

A Homotopy Category for Graphs

Tien Chih · Laura Scull

Received: date / Accepted: date

Abstract We show that the category of graphs has the structure of a 2-category with homotopy as the 2-cells. We then develop an explicit description of homotopies for finite graphs, and use it to create a homotopy category for finite graphs in the sense that our homotopy category satisfies the universal property for localizing homotopy equivalences. We then show that finite stiff graphs form a skeleton of this homotopy category.

Keywords graph homomorphism · homotopy · 2-category · homotopy category · skeleton of a category

Mathematics Subject Classification (2010) 05C60 · 55U35 · 18D05

1 Introduction

Homotopy traditionally studies continuous transformations of spaces and maps between them. Translating such a fundamentally continuous concept into a discrete setting such as graphs can be approached in several ways. The first strategy used was to create a 'Hom complex', a simplicial set which represents information about the morphisms between two graphs. This simplicial complex can then be turned into a topological space, and the homotopy of this space encodes information about homotopy of graphs [1, 2, 9, 15–18]. More recently, Dochtermann has shown that it is possible to define a homotopy for graphs, called \times -homotopy, using only categorical constructions inside of graphs, and get the same homotopy theory as that provided by studying the simplicial space [6]. Others have since developed results strictly within the graph category [7, 8, 11, 21]. We follow this

T. Chih
Montana State University-Billings, 1500 University Dr, Billings, MT 59101
Tel.: +1-406-657-2939
E-mail: tien.chih@msubillings.edu

L. Scull
Fort Lewis College, 1000 Rim Dr, Durango, CO 81301
Tel.: +1-970-247-7130
E-mail: scull_l@fortlewis.edu

second approach and study \times -homotopy, which we simply call homotopy, and work strictly with graphs and discrete constructions.

In homotopy of topological spaces, the existence of the homotopies and their structure gives rise to a 2-categorical structure on spaces, in which the homotopies form 2-cells. In this paper, we show that the category of graphs also has the structure of a 2-category with homotopies of morphisms as the 2-cells, and verify the necessary conditions. We then develop an explicit description of homotopy for finite graphs, based around our notion of 'spider moves'. Our spider moves can be seen as a generalization of the idea of folds, which have been linked to homotopy of graphs by [3, 5, 14]. Then we use our spider moves to define a homotopy category for finite graphs, in the sense that we produce a category which satisfies the universal property for localization of homotopy equivalences. Such a localization is often created via a Quillen model category, which offers extra structure for working with the homotopy category that is created. The existence of model structures for the category of graphs has been studied by [8, 20], where they examine a number of different model structures which localize with respect to various notions of graph homotopy. They do not produce a model structure for the \times -homotopy that we are studying, and in fact it is shown in [11] that no such model structure exists. Thus we simply provide a direct construction of the localized homotopy category. To give some handle on the structure of the localized category, we show that the subcategory formed by stiff graphs forms a skeleton of our homotopy category, and hence the stiff graphs give canonical representatives for finite graphs up to homotopy.

We begin in Section 2 by reviewing the basic definitions and properties of the graph category, including products, exponential objects and walks and their concatenations following [6, 10, 14, 21]. In Section 3, we establish that the category of graphs forms a 2-category. In Section 4 we give a concrete description of the structure of a homotopy of graph morphisms, showing that a homotopy with finite domain can be broken down into a sequence of simple 'spider moves' which move only one vertex at a time. In Section 5 we use our spider moves from Section 4 to show that the quotient of the 2-category constructed is a categorical homotopy category for finite graphs in the sense that it satisfies the universal property for localization of homotopy equivalences. In Section 6, we show that the finite stiff graphs form a skeleton for the new homotopy category and briefly discuss what can be said about the structure of this skeleton in the absence of any model categorical infrastructure.

2 Background

In this section, we give background definitions and notations. We include some basic results which seem like they should be standard, but we were unable to find specific references in the literature, so we include them here for completeness. We will use standard graph theory definitions and terminology following [4, 10, 14], and category theory definitions and terminology from [19, 22].

2.1 The Graph Category

We work in the category \mathbf{Gph} of finite undirected graphs, where we allow at most one edge connecting any pair of vertices. We do allow a (single) loop connecting a vertex to itself.

Definition 2.1 [14] The category of graphs \mathbf{Gph} is defined by:

- An object is a graph G , consisting of a set of vertices $V(G) = \{v_\lambda\}$ and a set $E(G)$ of edges connecting them, where each edge is given by an unordered set of two vertices. If two vertices

are connected by an edge, we will use notation $v_1 \sim v_2 \in E(G)$, or just $v_1 \sim v_2$ if the parent graph is clear.

- An arrow in the category **Gph** is a graph morphism $f : G \rightarrow H$. Specifically, this is given by a set map $f : V(G) \rightarrow V(H)$ such that if $v_1 \sim v_2 \in E(G)$ then $f(v_1) \sim f(v_2) \in E(H)$.

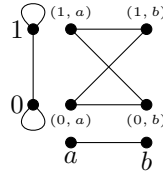
We will work in this category throughout this paper, and assume that 'graph' always refers to an object in **Gph**. When we have an invertible graph morphism $f : G \rightarrow H$ we will say that G and H are isomorphic and write $G \cong H$.

Definition 2.2 [21] Given a homomorphism $f : G \rightarrow H$, we define the image $Im(f)$ to be the subgraph of H where $V(Im(f)) = \{f(v) : v \in G\}$ and $E(Im(f)) = \{f(v) \sim f(w) : v \sim w \in E(G)\}$. Thus we specifically consider $Im(f)$ to contain only edges which are images of edges in G .

Definition 2.3 [14, 21] The (categorical) **product graph** $G \times H$ is defined by:

- A vertex is a pair (v, w) where $v \in V(G)$ and $w \in V(H)$.
- An edge is defined by $(v_1, w_1) \sim (v_2, w_2) \in E(G \times H)$ for $v_1 \sim v_2 \in E(G)$ and $w_1 \sim w_2 \in E(H)$.

Example 2.4 Let G be the graph on two adjacent looped vertices: $V(G) = \{0, 1\}$ and $E(G) = \{0 \sim 0, 1 \sim 1, 0 \sim 1, 0 \sim 1\}$. Let $H = K_2$ with $V(H) = \{a, b\}$ and $E(H) = \{a \sim b\}$. Then $G \times H$ is isomorphic to the cyclic graph C_4 :



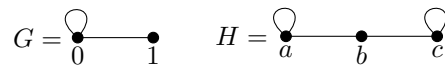
Lemma 2.5 [14] If $w \in V(H)$ is looped, i.e. $w \sim w \in E(H)$, then there is an inclusion $G \rightarrow G \times H$ given by $v \rightarrow (v, w)$ which is a graph morphism.

Proof If w is looped then $v \sim v'$ in G if and only if $(v, w) \sim (v', w)$ in $G \times H$. Thus the subgraph $G \times \{w\}$ is isomorphic to G .

Definition 2.6 [6] The **exponential graph** H^G is defined by:

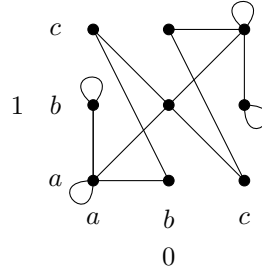
- A vertex in $V(H^G)$ is a **set** map $V(G) \rightarrow V(H)$ [not necessarily a graph morphism].
- There is an edge $f \sim g$ if whenever $v_1 \sim v_2 \in E(G)$, then $f(v_1) \sim g(v_2) \in E(H)$.

Example 2.7 Let G and H be the following graphs:



Then the exponential graph H^G is illustrated below, where the row indicates the image of 0 and the column the image of 1. So for example the vertex in the (a, c) spot represents the vertex map

$f(0) = a, f(1) = c.$



Observation 2.8 If f is looped in H^G , this means exactly that if $v_1 \sim v_2 \in E(G)$, then $f(v_1) \sim f(v_2) \in E(H)$. Thus a set map $f : V(G) \rightarrow V(H)$ is a graph morphism if and only if $f \sim f \in E(G^H)$.

Lemma 2.9 If $\phi : H \rightarrow K$ is a graph morphism and $f \sim g \in E(H^G)$ then $\phi f \sim \phi g \in E(K^G)$. So ϕ induces a graph morphism $\phi_* : H^G \rightarrow K^G$.

Proof Suppose that $f \sim g \in E(H^G)$. So for any $v_1 \sim v_2 \in E(G)$, we know that $f(v_1) \sim g(v_2) \in E(H)$. Since ϕ is a graph morphism, $\phi(f(v_1)) \sim \phi(g(v_2)) \in E(K)$. So $\phi f \sim \phi g$.

Lemma 2.10 If $\psi : K \rightarrow G$ is a graph morphism and $f \sim g \in E(H^G)$ then $f\psi \sim g\psi \in E(H^K)$. So ψ induces a graph morphism $\psi^* : H^G \rightarrow H^K$.

Proof Suppose that $v_1 \sim v_2 \in E(K)$; then we know that $\psi(v_1) \sim \psi(v_2) \in E(G)$. Since $f \sim g$ in H^G , $f(\psi(v_1)) \sim g(\psi(v_2))$. So $f\psi \sim g\psi$.

Proposition 2.11 [6] The category \mathbf{Gph} is cartesian closed. In particular, we have a bijection

$$\mathbf{Gph}(G \times H, K) \cong \mathbf{Gph}(G, K^H)$$

2.2 Walks and Concatenation

Definition 2.12 Let P_n be the path graph with $n + 1$ vertices $\{0, 1, \dots, n\}$ such that $i \sim i + 1$. Let I_n^ℓ be the looped path graph with $n + 1$ vertices $\{0, 1, \dots, n\}$ such that $i \sim i$ and $i \sim i + 1$.



Definition 2.13 A walk in G of length n is a morphism $\alpha : P_n \rightarrow G$. A **looped walk** in G of length n is a morphism $\alpha : I_n^\ell \rightarrow G$. If $\alpha(v_0) = x$ and $\alpha(v_n) = y$ we say α is a walk [resp. looped walk] from x to y .

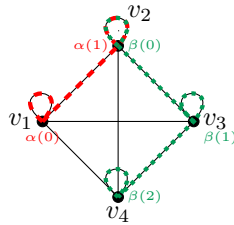
A walk can be described by a list of vertices $(v_0 v_1 v_2 \dots v_n)$ giving the images of the vertices $\alpha(i) = v_i$, such that $v_i \sim v_{i+1}$. Thus this definition agrees with the usual graph definition of walk. In the looped case, since $i \sim i \in E(I_n^\ell)$, we will have $v_i \sim v_i$ and so a looped walk is simply a walk where all the vertices along the walk are looped.

Definition 2.14 Given a walk $\alpha : P_n \rightarrow G$ from x to y , and a walk $\beta : P_m \rightarrow G$ from y to z , we define the **concatenation of walks** $\alpha * \beta : P_{m+n} \rightarrow G$ by

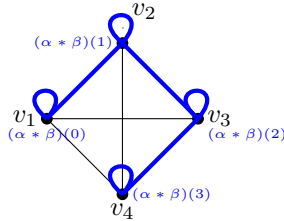
$$(\alpha * \beta)(i) = \begin{cases} \alpha(i) & \text{if } i \leq n \\ \beta(i - n) & \text{if } n < i \leq n + m \end{cases}$$

Since we are assuming that $\alpha(n) = y = \beta(0)$, $\alpha * \beta$ defines a length $n + m$ walk from x to z . In vertex list form, the concatenation $(xv_1v_2 \dots v_{n-1}y) * (yw_1w_2 \dots w_{m-1}z) = (xv_1v_2 \dots v_{n-1}yw_1 \dots w_{m-1}z)$. Contatenation of looped walks is defined in the same way.

Example 2.15 Consider the graph below, and let α be a length 1 looped walk (v_1v_2) and β a length 2 looped walk $(v_2v_3v_4)$.



Then $\alpha * \beta$ is a length 3 looped walk $(v_1v_2v_4v_3)$.



Observation 2.16 For any vertex x , there is a constant length 0 walk c_x from x to x defined by $c_x(0) = x$. Then for any other walk α from x to y , $c_x * \alpha = \alpha$ and $\alpha * c_y = \alpha$. If x is looped, we can similarly define a constant looped walk at x .

It is also straightforward to compare definitions and see both of the following:

Lemma 2.17 *Contatenation of [ordinary or looped] walks is associative: when the endpoints match up to make concatenation defined, we have $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma)$*

Lemma 2.18 *Contatenation of [ordinary or looped] walks is distributive: when ϕ and ψ are graph homomorphisms, then $\phi(g * h) = \phi g * \phi h$ and $(g * h)\psi = g\psi * h\psi$.*

3 Graphs as a 2-Category

In this section, we develop results on homotopies that allow us to show that **Gph** has the structure of a 2-category, with homotopies between morphisms as 2-cells.

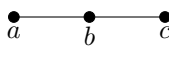
We define a homotopy between graph morphisms $G \rightarrow H$ via the graph $G \times I_n^\ell$. Because we use a looped interval graph, we have a graph inclusion $G \cong G \times \{k\} \hookrightarrow G \times I_n^\ell$ for each vertex k of I_n^ℓ .

Definition 3.1 [6] Given $f, g : G \rightarrow H$, we say that f is **homotopic** to g , written $f \simeq g$, if there is a map $\Lambda : G \times I_n^\ell \rightarrow H$ such that $\Lambda|_{G \times \{0\}} = f$ and $\Lambda|_{G \times \{n\}} = g$. We will say Λ is a length n homotopy.

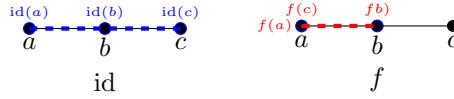
In [6], Definition 3.1 is referred to as \times -homotopy.

Observation 3.2 [6] By Propostion 2.11, a morphism $\Lambda : G \times I_n^\ell \rightarrow H$ is equivalent to a morphism $\Lambda : I_n^\ell \rightarrow H^G$. Since all the vertices of I_n^ℓ are looped, they can only be mapped to looped vertices in H^G which correspond to graph morphisms by Lemma 2.11. So the restriction of H to $G \times \{k\}$ always gives a graph morphism, and a length n homotopy corresponds to a sequence of graph morphisms $(f f_1 f_2 f_3 \dots f_{n-1} g)$ such that $f_i \sim f_{i+1} \in E(H^G)$. Thus we can think of a homotopy from f to g as a looped walk in the exponential object H^G . We will switch between these two views of homotopy as convenient.

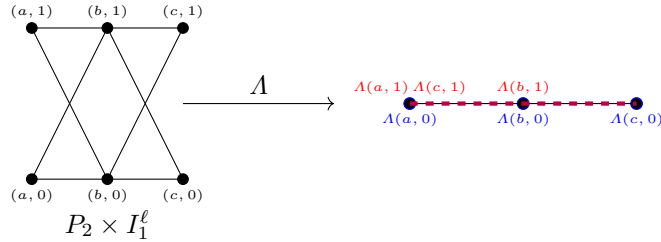
Observation 3.3 [6] $f \simeq g$ defines an equivalence relation on morphisms $G \rightarrow H$.

Example 3.4 Suppose we have the graph $G = P_2$ 

Consider the maps $\text{id}_G, f : G \rightarrow G$ where $f(a) = f(c) = a$ and $f(b) = b$. We abbreviate these morphisms by listing the images of vertices a, b , and c in order, so $\text{id}_G = abc$ and $f = aba$.



We can define a homotopy $\Lambda : G \times I_1^\ell \rightarrow G$ from id_G to f , where $\Lambda((x, 0)) = x$ and $\Lambda((x, 1)) = f(x)$. Since $0, 1$ are both looped in I_1^ℓ , the subgraphs $G \times \{0\}$ and $G \times \{1\}$ are both isomorphic to G . It is easy to verify that Λ is a graph homomorphism and thus is a length 1 homotopy.

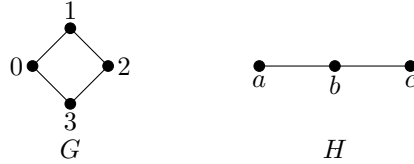


Lemma 3.5 Suppose that $g \simeq g' : G \rightarrow H$. If $h : H \rightarrow K$, then $hg \simeq hg'$; and if $f : F \rightarrow G$, then $gf \simeq g'f$.

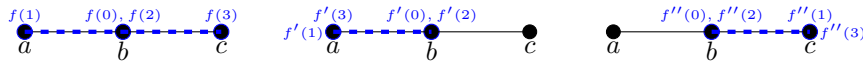
Proof Since $g \simeq g'$, there is a length n homotopy Λ from g to g' in H^G . Then $h_*\Lambda$ defines a length n homotopy from hg to hg' by Lemma 2.9. Similarly, $\Lambda(f \times id_{I_n^G})^*$ defines a length n homotopy from gf to $g'f$ by Lemma 2.10.

Definition 3.6 (Concatenation of Homotopies) Given $\Lambda_1 : f \simeq g$ and $\Lambda_2 : g \simeq h$, we define $\Lambda_1 * \Lambda_2 : f \simeq h$ using the concatenation of looped walks in G^H of Definition 2.14.

Example 3.7 Let $G = C_4$ and $H = P_2$ with vertices labeled as below.



Let $f : G \rightarrow H$ be defined by $f(0) = f(2) = b, f(1) = a, f(3) = c$. Again, we will abbreviate this morphism by listing the images of 0, 1, 2, 3 in order, so $f = babc$. Let $f' : G \rightarrow H$ be defined by $baba$, and let $f'' : G \rightarrow H$ be defined by $bcbc$. One can check that $f, f', f'' \in \text{Gph}(G, H)$.



Since $f \sim f' \in E(H^G)$ we have a length one homotopy $\alpha : I_1^\ell \rightarrow H^G$ defined by $\alpha(0) = f, \alpha(1) = f'$. Similarly, $f' \sim f'' \in E(H^G)$ and so we have a homotopy $\alpha' : I_1^\ell \rightarrow H^G$ defined by $\alpha'(0) = f', \alpha'(1) = f''$. Then $\alpha * \alpha' : I_2^\ell \rightarrow H^G$ is defined by the looped walk $(ff'f'')$ in H^G , depicted in Figure 1 below.

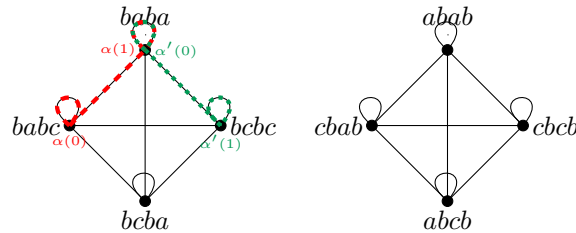


Fig. 3.1 Here we have depicted the walks α and α' . We have drawn only the subgraph of H^G induced by graph homomorphisms rather than the whole exponential graph.

Proposition 3.8 *The concatenation operation on homotopies is unital and associative.*

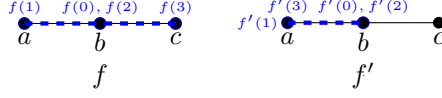
Proof The constant homotopy defines a unit by Observation 2.16, and associativity is given by Lemma 2.17.

We now define another composition of homotopies.

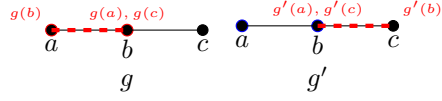
Definition 3.9 (Composition of Homotopies) Suppose that $f, f' : G \rightarrow H$ and $g, g' : H \rightarrow K$. Given $\alpha : f \simeq f'$ and $\beta : g \simeq g'$, we define $\alpha \circ \beta$ from gf to $g'f'$ as follows: let $g\alpha = g_*\alpha$ denote the homotopy from gf to gf' , and $(f')^*\beta = \beta f'$ denote the homotopy from gf' to $g'f'$, as defined in Lemma 3.5. Then

$$\alpha \circ \beta = g\alpha * \beta f'.$$

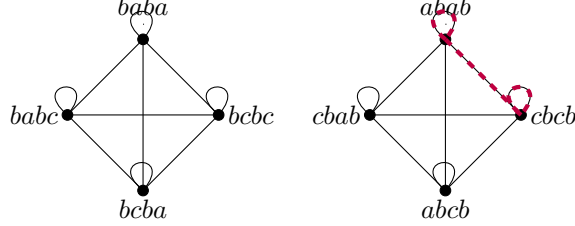
Example 3.10 As in Example 3.7, let $G = C_4, H = K = P_2$ and let $f : G \rightarrow H$ be defined by abc , and $f' : G \rightarrow H$ by $baba$, with α the length 1 homotopy (ff') .



Let $g : H \rightarrow K$ be defined by $g(a) = g(c) = b, g(b) = a$ and let $g' : H \rightarrow K$ be defined by $g'(a) = g'(c) = b, g'(b) = c$, with β the length 1 homotopy (gg') .



Then $\alpha \circ \beta$ is a length 2 homotopy $\beta_\alpha : I_2^\ell \rightarrow K^G$ defined by the looped walk $(gf gf' g'f')$. Concretely, both gf and $g'f'$ are given by the map $abab$ and $g'f'$ is defined by $cbcb$. Thus $\alpha \circ \beta$ is a length 2 homotopy defined by the walk $(abab abab cbcb)$.



We could equally well have chosen to define the composition as $\beta f * g'\alpha$. This is not the same homotopy; however, we will show that the two resulting homotopies are themselves homotopic. To make this notion precise, we observe that a homotopy α from f to g is defined as a looped walk in $(H^G)^{I_n^\ell}$ given by $(fh_1h_2h_3 \dots h_{n-1}g)$. Then for [looped or unlooped] walks, we define the notion of homotopy rel endpoints. The idea of fixing a subspace and allowing only homotopies which are constant on this subspace is a common one from homotopy theory, and when the fixed subspace is A , this is referred to as homotopy rel A [12]. In our case, we will take the subspace to be the end vertices of the path graph v_0, v_n .

Let G be any graph. Recall that a looped vertex of the exponential object G^{P_n} represents a length n walk in G , and similarly a looped vertex of $G^{I_n^\ell}$ represents a looped walk in G . Such an α is given by $(\alpha(0)\alpha(1)\alpha(2) \dots, \alpha(n)) = (v_0v_1v_2 \dots, v_n)$. Define $s, t : X^{P_n} \rightarrow X$ by $s(v_0v_1 \dots v_n) = v_0$ and $t(v_0v_1 \dots v_n) = v_n$. Note that these are NOT graph homomorphisms, just maps of vertex sets. Thus α is a walk from x to y if $s(\alpha) = x$ and $t(\alpha) = y$.

Definition 3.11 Suppose that α, β are walks in G from x to y . We say α and β are **homotopic rel endpoints** if they are homotopic in the subgraph

$$(G^{P_n})_{x,y} = \{\gamma \in G^{P_n} \mid s(\gamma) = x \text{ and } t(\gamma) = y\}$$

Thus two walks $\alpha = (xv_1 \dots v_{n-1}y)$ and $\beta = (xw_1 \dots w_{n-1}y)$ are homotopic rel endpoints if there is a looped walk of walks in G^{P_n} given by $\Lambda = (\alpha\lambda_1\lambda_2 \dots \lambda_{k-1}\beta)$ where each walk λ_i starts at x and ends at y .

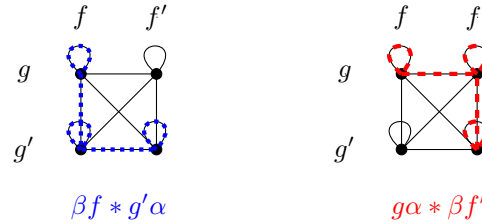
For looped walks, we make the same definitions in $G^{I_n^\ell}$.

Now we apply this notion to homotopies, viewed as looped walks in $(H^G)^{I_n^\ell}$.

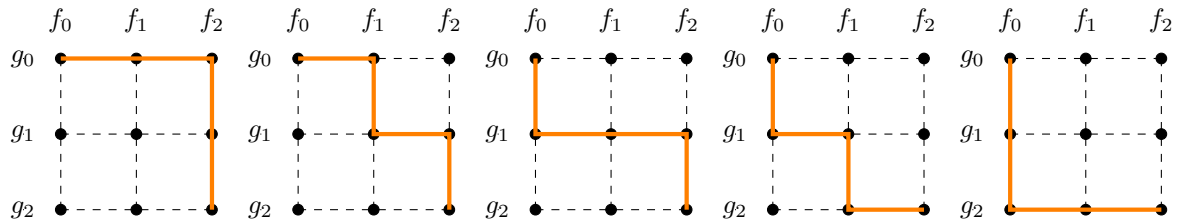
Definition 3.12 Two homotopies α, α' from f to g are themselves homotopic if they are homotopic rel endpoints viewed as looped walks in $(H^G)^{I_n^\ell}$.

Proposition 3.13 Suppose that $f, f' : G \rightarrow H$ and $g, g' : H \rightarrow K$. Given $\alpha : f \simeq f'$ and $\beta : g \simeq g'$, the two homotopies defined by $g\alpha * \beta f'$ and $\beta f * g'\alpha$ are homotopic.

Proof First, suppose that both α and β are length 1 homotopies, so that there are edges $f \sim f'$ and $g \sim g'$. We consider the two length 2 homotopies $g\alpha * \beta f' = (gf g'f' g'f')$, and $\beta f * g'\alpha = (gf g'f' g'f')$. We want to show that these are homotopic. In fact, we claim that they are connected by an edge in K^G . Since I_2^ℓ has edges connecting $0 \sim 1$ and $1 \sim 2$, this requires that $(g\alpha * \beta f')(i) \sim (\beta f * g'\alpha)(i+1)$ and $(g\alpha * \beta f')(i+1) \sim (\beta f * g'\alpha)(i)$ for $i = 0, 1$. So there are four conditions to check. Decoding them, they are: $gf \sim g'f', gf \sim g'f, g'f \sim g'f'$ and $g'f' \sim g'f'$. Each of these holds by Lemma 3.5. Lastly, we consider the loops $i \sim i$: for $i = 0, 2$ we have $\alpha(i) = \beta(i)$, and since these are looped vertices, $\alpha(i) \sim \beta(i)$. For $i = 1$, we have $\alpha(1) = g'f$ and $\beta(1) = g'f'$. If $v \sim w \in E(G)$, then $f(v) \sim f'(w) \in E(H)$ and hence $g'f(v) \sim g'f'(w) \in E(K)$, verifying the last condition. Observe that this length 1 homotopy fixes the endpoints, and thus we have a homotopy of homotopies (that is, the homotopies are homotopic rel endpoints).



Now if α and β are homotopies of length n and m , each of them is defined by a looped walk $(ff_1f_2f_3 \dots f_{n-1}f')$ and $(gg_1g_2g_3 \dots g_{m-1}g')$. Since each successive pair is connected, the outer edges of each square are connected by an edge, ie a length 1 homotopy, and we can repeatedly swap squares and get a length nm homotopy rel endpoints between $g\alpha * \beta f'$ and $\beta f * g'\alpha$.



Proposition 3.14 *The composition operation on homotopies is unital and associative.*

Proof Unital: If α is the constant homotopy at f , then $g\alpha$ is just constant at gf , and $g\alpha * \beta f' = \beta f'$ by Observation 2.16. Similarly if β is the constant homotopy at gf' , then $\beta f' = f'$ and $g\alpha * \beta f' = \beta f'$.

Associative: Suppose we have homotopies $\alpha : f \simeq f', \beta : g \simeq g'$ and $\gamma : h \simeq h'$. Then the distributive property of Lemma 2.18 and the associative property of Lemma 2.17 give:

$$\begin{aligned}
(\alpha \circ \beta) \circ \gamma &= (g\alpha * \beta f') \circ \gamma \\
&= h(g\alpha * \beta f') * \gamma g' f' \\
&= (hg\alpha * h\beta f') * \gamma g' f' \\
&= hg\alpha * (h\beta f' * \gamma g' f') \\
&= hg\alpha * (h\beta * \gamma g') f' \\
&= \alpha \circ (h\beta * \gamma g') \\
&= \alpha \circ (\beta \circ \gamma)
\end{aligned}$$

We will show that \mathbf{Gph} forms a 2-category [22]. We want our 2-cells to be defined by homotopies of morphisms, but this does not satisfy the required properties. However, since a homotopy α is defined by a looped walk given by a map $\alpha : I_n^\ell \rightarrow H^G$, we have a notion of when two such maps are themselves homotopic, as in Definition 3.12. In order to get a 2-category, we will define our 2-cells to be homotopy classes of homotopies.

We begin by showing that concatenation and composition operations are well defined up to homotopy. We will use the following more general result about homotopies of walks:

Lemma 3.15 *If f and g are looped walks of length n in G from x to y , and $f \simeq g$ are homotopic rel endpoints, then if h is a walk from y to z , then $f * h \simeq g * h$ rel endpoints; and if k is a walk from w to x , then $k * f \simeq k * g$ rel endpoints.*

Proof We have f and g representing vertices in $G^{I_n^\ell}$, and α a length m homotopy from f to g . So α is defined by a looped walk $(f f_1 f_2 \dots f_{n-1} g)$ in $G^{I_n^\ell}$. Now suppose that h is a walk from y to z . Define a sequence $(f_0 * h f_1 * h f_2 * h \dots f_n * h)$ in $G^{I_{n+m}^\ell}$, where each of these is a walk from x to z . We claim that each successive pair of these is connected by an edge in $G^{I_{n+m}^\ell}$. The requirement for this edge to exist is that given any edge $v_i \sim v_{i+1}$ in I_{n+m}^ℓ , we have $(f_k * h)(v_i) \sim (f_{k+1} * h)(v_{i+1})$. By definition of concatenation, if $i \leq n$ these are defined by $f_k(v_i)$ and $f_{k+1}(v_{i+1})$, which are connected in G since $f_k \sim f_{k+1}$; if $i > n$, these are defined by $h(v_i)$ and $h(v_{i+1})$, which are connected since h is a walk in G . Thus $f * h \simeq g * h$ rel endpoints. The other case follows by an analogous argument.

Corollary 3.16 *If $\alpha \simeq \alpha'$ are homotopic as homotopies (ie homotopic rel endpoints) and $\beta \simeq \beta'$ as homotopies, then $\alpha * \beta \simeq \alpha' * \beta'$.*

Lemma 3.17 *If $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ then $\alpha \circ \beta \simeq \alpha' \circ \beta'$.*

Proof Start with $\alpha \circ \beta = g\alpha * \beta f'$. Now by Lemma 3.5, we have a homotopy $g\alpha \simeq g\alpha'$, and hence by Lemma 3.15 a homotopy $g\alpha * \beta f' \simeq g\alpha' * \beta f'$. Then Lemma 3.15 also says that $\beta f' \simeq \beta' f'$, and so $g\alpha' * \beta f' \simeq g\alpha' * \beta' f' = \alpha' \circ \beta'$. Thus we have $\alpha \circ \beta \simeq \alpha' \circ \beta'$ as homotopies.

Theorem 3.18 *We can define a 2-category \mathbf{Gph} as follows:*

- Objects [0-cells] are given by objects of \mathbf{Gph} , the finite undirected graphs.

- Arrows [1-cells] are given by the arrows of \mathbf{Gph} , the graph morphisms
- Given $f, f' : G \rightarrow H$, a 2-cell from f to f' is a homotopy rel endpoints class $[\alpha]$ of homotopies $\alpha : I_n^{\ell} \rightarrow H^G$ such that $\alpha : f \simeq f'$.
- Vertical composition is defined using concatenation $[\alpha] * [\alpha'] = [\alpha * \alpha']$
- Horizontal composition is defined using composition $[\alpha] \circ [\beta] = [\alpha \circ \beta]$

Proof We have shown that vertical and horizontal composition are well-defined in Corollary 3.16 and Lemma 3.17, and that these operations are associative and unital in Propositions 3.8 and 3.14. Therefore what remains is to check the interchange law.

Our set-up is as follows: we have maps $f, f', f'' : G \rightarrow H$ and $g, g', g'' : H \rightarrow K$, with two cells $\alpha : f \simeq f', \alpha' : f' \simeq f''$ and $\beta : g \simeq g', \beta' : g' \simeq g''$:

We want to show that $(\alpha \circ \beta) * (\alpha' \circ \beta') \simeq (\alpha * \alpha') \circ (\beta * \beta')$. Unravelling the definitions here shows that $(\alpha \circ \beta) * (\alpha' \circ \beta') = (g\alpha * \beta f') * (g'\alpha' * \beta' f'')$, while $(\alpha * \alpha') \circ (\beta * \beta') = g(\alpha * \alpha') * (\beta * \beta')f'' = (g\alpha * g\alpha') * (\beta f'' * \beta' f'')$ using the distributivity of Lemma 2.18. Since concatenation is associative, we are comparing $g\alpha * \beta f' * g'\alpha' * \beta' f''$ with $g\alpha * g\alpha' * \beta f'' * \beta' f''$. Therefore it suffices to show that $\beta f' * g'\alpha' \simeq g\alpha' * \beta f''$. But this is exactly Proposition 3.13.

4 Structure of Homotopies for Finite Graphs

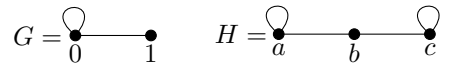
In this section, we develop a more explicit description of homotopies between graph morphisms when G is a finite graph. We show that such graph homotopies can always be defined 'locally', shifting one vertex at a time. We imagine a spider walking through the graph by moving one leg at a time.

Definition 4.1 Let $f, g : G \rightarrow H$ be graph morphisms. We say that f and g are a **spider pair** if there is a single vertex of G , say x , such that $f(y) = g(y)$ for all $y \neq x$. If x is unlooped there are no additional conditions, but if $x \sim x \in E(G)$, then we require that $f(x) \sim g(x) \in E(H)$. When we replace f with g we refer to it as a **spider move**.

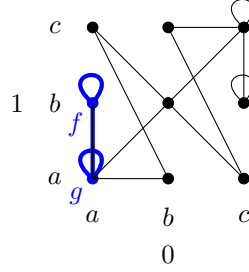
Lemma 4.2 *If f and g are a spider pair, then $f \sim g \in E(H^G)$.*

Proof For any $y \sim z \in E(G)$ we need to verify that $f(y) \sim g(z) \in E(H)$. If $y, z \neq x$ then $g(z) = f(z)$ and so this follows from the fact that f is a graph morphism. If $y \sim x$ for $y \neq x$, then $f(y) \sim g(x)$ since $f(y) = g(y)$ and g is a graph morphism; similarly, $f(x) \sim g(y)$. Lastly, if $x \sim x$, then we have asked that $f(x) \sim g(x)$. Therefore $f \sim g$ have an edge in the exponential graph H^G .

Example 4.3 Let G and H be the graphs from Example 2.7:



Let $f, g : G \rightarrow H$ be defined by $f(0) = a, f(1) = b$, and $g(0) = a, g(1) = a$. So f, g are a spider pair, and we see that the morphisms f, g are adjacent in the exponential object H^G .



We now prove that all homotopies with finite domain can be decomposed as a sequences of spider moves, moving one vertex at a time.

Proposition 4.4 (Spider Lemma) *If $f, g : G \rightarrow H$ and G is a finite graph, and $f \sim g \in E(H^G)$, then there is a finite sequence of morphisms $f = f_0, f_1, f_2, \dots, f_n = g$ such that each successive pair f_k, f_{k+1} is a spider pair.*

Proof Since G is a finite graph, we can label its vertices v_1, v_2, \dots, v_n . Then for $0 \leq k \leq n$, we define:

$$f_k(v_i) = \begin{cases} f(v_i) & \text{for } i \leq n - k \\ g(v_i) & \text{for } i > n - k \end{cases}$$

First we check that each f_k is a graph morphism. Suppose $v_i \sim v_j \in E(G)$; we need to show that $f_k(v_i) \sim f_k(v_j)$. If $i, j \leq n - k$ then $f_k = f$ for both vertices, and so since f is a morphism, $f(v_i) \sim f(v_j)$. Similarly if $i, j > n - k$ then $f_k = g$ on both vertices. Lastly, if $i \leq n - k$ and $j > n - k$, we know that $f \sim g$ in H^G , so by the structure of edges in the exponential object, $f(v_i) \sim g(v_j)$. Thus $f_k(v_i) \sim f_k(v_j)$.

It is clear that each pair f_k, f_{k+1} agrees on every vertex except v_{n-k} . So to show this is a spider pair, we only need to check that if v_{n-k} is looped, then $f_k(v_{n-k}) \sim f_{k+1}(v_{n-k})$. But since $f \sim g \in E(H^G)$, we know that if $v_{n-k} \sim v_{n-k}$ then $f(v_{n-k}) \sim g(v_{n-k})$.

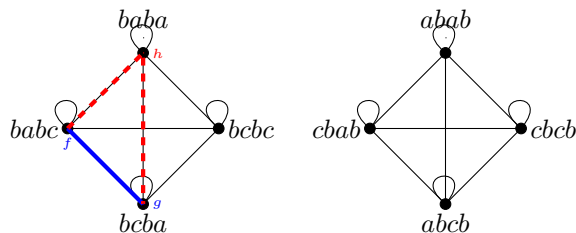
Corollary 4.5 *Whenever $f, g : G \rightarrow H$ with G finite and $f \simeq g$, there is a finite sequence of spider moves connecting f and g .*

Thus we can see explicitly what homotopies of graph morphisms between finite graphs can do.

Example 4.6 Let $G = C_4, H = P_2$ as in Example 3.7. The morphisms $f = abc$ and $g = bcba$ are adjacent in H^G . They are not a spider pair since $f(1) \neq g(1)$ and $f(3) \neq g(3)$. However, if we define $h = baba$; then there is a spider move f to h , and another from h to g , giving a sequence of spider moves from f to g , shown below.

A special case of the spider moves can be used to analyze homotopy equivalences. In the literature, homotopy has been linked to the idea of a **fold** or a **dismantling** [5, 6, 9, 14]. This can be seen as a special case of our more general spider moves.

Definition 4.7 If G is a graph, we say that a morphism $f : G \rightarrow G$ is a **fold** if f and the identity map are a spider pair.



Proposition 4.8 *If f is a fold, then $f : G \rightarrow Im(f)$ is a homotopy equivalence, where $Im(f)$ is as defined in Definition 2.2.*

Proof Since f and id_G form a spider pair, the map f is the identity on every vertex except one, call it v . If $f(v) = v$ then f is the identity and we are done.

If $f(v) = w \neq v$, then $Im(f) = G \setminus \{v\}$. Consider $\iota : Im(f) \rightarrow G$ to be the inclusion map. Then the composition $f\iota$ is the identity on $Im(f)$. Now consider $\iota f : G \rightarrow G$. Since ι is just the inclusion of the image, $\iota f = f$. By Lemma 4.2, $f \simeq id$.

We identify when we have a potential fold by a condition on neighborhood of vertices. In [5] [6] folds are defined using this condition. We denote the neighbourhood of a vertex v by $N(v) = \{w \in V(G) \mid w \sim v\}$.

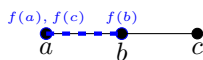
Proposition 4.9 *Suppose that $f : V(G) \rightarrow V(G)$ is a set map of vertices such that f is the identity on all vertices except one. Explicitly there exists a vertex $w \in V(G)$, and $f(x) = x$ for all $x \neq w$. Let $v = f(w)$. Then f is a fold if and only if $N(w) \subseteq N(v)$.*

Proof First, suppose that f is a fold, and hence a graph morphism. Let $y \in N(w)$; then $y \sim w$, and so $f(y) \sim f(w)$. If $y \neq w$, then $f(y) = y$ and $f(w) = v$, so $y \sim v$ and hence $y \in N(v)$. If $y = w$ then $w \sim w$ so by the looped condition for spider pair, we assume that $f(w) \sim id(w)$. So $v \sim w$ and $w \in N(v)$. Hence the neighbourhood condition is satisfied.

Conversely, suppose that f is a set map of vertices satisfying the neighbourhood condition. To show that f is a morphism, we check that it preserves all connections. If $x, x' \in V(G) \setminus \{w\}$ and $x \sim x'$, then $f(x) = x, f(x') = x'$, and so $f(x) \sim f(x')$. If $y \in V(G) \setminus \{w\}$ and $w \sim y$, then $y \in N(w) \subseteq N(v)$, so $v \sim y$ and hence $f(w) \sim f(y)$. Lastly, if w is looped, then $w \in N(w) \subseteq N(v)$, so $v \sim w$. But then $v \in N(w) \subseteq N(v)$, and consequently v must be looped as well. Thus $f(w) \sim f(w)$.

To see that f is a fold, we know that if $x \in V(G)$, we have that $f(x) = x$ if and only if $x \neq w$. So we just need to check that the extra condition on looped vertices holds. If $w \sim w$ then $w \in N(w) \subseteq N(v)$ and so $v \sim w$.

Example 4.10 Let $X = P_2$ and let $f : G \rightarrow G$ be defined by $f(a) = a, f(b) = b, f(c) = a$.



The vertex that f does not fix is c , and $N(c) = \{b\} = N(a)$. Hence the neighborhood condition of Proposition 4.9 holds here, and this is a fold map.

The fact that a fold, as defined using the neighbourhood condition, gives a homotopy equivalence is proved in [17] by looking at the simplicial Hom complex. Propositions 4.8 and 4.9 offer an alternate approach which is internal to graphs.

5 Defining a Homotopy Category for Finite Graphs

For this section, we restrict to finite graphs and consider the full sub-category FGph of Gph consisting of graphs with a finite set of vertices. We will be applying Proposition 4.4 to decompose homotopies as a sequence of spider moves, and so will require a finite set of vertices in our domains.

Since our 2-cells are defined by homotopies, known to be an equivalence relation on morphisms [6], we can make the following definition.

Definition 5.1 We define the **homotopy category** hFGph by modding out the 2-cells in the 2-category FGph . The objects of hFGph are the same as the objects of FGph , finite graphs, and the arrows of hFGph are given by equivalence classes $[f]$ of graph morphisms, where f and g are equivalent if they have a 2-cell between them, that is, if they are homotopic. This also defines a natural projection functor $\Psi : \text{FGph} \rightarrow \text{hFGph}$ which takes any graph G to G , and any morphism f to its homotopy class $[f]$.

Since all the 2-cells of FGph have become isomorphisms in hFGph , the result is an ordinary 1-category. We will show that this is a homotopy category for FGph in the sense that it satisfies the universal property for localizing homotopy equivalences as described in the following result.

Theorem 5.2 *Given any functor $F : \text{FGph} \rightarrow \mathcal{C}$ such that F takes homotopy equivalences to isomorphisms, then there is a unique functor $F' : \text{hFGph} \rightarrow \mathcal{C}$ such that $F'\Psi = F$.*

$$\begin{array}{ccc}
 \text{FGph} & & \\
 \downarrow \Psi & \searrow F & \\
 \text{hFGph} & \xrightarrow{\exists! F'} & \mathcal{C}
 \end{array}
 \quad =$$

Proof It is clear that $F' : \text{hFGph} \rightarrow \mathcal{C}$ needs to have $F'(G) = F(G)$ for any $G \in \text{Obj}(\text{hFGph})$ and $F'([f]) = F(f)$ for any $[f] \in \text{Hom}(\text{hFGph})$. Since $\text{Obj}(\text{Gph}) = \text{Obj}(\text{hFGph})$, we have that F' is well defined on $\text{Obj}(\text{hFGph})$. It remains to show that F' is well defined on $\text{Hom}(\text{hFGph})$: that is, given $f, f' \in [f]$, we always have $F(f) = F(f')$. By Proposition 4.4, it suffices to show that $F(f) = F(f')$ whenever f, f' are a spider pair.

Let $f, f' : G \rightarrow H$ be a spider pair. Then there is a vertex $v \in V(G)$ such that $f(w) = f'(w)$ for all $w \neq v$. Define a new graph \hat{G} as follows:

$$\begin{aligned}
 V(\hat{G}) &= V(G) \cup \{v^*\}. \\
 E(\hat{G}) &= \begin{cases} w_1 \sim w_2 & \text{when } w_1 \sim w_2 \in E(G), \\ v^* \sim w & \text{when } v \sim w \in E(G) \\ v^* \sim v^* & \text{when } v \sim v \in E(G) \end{cases}.
 \end{aligned}$$

Thus the new vertex v^* is attached to the same vertices as v , and is looped if and only if v is looped.

Let $\iota_1 : G \rightarrow \hat{G}$ be the inclusion defined by $\iota_1(w) = w$ for $w \in V(G)$. Let $\iota_2 : G \rightarrow \hat{G}$ be the inclusion defined by $\iota_2(w) = w$ for each $w \in V(G) \setminus \{v\}$ and $\iota_2(v) = v^*$. Since $N(v) = N(v^*)$ in \hat{G} , this is a graph morphism.

Define $\hat{f} : \hat{G} \rightarrow H$ by

$$\hat{f}(w) = \begin{cases} f(w) & \text{if } w \in V(G) \\ f'(v) & \text{if } w = v^* \end{cases}.$$

We claim that \hat{f} is a graph morphism: suppose $w_1 \sim w_2 \in E(\hat{G})$. If $w_1, w_2 \in V(G)$, then \hat{f} agrees with f , so since f is a graph morphism, $\hat{f}(w_1) \sim \hat{f}(w_2)$. Now suppose that $w_1 = v^*$ and $w_2 \in V(G)$. Then $\hat{f}(w_1) = f'(v)$ and $\hat{f}(w_2) = f(w_2) = f'(w_2)$. Then $f'(v) \sim f'(w_2)$ since f' is a graph morphism, so $\hat{f}(w_1) \sim \hat{f}(w_2)$. Lastly, if $w_1 = w_2 = v$ then $\hat{f}(w_i) = f'(v)$, which will be looped since v was looped and f' is a graph morphism.

It is clear from the definition that $f = \hat{f}\iota_1$ and $f' = \hat{f}\iota_2$. Define $\rho : \hat{G} \rightarrow G$ by

$$\rho(w) = \begin{cases} w & \text{if } w \in V(G) \\ v & \text{if } w = v^* \end{cases}$$

This is a fold by Proposition 4.9 since $N(v^*) = N(v)$. Moreover, $\rho\iota_1 = \rho\iota_2 = \text{id}_G$.

Replace with ref argument By Proposition 4.8, we know that ρ and ι_1, ι_2 are homotopy inverses, so $\iota_1\rho \simeq \text{id}_H \simeq \iota_2\rho$. Thus $F(\iota_1\rho) = F(\iota_1)F(\rho) = \text{id}_H = F(\iota_2)F(\rho) = F(\iota_2\rho)$. So $F(\rho)$ is an isomorphism and $F(\iota_1) = F(\iota_2)$.

Finally, we conclude that

$$\begin{aligned} F(f) &= F(\hat{f}\iota_1) \\ &= F(\hat{f})F(\iota_1) \\ &= F(\hat{f})F(\iota_2) \\ &= F(\hat{f}\iota_2) \\ &= F(f'). \end{aligned}$$

Thus, given $f, f' \in [f]$, we have that $F'([f]) = F(f) = F(f') = F'([f'])$ and F' is well defined.

6 A Skeleton for the Homotopy Category

Because we have created a homotopy category without a model structure, we look for another way to describe the structure of \mathbf{hFGph} . In this section, we will show that finite stiff graphs represent all finite graphs up to homotopy. More precisely, we show that the finite stiff graphs form a skeleton of the homotopy category \mathbf{hFGph} in the following sense.

Definition 6.1 [22] A full subcategory \mathbf{D} of a category \mathbf{C} is a **skeleton** of \mathbf{C} provided

- the inclusion $\mathbf{D} \hookrightarrow \mathbf{C}$ is essentially surjective, meaning that every object $C \in \mathbf{C}$ is isomorphic to an object $D \in \mathbf{D}$
- no two distinct objects of \mathbf{D} are isomorphic.

In the literature, graphs that cannot be folded are referred to as **stiff** graphs [5] [3].

Definition 6.2 We say that a graph G is **stiff** if there are no two distinct vertices v, w such that $N(v) \subseteq N(w)$.

Example 6.3 One large family of stiff graphs are **cores** [13, 14]. Since folds are graph morphisms, a core C cannot admit any folds and thus must be stiff. Therefore complete graphs, odd cycles, and all graphs where the only endomorphisms are automorphisms are minimal retracts.

Example 6.4 Another family of pleats is given by cycles of size 6 or greater. It is clear that C_4 will admit a fold, but for any greater cycle, distinct vertices can share at most 1 neighbor. The odd cycles are covered under Example 6.3; large even cycles are also stiff.

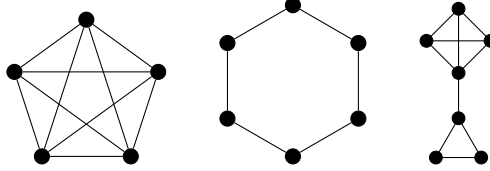


Fig. 6.1 Three examples of pleats. On the left a core K_5 , in the middle an even cycle C_6 , on the right a graph that is neither a core nor even cycle.

Let stGph refer to the full subcategory of finite stiff graphs in hFGph . Thus the objects of stGph are the finite stiff graphs, and the morphisms are homotopy classes of graph morphisms.

Theorem 6.5 stGph is a skeleton of hFGph in the sense of Definition 6.1.

We will consider the two conditions of Definition 6.1 separately.

Proposition 6.6 The inclusion $\text{stGph} \hookrightarrow \text{hFGph}$ is essentially surjective.

Proof We proceed via induction on $n := |V(G)|$. Note that if $n = 1$, G is necessarily stiff. Suppose $n > 1$ and G is not stiff, then there are distinct vertices v, w such that $N(v) \subseteq N(w)$, and we can define a fold map $\rho : G \rightarrow G - \{v\}$ which takes v to w and is the identity on all other vertices. By Propositions 4.8 and 4.9, this is a homotopy equivalence. By induction, $G - \{v\}$ is homotopy equivalent to a stiff graph, and thus G is as well.

To show the second condition, we note that any sequence of folds yields a unique graph up to isomorphism, proved in [3, 5, 14] in the context of cops and robbers on graphs. In [6] Proposition 6.6, Dochterman applies this and the interpretation of homotopy in the simplicial Hom complex to show that if G and H are stiff graphs then G and H are homotopy equivalent if and only if they are isomorphic, verifying the second condition for the skeleton. Here we offer an alternate proof which does not use the simplicial construction.

We start with the following.

Lemma 6.7 If G is stiff, then G is not homotopy equivalent to any proper subgraph of itself.

Proof We first show that for a stiff graph G , the identity is not homotopic to any other endomorphism. Suppose that $f \sim id_G$. Let $v \in V(G)$, and let $x \in N(v)$. Then $f(v) \sim id_G(x)$, i.e. $f(v) \sim x$ so $x \in N(f(v))$. So $N(v) \subseteq N(f(v))$. By the neighbourhood condition, we conclude that $v = f(v)$ and so $f = id$.

Then, suppose that G is homotopy equivalent to a subgraph of itself H . So we have $f : G \rightarrow H$ and $g : H \rightarrow G$ such that gf is homotopic to id_G . Then gf must actually be the identity on G . Hence G is isomorphic to H .

Lemma 6.8 *If $f : G \rightarrow G$ such that $f \sim id_G$ then G is homotopy equivalent to $Im(f)$.*

Proof Let ι denote the inclusion map $Im(f) \rightarrow G$. Then $\iota f = f$ which is homotopic to id_G . We need to show that $f \iota$ is homotopic to id_H where $H = Im(f)$. Suppose that $v \sim w \in E(H)$. Then by our definition of $Im(f)$ in Definition 2.2, this edge is the image of an edge $v' \sim w' \in E(G)$, where $f(v') = v, f(w') = w$. Since $f \sim id_G$, we know that $v' \sim f(w') \in E(G)$ and therefore $f(v') \sim f(f(w')) \in E(H)$. So $v \sim f \iota(w)$ whenever $v \sim w$, and so $id_H \sim f \iota$.

Theorem 6.9 ([6], Proposition 6.6) *If G, H are finite stiff graphs which are homotopy equivalent, then G and H are isomorphic.*

Proof Suppose we have graph morphisms $f : G \rightarrow H, g : H \rightarrow G$ such that $gf \simeq id_G$ and $fg \simeq id_H$. Thus by Proposition 4.4 we have a sequence of maps $id_G, k_1, k_2, \dots, k_n = gf$ such that each successive pair is a spider pair. So by Lemma 6.8 $Im(gf)$ is homotopy equivalent to G . Since G is stiff, it follows that $Im(gf) = G$. Similarly $Im(fg) = H$, and f, g are isomorphisms.

This completes the proof of Theorem 6.5.

Observation 6.10 Any graph which is not stiff may be folded. Thus, we obtain a homotopy equivalent graph by continuously applying folds as in Lemma 6.6. A consequence of Theorem 6.9 is that one may apply these folds in any arbitrary fashion, and the resulting pleats will be isomorphic. See Figure 4.1 below.

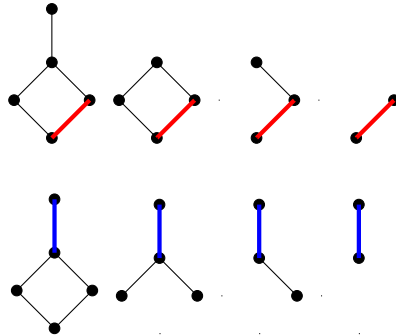


Fig. 6.2 Any series of folds of will eventually terminate with a subgraph isomorphic to K_2 , although not necessarily the same subgraph.

We end with some observations about the structure of the skeleton category $stGph$, and show that it respects some basic graph operations.

Definition 6.11 We will refer to a stiff graph obtained from applying folds to a graph G , which is homotopy equivalent to G , as a **pleat** of G , with notation $\mathcal{Pl}(G)$. Note that this is only defined up to isomorphism.

Corollary 6.12 *Two graphs G, H are homotopy equivalent if and only if their pleats $\mathcal{Pl}(G)$ and $\mathcal{Pl}(H)$ are isomorphic.*

Observation 6.13 If we have a graph G with an unlooped isolated vertex v , then $N(v) = \emptyset$, and thus v may be folded into any other vertex; and $\mathcal{P}\ell(G) = \mathcal{P}\ell(G \setminus \{v\})$ (unless $G \setminus \{v\} = \emptyset$). Thus unlooped isolated vertices are homotopically null.

Observation 6.14 If we consider graphs with no isolated vertices, then the disjoint union (or categorical coproduct [14, 21]) of pleats is itself a pleat, since it will be stiff. More generally, given any graph G without isolated vertices, $\mathcal{P}\ell(G \coprod H) \cong \mathcal{P}\ell(G) \coprod \mathcal{P}\ell(H)$.

We show that pleating also respects products in the absence of unlooped isolated vertices.

Proposition 6.15 *Let $G, H \in \text{FGph}$ be graphs with no unlooped isolated vertices. Then $\mathcal{P}\ell(G) \times \mathcal{P}\ell(H)$ is stiff.*

Proof Suppose $(v, w), (v', w') \in V(\mathcal{P}\ell(G) \times \mathcal{P}\ell(H))$ such that $N(v, w) \subseteq N(v', w')$. Since we do not have any isolated vertices, there exists $y \in N(w)$. Then for any $x \in N(v)$, $(x, y) \sim (v, w) \in E(\mathcal{P}\ell(G) \times \mathcal{P}\ell(H))$, and thus $(x, y) \sim (v', w') \in E(\mathcal{P}\ell(G) \times \mathcal{P}\ell(H))$. This implies that $x \in N(v')$. Therefore $N(v) \subseteq N(v')$. Since $\mathcal{P}\ell(G)$ is stiff, $v = v'$. Similarly, $w = w'$.

Lemma 6.16 *Let $v \in V(G)$ such that there is a fold $\rho : G \rightarrow G \setminus \{v\}$. Then $G \times H$ is homotopy equivalent to $G \setminus \{v\} \times H$.*

Proof Since there is a fold $\rho : G \rightarrow G \setminus \{v\}$, there is a $v' \in V(G)$ where $N(v) \subseteq N(v')$ and $\rho(v) = v'$ by Proposition 4.9. Let $w \in V(H)$. Given any $x \in N(v)$ and $y \in N(w)$, we have that $x \in N(v')$ and thus $(x, y) \in N(v', w)$. Thus $N(v, w) \subseteq N(v', w)$, and thus there is a fold from $\hat{\rho} : G \times H \rightarrow G \times H \setminus \{(v, w)\}$ where $\hat{\rho}(v, w) = (v', w)$.

We also note that if $N(v, w) \subseteq N(v', w')$, then $N_{G \setminus \{(v, w)\}}(v, w) \subseteq N_{G \setminus \{(v, w)\}}(v', w')$ since deletion of vertices preserves neighborhood containment. Thus, through a series of folds for each $w \in V(H)$, we have a homotopy equivalence between $G \times H$ and $G \setminus \{v\} \times H$.

Theorem 6.17 *Let $G, H \in \text{FGph}$ be graphs with no unlooped isolated vertices. Then $\mathcal{P}\ell(G \times H) \cong \mathcal{P}\ell(G) \times \mathcal{P}\ell(H)$.*

Proof Since $\mathcal{P}\ell(G)$ is the pleat of G , by Theorem 6.9, there is a series of folds from G to $\mathcal{P}\ell(G)$. Thus by repeated applications of Lemma 6.16, we have that $G \times H$ is homotopy equivalent to $\mathcal{P}\ell(G) \times H$, and then to $\mathcal{P}\ell(G) \times \mathcal{P}\ell(H)$. Since $\mathcal{P}\ell(G) \times \mathcal{P}\ell(H)$ is stiff by Proposition 6.15, Theorem 6.9 says that it is the pleat of $G \times H$.

Corollary 6.18 *If G, G' are homotopy equivalent and H, H' are homotopy equivalent, and none of these graphs have unlooped isolated vertices, then $G \times H$ is homotopy equivalent to $G' \times H'$.*

Proof We know that $\mathcal{P}\ell(G) \cong \mathcal{P}\ell(G')$ and $\mathcal{P}\ell(H) \cong \mathcal{P}\ell(H')$, so $\mathcal{P}\ell(G \times H) \cong \mathcal{P}\ell(G) \times \mathcal{P}\ell(H) \cong \mathcal{P}\ell(G') \times \mathcal{P}\ell(H') \cong \mathcal{P}\ell(G' \times H')$. Since $G \times H$ and $G' \times H'$ are both homotopy equivalent to $\mathcal{P}\ell(G) \times \mathcal{P}\ell(H)$, they are homotopy equivalent to each other.

Acknowledgements The authors are grateful to Dr. Demetri Plessas for his previous forays into categorical graph theory, and for his feedback. We also want to thank Dr. Jeffery Johnson for helping us with some of the terminology in this paper.

References

1. Babson, E., Barcelo, H., de Loungeville, M., Laubenbacher, R.: Homotopy theory of graphs. *Journal of Algebraic Combinatorics* **24**(1), 31–44 (2006)
2. Babson, E., Kozlov, D.N.: Proof of the Lovász conjecture. *Annals of Mathematics* **165**(3), 965–1007 (2007)
3. Bonato, A., Nowakowski, R.: *The Game of Cops and Robbers on Graphs*. Student mathematical library. American Mathematical Society (2010)
4. Bondy, J., Murty, U.: *Graph theory*. 2008. Grad. Texts Math (2008)
5. Brightwell, G.R., Winkler, P.: Gibbs measures and dismantlable graphs. *Journal of Combinatorial Theory, Series B* **78**(1), 141–66 (2000)
6. Dochtermann, A.: Hom complexes and homotopy theory in the category of graphs. *European Journal of Combinatorics* **30**(2), 490–509 (2009)
7. Dochtermann, A.: Homotopy groups of hom complexes of graphs. *Journal of Combinatorial Theory, Series A* **116**(1), 180–194 (2009)
8. Droz, J.M.: Quillen model structures on the category of graphs. *Homology, Homotopy and Applications* **14**(2), 265–284 (2012)
9. Fieux, E., Lacaze, J.: Foldings in graphs and relations with simplicial complexes and posets. *Discrete Mathematics* **312**(17), 2639–2651 (2012)
10. Godsil, C., Royle, G.: *Algebraic Graph Theory*. Springer (2001)
11. Goyal, S., Santhanam, R.: (lack of) model structures on the category of graphs (2005). URL <https://arxiv.org/abs/1902.09182>
12. Hatcher, A.: *Algebraic Topology*. Cambridge University Press (2001)
13. Hell, P., Nešetřil, J.: The core of a graph. *Discrete Mathematics* **109**, 117–126 (1992)
14. Hell, P., Nešetřil, J.: *Graphs and homomorphisms, Oxford Lecture Series in Mathematics and its Applications*, vol. 28. Oxford University Press, Oxford (2004)
15. Kozlov, D.N.: Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes (2005). URL <https://arxiv.org/abs/math/0505563>
16. Kozlov, D.N.: Collapsing along monotone poset maps. *International Journal of Mathematics* **8** (2006)
17. Kozlov, D.N.: Simple homotopy types of Hom-complexes, neighborhood complexes, Lovász complexes, and atom crosscut complexes. *Topology Applications* **14**, 2445–2454 (2006)
18. Kozlov, D.N.: A simple proof for folds on both sides in complexes of graph homomorphisms. *Proceedings of the American Mathematical Society* **134**(5), 1265–1270 (2006)
19. Mac Lane, S.: *Categories for the working mathematician, Graduate Texts in Mathematics*, vol. 5, second edn. Springer-Verlag, New York (1998)
20. Matsushita, T.: Box complexes and homotopy theory of graphs. *Homology, Homotopy and Applications* **19**(2), 175–197 (2017)
21. Plessas, D.: *The Categories of Graphs*. Ph.D. thesis, The University of Montana (2012)
22. Riehl, E.: *Category Theory in Context*. Aurora: Dover Modern Math Originals. Dover Publications (2017)