_2 s_i r_i$ are in $\mathfrak{W}$ we can use conditions **[WB3]**, **[WB4]** and **[WB2]** to obtain a square with an invertible 2-cell,



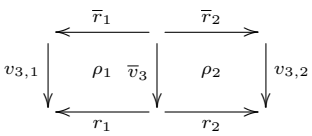
with $w_1\overline{w}_2 s_i r_i \overline{v}_{3,i} \in \mathfrak{W}$. We apply the same conditions then to $w_1\overline{w}_2 s_1 r_1 v'_{3,1}$ and $w_1\overline{w}_2 s_2 r_2 v'_{3,2}$ to obtain a square with an invertible 2-cell,



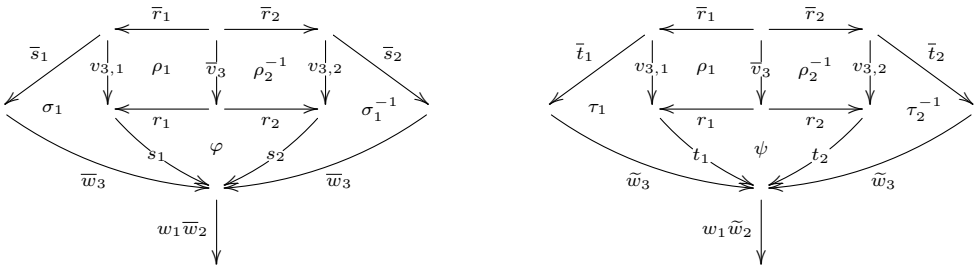
such that $w_1\overline{w}_2 s_1 r_1 v'_{3,1} u_1 \in \mathfrak{W}$. Now write $\rho_1 := \rho'_1 u_1$, $\overline{r}_1 := r'_1 u_1$, $\overline{v}_3 := v'_{3,1} u_1$, ρ_2 for the pasting of



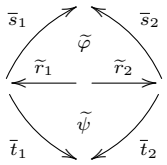
and $\overline{r}_2 := r'_2 u_2$. Then we obtain the following diagram,



Now consider the following two pasting diagrams,



Use condition **[WB4]** to lift the first pasting with respect to $w_1\overline{w}_2\overline{w}_3$ to obtain $\varphi' \colon \overline{s}_1\overline{r}_1u \Rightarrow \overline{s}_2\overline{r}_2u$; similarly, apply condition **[WB4]** to the pasting of the second diagram composed with u and lift with respect to $w_1\widetilde{w}_2\widetilde{w}_3$ to obtain $\widetilde{\psi} \colon \overline{t}_1\overline{r}_1uu' \Rightarrow \overline{t}_2\overline{r}_2uu'$. Now write $\widetilde{r}_1 = \overline{r}_1uu'$, $\widetilde{r}_2 = \overline{r}_2uu'$, and $\widetilde{\varphi} = \varphi'u'$. Then the reader may check that the 2-cells

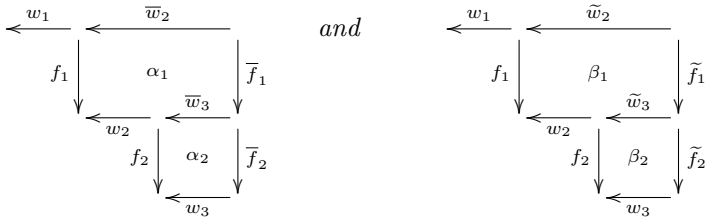


witness to the 2-cell diagrams in (29) being equivalent. □

The following proposition is the dual to the previous one; however, the proof is unfortunately not dual due to the special role played by arrows in $\mathfrak{W}$.

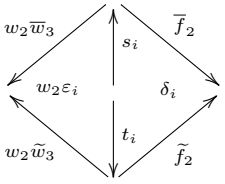
Proposition B.2. *Suppose we have two diagrams in $\mathcal{B}$,*

(30)



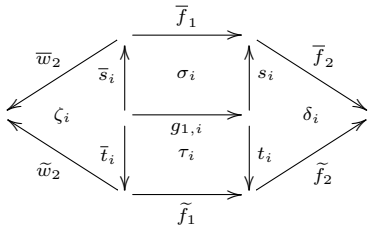
and

with two 2-cell diagrams



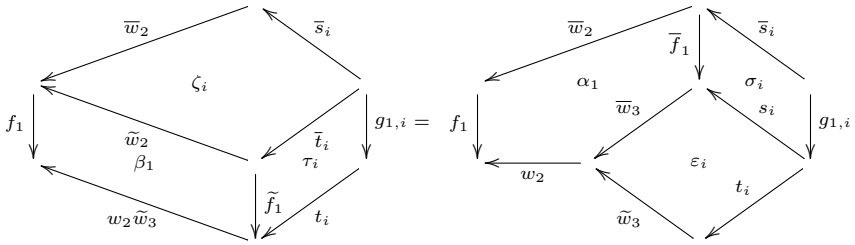
for $i = 1, 2$,

both connecting α_2 and β_2. Suppose that there are 2-cells σ_i, τ_i and ζ_i for $i = 1, 2$ as in,



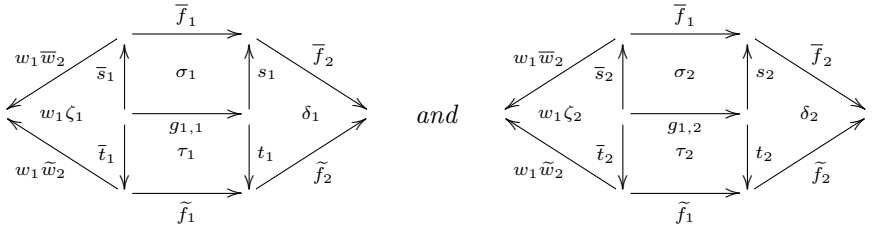
such that

(31)



for $i = 1, 2$. Then the 2-cell diagrams,

(32)

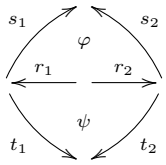


are equivalent.

Proof. By Proposition A.1 we know that

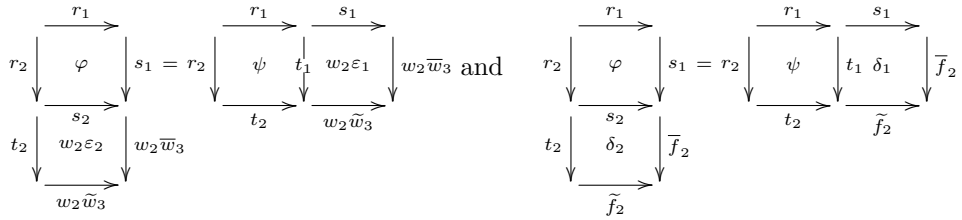
are equivalent 2-cell diagrams as they both connect the same pair of squares. So there are 2-cells

(33)

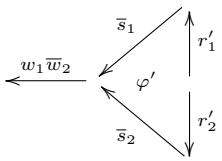


such that

(34)

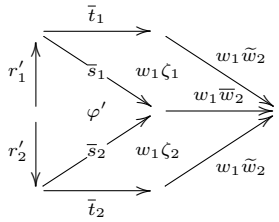


Note that the composites $w_1\overline{w}_2\overline{s}_i \in \mathfrak{W}$ for both $i = 1$ and $i = 2$. So we can use conditions **[WB3]**, **[WB4]** and **[WB2]** to obtain an invertible 2-cell φ' as in

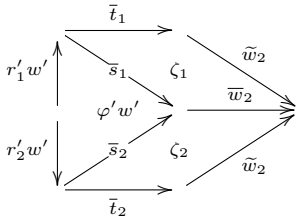


with $w_2\overline{s}_i\overline{r}_i \in \mathfrak{W}$. We want to define a corresponding cell ψ'. So consider the diagram,

(35)

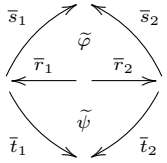


Since $w_1\widetilde{w}_2 \in \mathfrak{W}$, we apply conditions **[WB4]** and **[WB2]** to lift the pasting of this diagram with respect to $w_1\widetilde{w}_2$ to obtain $\psi' \colon \overline{t}_1 r'_1 w' \Rightarrow \overline{t}_2 r'_2 w'$. Now note that $\widetilde{w}_2\psi'$ and the composite of

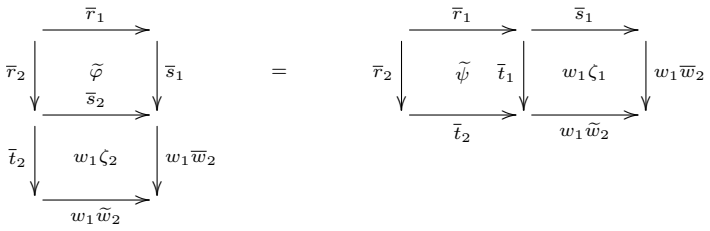


are both liftings of the pasting of (35) with respect to w_1. So by condition**[WB4]** there is an arrow w'' such that $\psi'w''$ is equal to the composition of this last pasting with w''. We will need this in our calculations, so we write $\overline{r}_i = r'_i w' w''$, $\widetilde{\varphi} = \varphi' w' w''$, and $\widetilde{\psi} = \psi' w''$. This gives us the following diagram

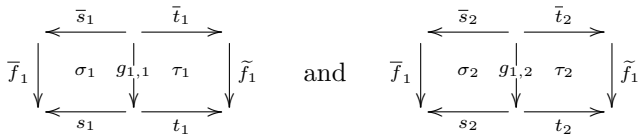
(36)



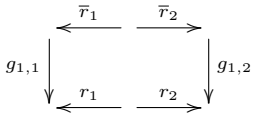
These cells satisfy the required equation with the ζ_i by construction:



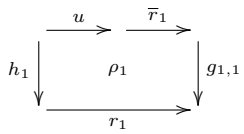
We will next see that after precomposing with an appropriate arrow they will also satisfy the equation for the composites of the right hand sides of (32). Since the cells φ and ψ satisfy the equation with the δ_i as stated in (34), we will focus on the cylinder with the diagram (33) as bottom and (36) as top. The sides of this cylinder are given by



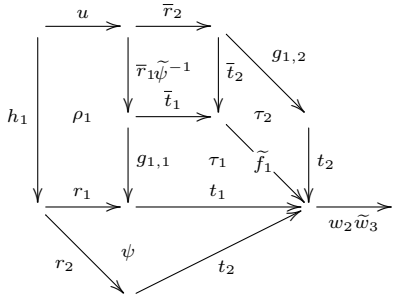
Before we can discuss the commutativity of this cylinder, we need to build cells to fill in the following frame,



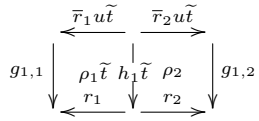
Since $w_1\overline{w}_2s_1r_1 \in \mathfrak{W}$, we can use conditions **[WB3]**, **[WB4]** and **[WB2]** to construct an invertible 2-cell ρ_1 as in



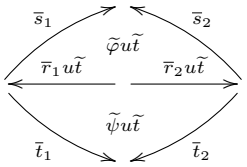
where $w_1\overline{w}_2\overline{s}_1\overline{r}_1u \in \mathfrak{W}$. Use this to construct a left hand square in the frame. To obtain a cell to fill the remaining right hand square, we consider the following pasting diagram,



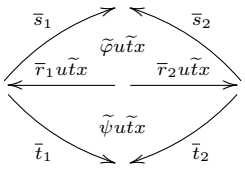
Now lift with respect to $w_2\widetilde{w}_3t_2$ to obtain $\rho_2\colon g_{1,2}\overline{r}_2u\widetilde{t} \Rightarrow r_2h_1\widetilde{t}$. So the middle frame gets filled as follows:



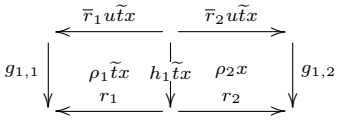
Furthermore, we have adjusted the top of the cylinder to become



We have defined ρ_2 in such a way that if the half of the cylinder that contains the $\psi, \widetilde{\psi}, \tau_1$ and τ_2 gets composed with $w_2\widetilde{w}_3$ it commutes. Condition **[WB4]** now gives that there is an arrow x such that if we precompose the top of the cylinder and the middle frame both with x, this half of the cylinder commutes. So now the top and the middle frame are respectively,

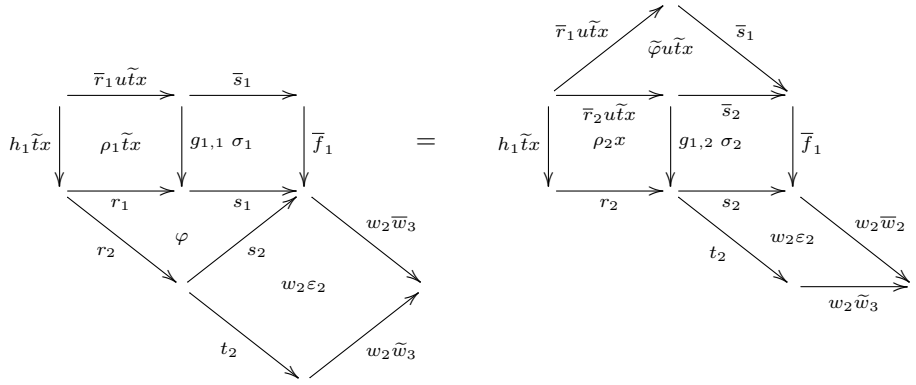


and

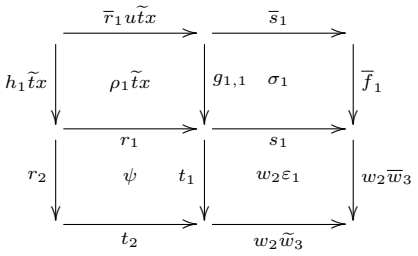


To investigate the commutativity of the other half of the cylinder, we will show that

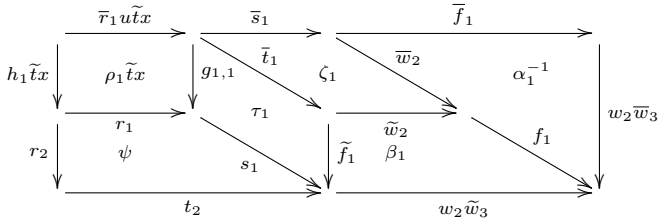
(37)



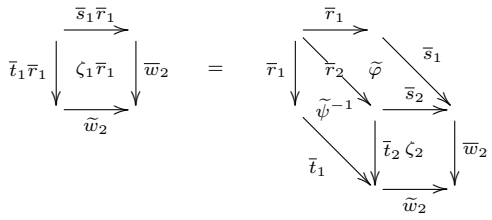
We begin by rewriting the left hand side. By (34) this pasting is equal to the pasting of



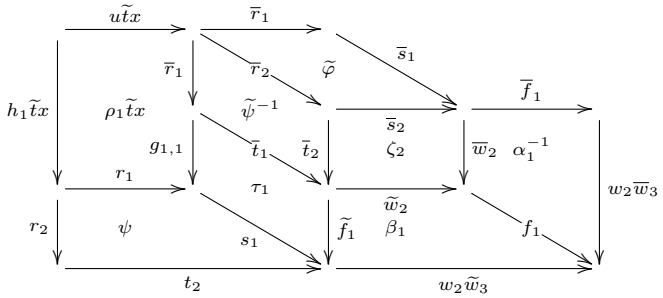
We use (34) to rewrite the right two 2-cells in this diagram to get



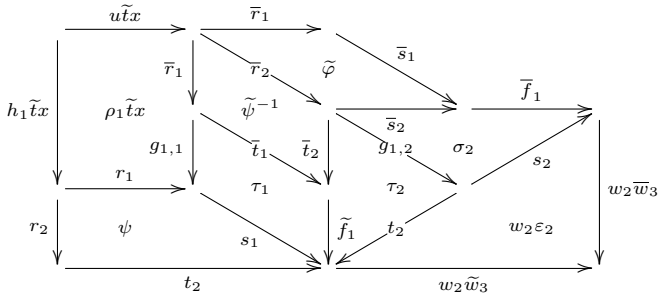
Now note that we have constructed $\widetilde{\varphi}$ and $\widetilde{\psi}$ such that



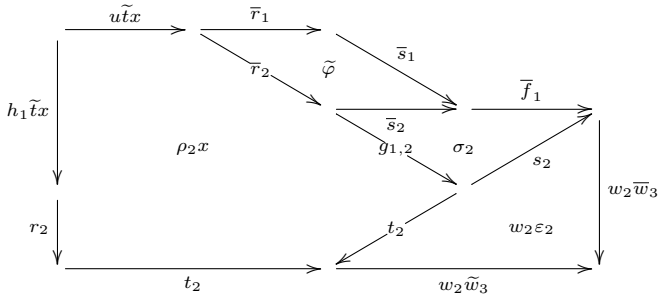
so we make this replacement in the diagram above to obtain,



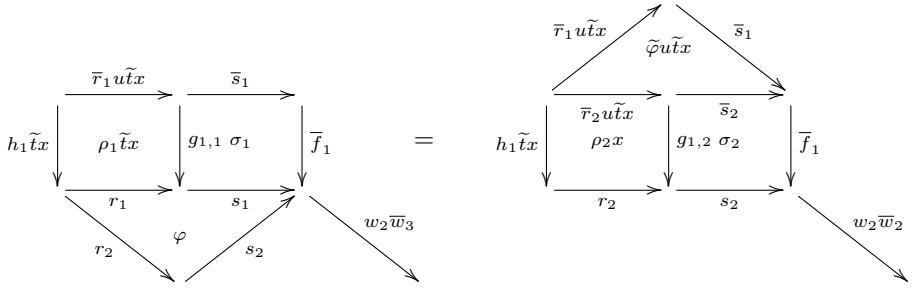
We use (34) again; this time to rewrite the bottom right hand corner of the diagram:



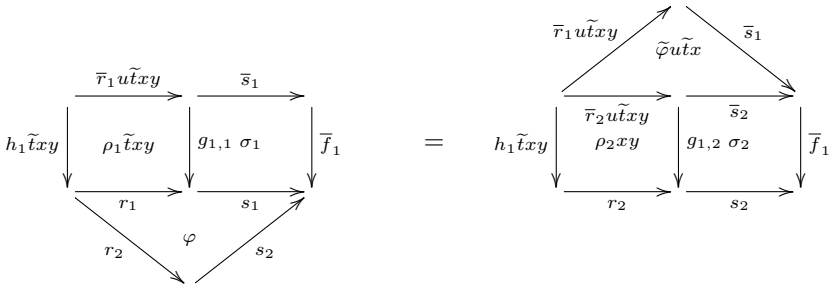
and by the definition of ρ_2, this is equal to



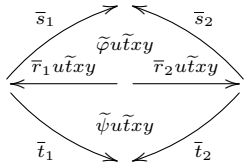
This completes our proof of equation (37). Since ε_2 is invertible it follows that



It follows from condition **[WB4]** that there is an arrow y such that



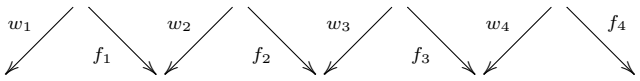
Hence, it follows from the arguments above that the cells



witness to the equivalence of the 2-cell diagrams in (equivcells2). □

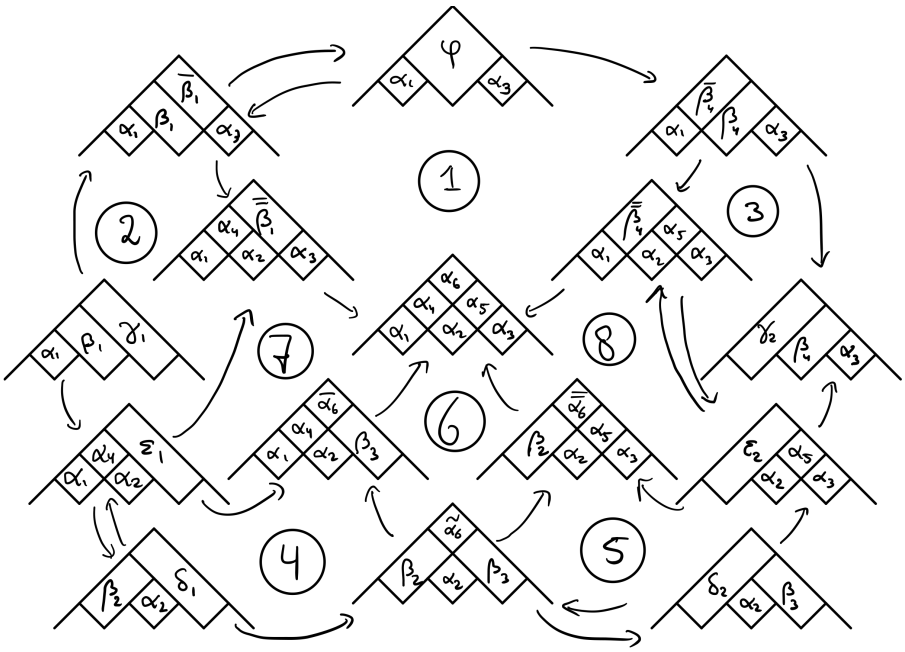
Remark B.3. Analogous to the situation in the Proposition A.1, we say that the 2-cell diagrams in (29) (respectively in (32)) *connect* the 2-cell configurations in (28) (respectively (30)). Propositions B.1 and B.2 only state uniqueness results, but it is not hard to prove existence as well. Since we will only need uniqueness in the proof of associativity coherence, we will not include the proof of existence.

Proposition B.4. *For any composable path of four spans,*



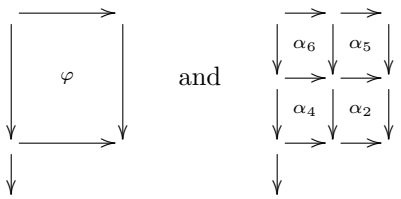
the associativity 2-cells defined in Propostion A.3 make the associativity coherence pentagon commute.

Proof. The following diagram shows the associativity coherence pentagon.



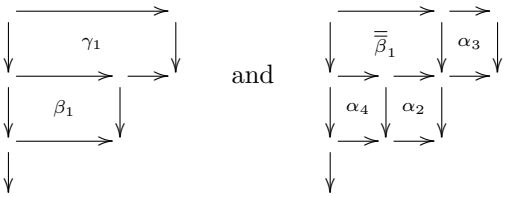
We have divided the pentagon into regions corresponding to various subdivisions, and we will show that each region commutes by one of the three results in Theorems A.1, B.1 and B.2. We sketch the argument for each region, leaving the details for the reader.

For region ① both composites provide a whiskering of a 2-cell that connects the squares



Since there is only one such 2-cell by Proposition A.1, this region commutes.

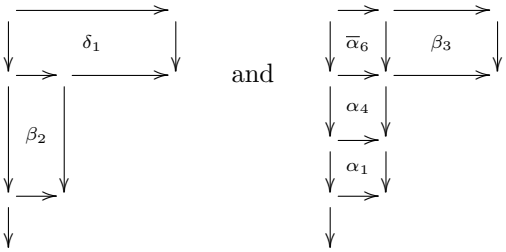
For region ② the two compositions connect the diagrams



as in Proposition B.1.

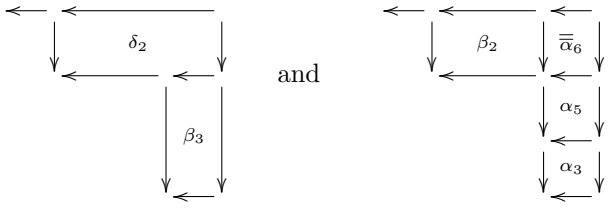
Region ③ is the dual of region ② and follows from Proposition B.2.

For region ④ commutativity is obtained from Proposition B.1 applied to



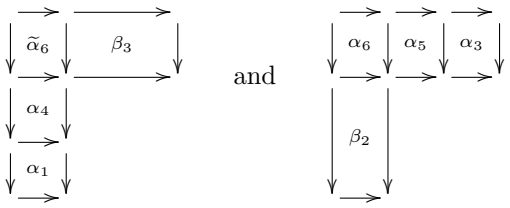
where we view the pasting of α_1 and α_4 as a single cell.

Region ⑤ is the dual of region ④ and commutativity can be obtained by applying Proposition B.2 to



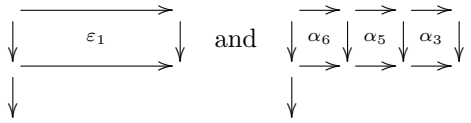
where we view the pasting of α_5 and α_3 as a single cell and the pasting of $\overline{\alpha}_6$ and β_3 as a single cell.

Region ⑥ could be done with an application of either Proposition B.1 or Proposition B.2. If we use Proposition B.1, we focus on the diagrams,



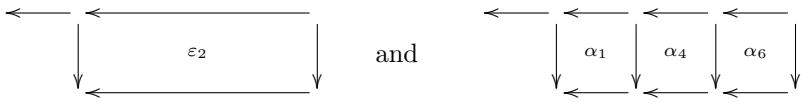
Here we consider the pasting of α_4 and α_1 as a single cell, the pasting of $\widetilde{\alpha}_6$ as a single cell, and the pasting of α_6, α_5 and α_3 as a single cell.

For region ⑦ the two ways of composing provide to 2-cells that connect the rectangles,



and there is only one such cell by Proposition A.1, so this region commutes.

Region ⑧ is the dual of region ⑦ whose two compositions give the 2-cell connecting the rectangles,



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