Contemporary Mathematics

Formality and S^1-Equivariant Algebraic Models

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ABSTRACT. We define a notion of S^1-equivariant formality using S^1-equivariant algebraic models. In order to apply the existing S^1-models, we show the existence of injective envelopes for the cohomology of S^1-minimal models.

1. Introduction

The idea of a space being formal is that its rational homotopy type is determined by its rational cohomology ring, simplifying many calculations. To make this precise, we can use the minimal model $\mathcal{M}_X$ of a space X defined by Sullivan in [**2, 10**], which is a commutative differential graded algebra (CDGA). A commutative differential graded algebra A is formal if it is quasi-isomorphic to its cohomology, that is, if there is a chain of quasi-isomorphisms $A \to \cdot \leftarrow \cdot \to \cdots \to H^*(A)$. For minimal algebras this is equvialent to requiring a single quasi-isomorphism $A \to H^*(A)$, and a space X is formal if its minimal model is formal. The algebra $\mathcal{M}_X$ encodes all rational homotopy information, including homotopy and homology groups, and for a formal space we get a quasi-isomorphism $\mathcal{M}_X \to H^*(\mathcal{M}_X) = H^*(X;\mathbb{Q})$; therefore rational homotopy type of a formal space can be recovered from the cohomology ring. Many classes of interesting spaces turn out to be formal, including Lie groups, classifying spaces and compact 1-connected Kähler manifolds; see for example the discussion in [**4**].

For actions of groups where an equivariant analogue of minimal models exist we can use them to define a notion of equivariant formality. Equivariant minimal models have been developed by Triantafillou for actions of finite groups [**11**], and by the author for the circle group $S^1 = \mathbb{T}$ [**9**]. The difficulty in defining formality in both cases is that not all objects in the algebraic category have minimal models; models only exist for objects which are "injective" in a certain sense. In particular, although the space X has a minimal model its cohomology ring may not. To get around this in the finite case, Fine and Triantafillou produce injective envelopes [**5, 6**] and use them in the definition of formality. This paper contains an analogous result for the $\mathbb{T}$-minimal models and develops the necessary technical results to be

1991 *Mathematics Subject Classification.* Primary 55P62; Secondary 55P91, 18G05.

Key words and phrases. Equivariant homotopy, minimal model, circle actions, rationalization, formality.

The author was supported in part by NSERC.

able to define $\mathbb{T}$-equivariant formality using minimal models. We illustrate this definition by showing that $\mathbb{T}$-equivariant Eilenberg-MacLane spaces are formal and using this to provide a couple of examples of formal spaces. Further properties of this definition and a comparison to alternate approaches to equivariant formality will be explored in a forthcoming paper.

2. $\mathbb{T}$-minimal models

This section contains a brief summary of the major results of the author [**9**] on the existence of $\mathbb{T}$-minimal models. For these results, $\mathbb{T}$-spaces are assumed to be $\mathbb{T}$-CW complexes; note that this ensures that all orbit spaces and related constructions, such as Borel spaces, are also CW complexes. In addition, all $\mathbb{T}$-spaces are assumed to have finitely many orbit types. We also assume that all $\mathbb{T}$-spaces are $\mathbb{T}$-simply connected in the sense that the fixed point subspaces X^H are all connected and simply connected (and also non-empty). Lastly, we assume that the rational cohomology of each fixed point subspace X^H is of finite type. We refer to spaces satisfying all of these conditions as $\mathbb{Q}$-good.

The equivariant homotopy type of a G-space X depends not only on the homotopy type of the space itself but also on the homotopy type of all the fixed point subspaces X^H for closed subgroups $H \leq G$. Together with the natural inclusions and maps induced by the action of G, these form a diagram of spaces where the shape of the diagram is described by the orbit category $\mathcal{O}_G$. This category has objects the canonical orbits G/H, with morphisms given by the equivariant maps between them. The fixed point subspaces X^H form a functor from $\mathcal{O}_G$ to spaces which completely describes the equivariant homotopy type, as shown by Elmendorf in [**3**]. When defining discrete algebraic invariants, we look at functors from the discrete homotopy category $h\mathcal{O}_G$ instead, which has the same objects G/H with homotopy classes of maps between them. Note that the objects of $\mathcal{O}_G$ will be abbreviated from G/H to H for simplicity of notation. Observe that if X is a G space and H is not an isotropy type of X, then the fixed set X^H is determined by the fixed sets X^K contained in it for $H \supset K$. Since the value of the functor at a subgroup H reflects H-fixed information, a space X corresponds to an algebraic functor such that for all subgroups H which are not isotropy subgroups of X, the value of the functor $A(H)$ is given by $A(H) = \lim_{K \supset H}(K)$. If the space has finitely many orbit types, the value of $A(H)$ will be determined in this way for all but a finite number of subgroups; such a functor will be said to have finitely many orbit types.

For the group in question $G = \mathbb{T}$, this indexing category can be described quite simply. Objects are orbits $\mathbb{T}/H$ for subgroups $H = \mathbb{Z}/n$. Group theory tells us that any equivariant map between orbits $\mathbb{T}/H \to \mathbb{T}/K$ is of the form $\hat{a} : gH \to gaK$ for some $a \in \mathbb{T}$ for which $a^{-1}Ha \subseteq K$; since $\mathbb{T}$ is abelian this is equivalent to $H \subseteq K$. Two such maps $\hat{a}$ and $\hat{b}$ are the same if and only if $aK = bK$, that is, $ab^{-1} \in K$. Thus the orbit category $\mathcal{O}_\mathbb{T}$ has morphisms

$$\text{Hom } (H, K) = \begin{cases} \mathbb{T}/K & \text{if } H \subseteq K \\ \emptyset & \text{otherwise} \end{cases}$$

All the equivariant maps from $\mathbb{T}/H$ to $\mathbb{T}/K$ are homotopic, since $\mathbb{T}$ is connected; so the homotopy orbit category $h\mathcal{O}_\mathbb{T}$ has exactly one morphism from H to K if

$H \subseteq K$ and no morphisms otherwise. This gives the shape of the diagram category we use to study $\mathbb{T}$-spaces.

The algebraic category used to model the rational homotopy of $\mathbb{T}$-spaces is a category of functors from $h\mathcal{O}_{\mathbb{T}}$ to commutative differential graded algebras. We work with CDGAs which are modules over $H^*(B\mathbb{T}) = \mathbb{Q}[c]$, the polynomial ring with a single generator of degree 2. Precisely, the category is given by the following.

Definition 2.1. ([**9**], Defn. 5.18) A $\mathbb{T}$-system consists of

(1) A covariant functor $\mathcal{A}$ from $hO_{\mathbb{T}}$ to the category of finitely generated CDGAs under $\mathbb{Q}[c]$ such that $\mathcal{A}$ has finitely many orbit types, and such that the functor is *injective* in a categorical sense when regarded by neglect of structure as a functor to rational vector spaces.
(2) A distinguished sub-CDGA $\mathcal{A}_{\mathbb{T}}$ of $\mathcal{A}(\mathbb{T})$ such that the map $\mathcal{A}_{\mathbb{T}} \otimes \mathbb{Q}[c] \to \mathcal{A}(\mathbb{T})$ is a quasi-isomorphism.

A morphism between $\mathbb{T}$-systems $\mathcal{A}$ and $\mathcal{B}$ is a natural transformation such that $\mathcal{A}_{\mathbb{T}}$ lands in $\mathcal{B}_{\mathbb{T}}$.

The restriction to injective objects is important because it allows us to define a minimal model. Minimal $\mathbb{T}$-systems are particularly nice $\mathbb{T}$-systems which are built up in stages from elementary extensions which are freely generated by a diagram of vector spaces; for further details see [**9**].

The most important properties of the minimal $\mathbb{T}$-systems are given by the following.

Proposition 2.2. *([**9**], Prop. 5.26) If $\mathcal{A}$ is a $\mathbb{T}$-system, then there is a minimal $\mathbb{T}$-system $\mathcal{M}$ with a quasi-isomorphism $\mathcal{M} \to \mathcal{A}$.*

Theorem 2.3. *([**9**], Prop. 5.23) If $f : \mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism between two minimal systems, then $f \simeq g$ where g is an isomorphism.*

Corollary 2.4. *([**9**], Cor. 5.24) If $\mathcal{M}$ and $\mathcal{M}'$ are two minimal systems and $\rho : \mathcal{M} \to \mathcal{A}$ and $\rho' : \mathcal{M}' \to \mathcal{A}$ are quasi-isomorphisms, then there is an isomorphism $f : \mathcal{M} \cong \mathcal{M}'$ such that $\rho' f \simeq \rho$.*

This shows that minimal models serve as preferred quasi-isomorphism class representatives, and allows us to make the following definition.

Definition 2.5. ([**9**], Defn. 5.25) If $\mathcal{M}$ is minimal and $\rho : \mathcal{M} \to \mathcal{A}$ is a quasi-isomorphism, we say that $\mathcal{M}$ is the *minimal model* of the system $\mathcal{A}$.

To get a $\mathbb{T}$-system associated to a $\mathbb{T}$-space X we use the Borel bundle construction combined with a suitable version of the functor of de Rham differential forms ([**9**], Defn. 4.12), denoted here by A.

Definition 2.6. ([**9**], Defn. 6.27) Let X be a $\mathbb{Q}$-good $\mathbb{T}$-space, and consider the Borel construction $X \times_{\mathbb{T}} E\mathbb{T}$. Let $\underline{\mathcal{E}}_{\mathbb{T}}(X)$ be the $\mathbb{T}$-system defined by

$$\underline{\mathcal{E}}_{\mathbb{T}}(X)(H) = \mathbf{A}(X^H \times_{\mathbb{T}} E\mathbb{T}),$$

with special sub-CDGA

$$\mathcal{E}_{\mathbb{T}} = \mathbf{A}(X^{\mathbb{T}}) \subset \mathbf{A}(X^{\mathbb{T}} \times B\mathbb{T}) = \underline{\mathcal{E}}_{\mathbb{T}}(X)(\mathbb{T}),$$

where the inclusion $\mathbf{A}(X^{\mathbb{T}}) \subset \mathbf{A}(X^{\mathbb{T}} \times B\mathbb{T})$ is induced by the projection $p_1 : X^{\mathbb{T}} \times B\mathbb{T} \to X^{\mathbb{T}}$.

This is a $\mathbb{T}$-system according to the definition above, as it is injective as a functor to vector spaces. Thus it has a minimal model $\mathcal{M}_X$, the minimal model of X. There is a quasi-isomorphism $\mathcal{M}_X \to \underline{\mathcal{E}}_{\mathbb{T}}(X)$, and $\mathcal{M}_X$ encodes rational homotopy information. The main theorem of [**9**] shows

Theorem 2.7. *([**9**], Thm. 6.28) Let X be a $\mathbb{Q}$-good $\mathbb{T}$-space, and $\mathcal{M}_X$ be the minimal model of $\underline{\mathcal{E}}_{\mathbb{T}}(X)$. Then the correspondence $X \to \mathcal{M}_X$ induces a bijection between rational homotopy types of $\mathbb{Q}$-good spaces and isomorphism classes of minimal $\mathbb{T}$-systems.*

Furthermore, $\mathcal{M}_X$ computes geometric information in the following way ([**9**], Sec. 6; 14-18).

- the cohomology of $\mathcal{M}_X(H)$ is equal to the rational cohomology of the Borel construction of the fixed set X^H, as a module over $H^*(B\mathbb{T}) = \mathbb{Q}[c]$
- $\mathcal{M}_X$ is generated by free $\mathbb{Q}[c]$-CDGA's on the diagram of vector spaces $\underline{\pi}^*$, the duals of the rational homotopy groups $\underline{\pi}_n(X^H_{\mathbb{Q}})$
- the Grivel-Halperin-Thomas theorem [**4**] implies that the non-equivariant minimal model of $X \times_{\mathbb{T}} E\mathbb{T}$ is given by $\mathcal{N} \otimes \mathbb{Q}[c]$, where $\mathcal{N}$ is the Sullivan minimal model of X. Therefore for any fixed set X^H, we can recover the Sullivan model $\mathcal{M}_{X^H}$ by taking a minimal model for $\mathcal{M}_X(H)/(c)$ where (c) is the ideal generated by c; and the cohomology of $\mathcal{M}_X(H)/(c)$ is the rational cohomology of X^H.

Thus the quasi-isomorphism class of the $\mathbb{T}$-system determines the rational homotopy type of the $\mathbb{T}$-space, and the minimal model provides a concrete way of calculating many rational geometric invariants.

3. Injective Envelopes

The most problematic technical consideration for working with $\mathbb{T}$-systems is the requirement that they be injective objects in the category of functors to rational vector spaces. An analysis of what such injectives look like was done in [**9**].

Proposition 3.1. *([**9**], Prop 8.37) A functor $\underline{V}$ from $h\mathcal{O}_{\mathbb{T}}$ to vector spaces is injective if the map $\underline{V}(H) \to \lim_{K \supset H} \underline{V}(K)$ induced by the structure maps of A is* surjective *for all H.*

We construct the injective systems from rational vector spaces as follows.

Definition 3.2. The injective functor from $h\mathcal{O}_{\mathbb{T}}$ to vector spaces generated by the vector space V at the subgroup H is

$$\underline{V}_H(G/K) = \begin{cases} V & \text{if } K \subseteq H \\ 0 & \text{otherwise} \end{cases}$$

with structure maps equal to either the identity or 0, as appropriate.

Observe that these meet the criterion given in Proposition 3.1. We use these as the basic building blocks for all injective systems.

Proposition 3.3. *([**9**], Prop 8.39) A functor $\underline{A}$from $h\mathcal{O}_{\mathbb{T}}$ to rational vector spaces is injective if and only if it is of the form $\underline{A} = \oplus_H \underline{V}_H$ for some collection of vector spaces V_H.*

Corollary 3.4. *([**9**], Cor 8.41) Any functor from $h\mathcal{O}_{\mathbb{T}}$ to rational vector spaces can be embedded in an injective system.*

We produce the injective by defining

$$V_H = \ker [\underline{A}(H) \to \lim_{K \supset H} \underline{A}(K)]$$

and using the natural map $\underline{A} \to \oplus \underline{V}_H$. When constructing the minimal model of the space, which is a $\mathbb{T}$-system and therefore injective, this procedure is used to create injective envelopes for the diagrams of vector spaces $\underline{\pi}^*$ given by the duals of the rational homotopy groups $\underline{\pi}_n^*(X)(H) = (\pi_n(X^H) \otimes \mathbb{Q})^*$.

Now we consider functors to algebras; we would like to construct an injective envelope which respects the multiplicative structure, and use this to get a minimal model. Precisely, we consider the following.

Definition 3.5. A coh-$\mathbb{T}$-system is a covariant functor from $h\mathcal{O}_{\mathbb{T}}$ to commutative graded $\mathbb{Q}[c]$-algebras, such that the functor has finitely many orbit types; together with a distinguished sub-algebra which is a $\mathbb{Q}[c]$-basis for the value of the functor at $\mathbb{T}$.

This is exactly the structure given by taking the cohomology of a $\mathbb{T}$-system.

Theorem 3.6. *Suppose $\mathcal{A}$ is a coh-$\mathbb{T}$-system. Then there is a $\mathbb{T}$-system $\mathcal{I}$ and an inclusion $\mathcal{A} \to \mathcal{I}$ which is a quasi-isomorphism when $\mathcal{A}$ is regarded as a functor to $\mathbb{Q}[c]$-CDGA's with zero differential.*

The $\mathbb{T}$-system $\mathcal{I}$ is called the injective envelope of $\mathcal{A}$. The strategy will be to create $\mathcal{I}$ by adding acyclic pieces to $\mathcal{A}$, designed to make the result injective without changing cohomology. The basic construction used is the following.

Definition 3.7. Suppose $\mathcal{A}$ is a coh-$\mathbb{T}$-system, and let V be a vector space with a map $V \to \lim_{K \supset H} \mathcal{A}(K)$. Let $\underline{V}_H$ be the injective system of vector spaces generated by V at H, as defined in 3.2. Let $s\underline{V}_H$ be a copy of $\underline{V}_H$ shifted up one degree, and define Λ_H to be the free acyclic $\mathbb{Q}[c]$-CDGA generated by $\underline{V}_H \oplus s\underline{V}_H$ with $d(\underline{V}_H) = s\underline{V}_H$. The enlargement $\mathcal{I}_H$ of $\mathcal{A}$ by V at H is defined to be

$$\mathcal{I}_H(K) = \begin{cases} \mathcal{A}(K) \otimes_{\mathbb{Q}[c]} \Lambda_H(K) & \text{if } K \subseteq H \\ \mathcal{A}(K) & \text{otherwise} \end{cases}$$

To define structure maps out of $\Lambda_H(H)$, we use the map $V \to \lim_{K \supset H} \mathcal{A}(K)$ to produce maps $V \to \mathcal{A}(K)$ for $K \supset H$; and send $sV(H)$ to zero. This can be extended over $\Lambda_H(H)$ by freeness. Since $d = 0$ on $\lim_{K \supset H} \mathcal{A}(K)$, this gives morphisms of $\mathbb{Q}[c]$-CDGAs. The structure maps on the rest of $\mathcal{I}_H$ are the obvious maps induced by these maps and the structure maps of $\mathcal{A}$ and Λ_H.

Observe that the inclusion $\mathcal{A} \to \mathcal{I}_H$ is a cohomology isomorphism since Λ_H is acyclic. The strategy will be to produce the injective envelope by enlarging as necessary, choosing vector spaces V_H to make the maps $\mathcal{I}_H \to \lim_{K \supset H} \mathcal{I}_H(K)$ onto.

Proof of Theorem 3.6. We create $\mathcal{I}$ by performing the enlargement defined above to create an injective system. By assumption $\mathcal{A}$ has finitely many orbit types, and so the map $\mathcal{A}(H) \to \lim_{K \supset H} \mathcal{A}(K)$ is an isomorphism for all but finitely many H. Let $\{H_i\}$ be the list of all problem subgroups for which this map fails to be surjective; we will perform an enlargement for each one.

We work by induction. Choose a subgroup H from the list of H_i which is not contained in any of the others; note that this means that for any $H' \subset H$, the map $\mathcal{A}(H') \to \lim_{K \supset H'} \mathcal{A}(K)$ is onto. Let $C_H = coker[\mathcal{A}(H) \to \lim_{K \supset H} \mathcal{A}(K)]$. This is a commutative graded $\mathbb{Q}[c]$-algebra; choose a set of $\mathbb{Q}[c]$-module generators and let V_H be the vector space they span. There is a vector space splitting $\lim_{K \subset H} \mathcal{I}_H(H) = im \oplus C_H$, and also $C_H = V_H \oplus W$. We use this to produce a map $V_H \to \lim_{K \subset H} \mathcal{I}_H(H)$, which we then use to create an enlargement $\mathcal{I}_H$. Observe that since V_H contains a generating set for C_H, and the enlargement contains a free $\mathbb{Q}[c]$-CDGA generated by V_H, the induced map $\mathcal{I}_H \to \lim_{K\supset H} \mathcal{I}_H(K)$ will be onto. All other structure maps in $\mathcal{I}_H$ are induced from $\mathcal{A}$ and the injective system Λ_H, and so away from H the maps $\mathcal{I}_H(H') \to \lim_{K\supset H'} \mathcal{I}_H(K)$ will fail to be onto in the same places that $\mathcal{A}$ failed; precisely, all H_i on the original list other than the chosen H.

We continue inductively through the finite list of H_i, adding an enlargement at each one; each enlargement removes one subgroup from the list of problem subgroups, and the end result is $\mathcal{I}$. The resulting system satisfies the condition of Proposition 3.1 and so is injective. Moreover, since each Λ_{H_i} is acyclic the inclusion map $\mathcal{A} \to \mathcal{I}$ is a quasi-isomorphism. □

The main application is the following.

Corollary 3.8. *If $\mathcal{A}$ is a $\mathbb{T}$-system and $\underline{H}$ is the functor obtained by taking the cohomology of $\mathcal{A}$, then $\underline{H}$ has an injective envelope.*

The injective envelope constructed above specifies a unique quasi-isomorphism type, as can be shown using the following lifting property.

Lemma 3.9. *If $\mathcal{A}$ is a coh-$\mathbb{T}$-system and $\mathcal{B}$ is a $\mathbb{T}$-system, and there is a morphism $\mathcal{A} \to \mathcal{B}$ then the morphism may be extended to a map of $\mathbb{T}$-systems from the injective envelope $\mathcal{I}$ of $\mathcal{A}$ making the following diagram commute.*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \\ \mathcal{I} \arrow[ur, dashed] & \end{tikzcd}$$

Proof. The morphism $\mathcal{A} \to \mathcal{B}$ induces commuting maps

$$\begin{tikzcd} \mathcal{A}(H)/(c) \arrow[r] \arrow[d] & \mathcal{B}(H)/(c) \arrow[d] \\ \lim_{K\supset H} \mathcal{A}(K)/(c) \arrow[r] & \lim_{K\supset H} \mathcal{B}(K)/(c) \end{tikzcd}$$

Since $\lim_{K\supset H} \mathcal{A}(K)/(c) = V_H \oplus W_H$ we can restrict to get a map $V_H \to \lim_{K\supset H} \mathcal{B}(K)/(c)$. Now $\mathcal{B}$ is a $\mathbb{T}$-system and therefore injective, so the vertical map on the right is onto. Therefore we can choose a lift and get a vector space map $V_H \to \mathcal{B}(H)/(c)$; and a vector space splitting $\mathcal{B}(H) = \mathcal{B}(H)/(c) \oplus W$ will allow us to extend this to a map $\phi : V_H \to \mathcal{B}(H)$. We then define a map $sV_H \to \mathcal{B}$ that commutes with the differential on $\mathcal{B}$; since $dV_H = sV_H$ this is determined by $\phi(sv) = d\phi(v)$. Using the injective property we can extend this to a map of functors of vector spaces $\underline{V}_H \oplus \underline{sV}_H \to \mathcal{B}$. Now for each H_i where an enlargement was done, Λ_{H_i} is a free acyclic $\mathbb{Q}[c]$-CDGA generated by a vector space $\underline{V}_{H_i} \oplus \underline{sV}_{H_i}$, and so ϕ extends to a map of systems of $\mathbb{Q}[c]$-CDGA's $\Lambda_H \to \mathcal{B}$. Repeating this extension process for each H_i gives the desired map from $\mathcal{I}$. □

Corollary 3.10. *If $\mathcal{A}$ is a coh-$\mathbb{T}$-system and $\mathcal{B}$ is a $\mathbb{T}$-system, and there is a quasi-isomorphism $\mathcal{A} \to \mathcal{B}$ then $\mathcal{B}$ is quasi-isomorphic to the injective envelope $\mathcal{I}$.*

Proof. We use the lifting property of Lemma 3.9 to produce a map $\mathcal{I} \to \mathcal{B}$ which must be a quasi-isomorphism by the commutativity of the diagram. □

4. $\mathbb{T}$-equivariant Formality

We now use the results of the previous section to give a definition for $\mathbb{T}$-equivariant formality using $\mathbb{T}$-minimal models. We begin by considering formal $\mathbb{T}$-systems. We cannot mimic the non-equivariant definition exactly because the functor given by the cohomology of a $\mathbb{T}$-system may not itself be a $\mathbb{T}$-system; but we can use the injective envelope of Theorem 3.6 instead.

Definition 4.1. A minimal $\mathbb{T}$-system is formal if there is a quasi-isomorphism of $\mathbb{T}$-systems $\mathcal{M} \to \mathcal{I}$, where $\mathcal{I}$ is an injective envelope of $\underline{H}^*(\mathcal{M})$.

We use the minimal model of a $\mathbb{T}$-space X to define formality for spaces.

Definition 4.2. A $\mathbb{T}$-space X is formal if its minimal model $\mathcal{M}_X$ is formal.

Recall that the cohomology of the minimal model computes the cohomology of the Borel space of the various fixed sets $\underline{H}^*(\mathcal{M}_X) = \underline{H}^*(X^H \times_{\mathbb{T}} E\mathbb{T}; \mathbb{Q})$. Therefore the condition that X is formal is equivalent to the existence of a quasi-isomorphism of $\mathbb{T}$-systems $\mathcal{M}_X \to \mathcal{I}_X$, where $\mathcal{I}_X$ is the injective envelope of $\underline{H}^*(X^H \times_{\mathbb{T}} E\mathbb{T}, \mathbb{Q})$. Because the $\mathbb{T}$-minimal model encodes the rational $\mathbb{T}$-homotopy type, this notion of formality can be interpreted as meaning that the equivariant rational homotopy type of the space is determined by its equivariant cohomology, as long as we use a suitable interpretation of equivariant cohomology. In this case, we take equivariant cohomology to refer to the functor determined by taking the Borel cohomology $H^*(X^H \times_{\mathbb{T}} E\mathbb{T}; \mathbb{Q})$, as a $\mathbb{Q}[c]$-module, of the diagram of fixed point sets $\{X^H\}$.

There are several alternate definitions of equivariant formality in use, which make sense for the action of more general groups. In [**8**], Lillywhite defines equivariant formality of a G-space X by also looking at the Borel bundle $X_G = X \times_G EG \to BG$ and taking the CDGA of differential forms $A(X_G)$; this is an object in the category of augmented CDGA's under $H^*(BG)$. A formal space is defined to be one which admits a quasi-isomorphism of $H^*(BG)$-CDGA's between the minimal model of the Borel construction $\mathcal{M}_{X \times_G EG}$ and $H^*(X \times_G EG, \mathbb{Q})$ in this category. Because this takes into account the action of $H^*(BG)$, this is stronger than simply requiring that the space $X \times_{\mathbb{T}} E\mathbb{T}$ be non-equivariantly formal; this definition could be interpreted to mean that the rational homotopy type of the Borel construction, as a bundle, is determined by its cohomology as an $H^*(BG)$-module. However, the Borel construction does not determine the equivariant homotopy type of the original space, even when considered as a bundle instead of simply a space. In order to capture the equivariant homotopy type of a G-space, we need information about the fixed sets. So this definition does not translate into capturing the equivariant homotopy type of X.

Yet another definition of G-equivariant formality has been given by Goresky-Kottwitz-Macpherson in [**7**], where the authors interpret the collapse of the spectral sequence associated to the Borel construction of the space as a type of equivariant formality. A future paper will develop further comparisons concerning these various definitions in the case of the circle group $\mathbb{T}$.

It is natural to ask about the equivariant formality of various classes of spaces such as H-spaces or Kähler manifolds. This is work in progress; it is expected that Kähler manifolds are in fact formal under some reasonable conditions. The proof of this poses some difficulty, however, and even for the case of finite group actions it is not yet proven. The proof given in [**6**] was based on some incorrect results in [**11**] and has not currently been fixed.

One class of spaces which are easily shown to be formal are equivariant Eilenberg-MacLane spaces.

THEOREM 4.3. *If $K = K(\underline{\pi}, n)$ is a $\mathbb{T}$-equivariant Eilenberg-MacLane space, then K is formal.*

PROOF. By the definition of $\mathbb{T}$-equivariant Eilenberg-MacLane space, each fixed point set $K^H = K(\underline{\pi}(H), n)$ is an ordinary non-equivariant Eilenberg-MacLane space, so $H^*(K^H, \mathbb{Q}) = \mathbb{Q}(\underline{\pi}(H))$; and the maps $i^* : H^*(K^{H'}) \to H^*(K^H)$ are induced by the structure maps of the dual coefficient system $\underline{\pi}^*$. When we consider the Borel construction, we see that the inclusions induce maps of the fibrations over $B\mathbb{T}$ and the structure maps for the functor $\underline{H}^*(K^H \times_{\mathbb{T}} E\mathbb{T})$ are given by $i^* \otimes id : H^*(K^{H'}) \otimes \mathbb{Q}[c] \to H^*(K^H) \otimes \mathbb{Q}[c]$. So as a functor,

$$\mathcal{H} = \underline{H}^*(K^H \times_{\mathbb{T}} E\mathbb{T}) = \mathbb{Q}(\underline{\pi}^*) \otimes \mathbb{Q}[c].$$

To produce a minimal model $\mathcal{M}$ for K, we take an injective resolution of $\underline{H}^n = \underline{\pi}^*$ given by

$$\underline{\pi}^* \to \underline{V}_0 \to \underline{V}_1 \to \dots,$$

and let

$$\mathcal{M} = \otimes_i \mathbb{Q}(\underline{V}_i) \otimes \mathbb{Q}[c]$$

be the $\mathbb{T}$-system with differential $d = v_i$ induced by the resolution, with sub-DGA $\mathcal{V}_{\mathbb{T}} = \otimes_i \mathbb{Q}(\underline{V}_i(\mathbb{T}/\mathbb{T})) \otimes \mathbb{Q}$. (See [**9**], Section 9 for further discussion.)

To show formality, observe that there is a map $\mathcal{H} \to \mathcal{M}$ induced by the inclusion $\underline{\pi}^* \to \underline{V_0}$; and Lemma 3.9 gives an extension to the injective envelope $\mathcal{I} \to \mathcal{M}$ which must be a quasi-isomorphism by commutativity. The uniqueness of the minimal model given by Corollory 2.4 shows that $\mathcal{M}$ is the minimal model of $\mathcal{I}$ and so K is $\mathbb{T}$-formal. □

We end by using this result to provide the following examples of formal spaces.

EXAMPLE 4.4. Let $\underline{\pi}$ be the diagram of vector spaces given by $\underline{\pi}(e) = \mathbb{Q}$ and $\underline{\pi}(H) = 0$ for all other subgroups H. Then $K(\underline{\pi}, 3)$ has contactible fixed sets $X^H \simeq *$ for all $H \neq \mathbb{T}$, and $X = K(\mathbb{Q}, 3) \simeq S^3$. We can construct such a space by defining $X = S^3 \wedge S^\infty_+$ where S^3 has a trivial $\mathbb{T}$-action and S^∞ has a free $\mathbb{T}$-action. The system of vector spaces $\underline{\pi}^*$ is injective, and so the minimal model of the space is given by $\mathbb{Q}(\underline{\pi}^*) \otimes \mathbb{Q}[c]$; in this case, $\mathcal{M}_X(H) = \mathbb{Q}[c]$ for all subgroups $H \neq e$, and $\mathcal{M}_X(e) = \mathbb{Q}(x) \otimes \mathbb{Q}[c]$ where x has degree 3 and the differential $d = 0$. Observe that in this case the cohomology of this $\mathbb{T}$-system $\mathcal{H} \cong \mathcal{M}_X$, and no injective envelope is needed.

EXAMPLE 4.5. Let $\underline{\pi}$ be the diagram of vector spaces given by $\underline{\pi}(e) = 0$ and $\underline{\pi}(H) = \mathbb{Q}$ for all other subgroups H. Then $K(\pi, 3)$ has $X^H \cong K(\mathbb{Q}, 3) \simeq S^3$ for all $H \neq e$, and X itself is contractible. We can construct such a space by defining $X = S^\infty$ with $\mathbb{T}$-action $\lambda[z_0, z_1, z_2, z_3, z_4, \dots] = [z_0, z_1, \lambda z_2, \lambda z_3, \lambda z_4, \dots]$. Observe that in this case $\underline{\pi}^*$ is *not* injective as a vector space, and so to produce the minimal model we take the injective resolution

$$\begin{tikzcd}
\underline{\pi}^*(e) = 0 \arrow[r, hook] \arrow[d] & \mathbb{Q} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\
\underline{\pi}^*(H) = \mathbb{Q} \arrow[r, hook] & \mathbb{Q} \arrow[r] & 0
\end{tikzcd}$$

Thus the minimal model is given by

$$\begin{array}{c} \mathcal{M}_X(e) = \mathbb{Q}(x,y) \otimes \mathbb{Q}[c] \\ \downarrow \\ \mathcal{M}_X(H) = \mathbb{Q}(x) \otimes \mathbb{Q}[c] \end{array}$$

where x has degree 3 and $d(x) = y$ in $\mathcal{M}_X(e)$. The homology of the system is given by

$$\begin{array}{c} \mathcal{H}(e) = \mathbb{Q}[c] \\ \downarrow \\ \mathcal{H}(H) = \mathbb{Q}(x) \otimes \mathbb{Q}[c] \end{array}$$

which is again not injective; the map $\mathcal{H}(e) \to \lim_{K \supset e} \mathcal{H}(K)$ is not surjective, and $coker\{\mathcal{H}_X(\mathbb{T}) \to \lim_{K \supset \mathbb{T}} \mathcal{H}_X(K)\}$ is generated by the elements xc^m. To create the injective envlelope we can choose x to be a $\mathbb{Q}[c]$-generator and define the enlargement $\mathcal{H}_X \otimes \Lambda_{\mathbb{T}}(x)$ where $\Lambda_{\mathbb{T}}(x)(H) = 1$ and $\Lambda_{\mathbb{T}}(x)(T)$ is the free acyclic $\mathbb{Q}[c]$-CDGA generated by x and sx, with $d(x) = sx$. Thus we see that in this case the injective envelope of $\mathcal{H}_X$ is isomorphic to the minimal model $\mathcal{M}_X$.

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