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On the equivariant formality of Kähler manifolds with torus group actions

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Abstract We consider simply connected compact Kähler manifolds which have a holomorphic action of a torus group. We use the existing equivariant models for rational homotopy to show that these spaces satisfy an equivariant formality condition over the complex numbers.

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1 Introduction

A space is formal if its rational homotopy type is determined by its rational cohomology ring. To make this precise, formality is defined using the category of commutative differential graded algebras (CDGAs). A CDGA is formal if it is quasi-isomorphic to its cohomology ring, regarded as a CDGA with differential $d = 0$. This means that $\mathcal{A}$ is formal if there is a zig-zag of quasi-isomorphisms $\mathcal{A} \to \mathcal{A}_1 \leftarrow \mathcal{A}_2 \to \cdots \to H^*(\mathcal{A})$. For CDGAs which are minimal in the sense of Sullivan ([8], [19]), formality can be described more simply, since the zig-zag of quasi-isomorphisms in this case is equivalent to the existence of a single quasi-isomorphism $\mathcal{A} \to H^*(\mathcal{A})$.

To define formality for spaces, we use the minimal model $\mathcal{M}_X$ of a space X from ([8], [19]), which is a minimal CDGA. A space X is formal if its minimal model $\mathcal{M}_X$ is formal. The CDGA $\mathcal{M}_X$ encodes all rational homotopy information, including homotopy and homology groups; in particular, the cohomology $H^*(X;\mathbb{Q}) = H^*(\mathcal{M}_X)$. If a space X is formal then there is a quasi-isomorphism $\mathcal{M}_X \to H^*(\mathcal{M}_X) = H^*(X;\mathbb{Q})$, and we may compute the model $\mathcal{M}_X$ of X as the minimal model for the rational cohomology ring $H^*(X;\mathbb{Q})$. Therefore the

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rational homotopy type of a formal space can be recovered from the cohomology ring. Since cohomology is generally more accessible than other invariants, this can make calculations much easier. Several interesting classes of spaces turn out to be formal, and the paper of Deligne, Griffith, Morgan, and Sullivan [8] which introduced Sullivan's minimal models used them to show the formality of simply connected compact Kähler manifolds as its primary application.

Equivariantly, where algebraic models exist which describe the rational homotopy type of G-spaces, they can be used to define an analogous notion of formality. For actions of finite groups, Triantafillou and Fine [10] have shown that simply connected compact Kähler manifolds are equivariantly formal over the complex numbers. In this paper, we extend this result and consider actions of torus groups $\mathbb{T}$. Working with complex coefficients, we prove that simply connected compact Kähler manifolds with holomorphic $\mathbb{T}$-actions are equivariantly formal. This result uses the definition of equivariant formality from [17], which is based on the algebraic models for $\mathbb{T}$-spaces developed in [14] and [15].

Throughout this paper, the $\mathbb{T}$-spaces under consideration will be compact manifolds with a smooth action of a torus $\mathbb{T}$; in particular, this implies that they are $\mathbb{T}$-CW complexes, and all orbit spaces and related constructions, such as Borel spaces, are also CW complexes. In addition, compactness assures that all $\mathbb{T}$-spaces have finitely many orbit types and that the rational cohomology of each fixed point subspace $X^H = \{x \in X \mid hx = x \text{ for all } h \in H\}$ is of finite type for all closed subgroups $H \subseteq \mathbb{T}$. We will also assume that all $\mathbb{T}$-spaces are based, with basepoint fixed by the $\mathbb{T}$-action, and $\mathbb{T}$-simply connected in the sense that the connected components of the fixed point subspaces X^H are all simply connected. These will be standing assumptions in what follows; note that we are not assuming that the fixed point sets are also connected, as needed for the minimal models of [15], but instead are allowing disconnected fixed sets as in [14], as is necessary for the study of Kähler manifolds.

Section 2 gives a quick sketch of the background material on equivariant formality. Section 3 contains the proof of the equivariant formality of compact Kähler $\mathbb{T}$-manifolds. Section 4 has two simple examples of computations for a linear circle action on complex projective spaces, and Section 5 contains the proof of an important but somewhat involved proposition used to translate from the usual equivariant model to a more convenient one in the proof of the main theorem.

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2 Equivariant Models and Formality

The equivariant homotopy type of a space X with the action of a group G depends not only on the homotopy type of the space itself but also on the homotopy type of the fixed point subspaces X^H for all closed subgroups $H \subseteq G$. Together with the natural inclusions and maps induced by the action of G, these form a diagram of spaces indexed by the closed subgroups of G. Much of equivariant homotopy theory makes use of this diagram, and we often define algebraic invariants by constructing diagrams of algebras reflecting the fixed point data.

In defining equivariant algebraic models for the rational homotopy of spaces with actions of torus groups, we use the diagram category defined in [14]. Because we are dealing with actions of connected torus groups, we do not need the full generality of the diagrams defined in [14], Definition 1.2, but can instead use the simplified diagrams discussed in [14], Section 4. Therefore we define our indexing category $\mathcal{D}$ as follows. The objects of $\mathcal{D}$ are pairs of closed subgroups $\{K[H] \,|\, H \subseteq K, H \text{ connected }\}$ of $\mathbb{T}$, and maps are defined by

$$\mathcal{D}(K_1[H_1], K_2[H_2]) = \begin{cases} \pi_0(\mathbb{T}/H_1) & \text{if } H_2 \subseteq H_1 \subseteq K_1 \subseteq K_2 \\ \emptyset & \text{otherwise} \end{cases}$$

Such a diagram is designed to give a framework for information about the fixed sets X^K of a $\mathbb{T}$-space X, and the extra indexing subgroup allows us to consider a fixed set X^K as a $\mathbb{T}/H$ space for any connected $H \subseteq K$, allowing for the necessary comparison maps.

To get from a space to a $\mathcal{D}$-diagram, we construct the diagram whose entries are the Borel spaces $\{X^K \times_{\mathbb{T}/H} E(\mathbb{T}/H)\}$ of the various fixed sets associated to pairs of subgroups $K[H]$. The inclusions of the fixed sets $X^{K_2} \subseteq X^{K_1}$ for $K_1 \subseteq K_2$ and the projection maps $\mathbb{T}/H_2 \to \mathbb{T}/H_1$ for $H_2 \subseteq H_1$ induce the necessary structure maps to give a (contravariant) diagram of shape $\mathcal{D}$. Note that just as all spaces have a map to the terminal one-point object in the category of spaces, these diagrams have natural maps to the diagram of classifying spaces $B(\mathbb{T}/H)$ associated with the one-point space. The map $B(\mathbb{T}/H_2) \to B(\mathbb{T}/H_1)$ is a rational equivalence when the index of H_2 in H_1 is finite; moreover, so is the projection $X^K \times_{\mathbb{T}/H_2} E(\mathbb{T}/H_2) \to X^K \times_{\mathbb{T}/H_1} E(\mathbb{T}/H_1)$. This is what allows us to index all our diagrams on the smaller indexing category discussed here, which has as its objects pairs $K[H]$ for connected subgroups H in K, rather than indexing on all pairs of nested subgroups.

By considering these diagrams of bundles and applying a suitable version of de Rham differential forms functor Ω, we obtain a (covariant) diagram of CDGAs; we refer to such diagrams as $\mathcal{D}{-}CDGA$s. The diagram obtained from the Borel bundles of a space also comes with a map from an initial diagram object $\underline{P}$, where the object $\underline{P}$ is obtained by applying Ω to the diagram associated with the one point space, the diagram of classifying spaces $B(\mathbb{T}/H)$. In fact we can simplify this by taking the entries of $\underline{P}$ to be the minimal models for these CDGAs; since the classifying spaces are formal, this $\underline{P}$ has entries which are the CDGAs given by the rational cohomologies of the classifying spaces $B(\mathbb{T}/H)$. The diagram $\underline{\Omega}(X)$ obtained by taking differential forms of the Borel bundles is an object in the category of $\mathcal{D}{-}CDGAs$ under $\underline{P}$, and is an algebraic model which encodes $\mathbb{T}$-equivariant rational homotopy information. This describes the mechanism for passing from a $\mathbb{T}$-space to an algebraic object, and the algebraic category used for the models is the homotopy category of $\mathcal{D}{-}CDGAs$ under $\underline{P}$.

Note that when considering diagrams of spaces, it makes sense (and is sometimes necessary) to consider diagrams with structure maps between entries induced by all equivariant maps $\mathbb{T}/H_2 \to \mathbb{T}/H_1$. Once we pass to an algebraic category like CDGAs, however, we need only consider homotopy classes of structure maps, and so use the discrete indexing category $\mathcal{D}$. It is worth making a couple of observations about this category, which gives the basic shape of the algebraic diagrams considered. First, for a torus group $\mathbb{T}$, the morphisms in the indexing category $\mathcal{D}$ are defined by connected torus groups $\mathbb{T}/H$. So in the discrete indexing category, the structure maps between entries are unique where they exist. Moreover, there are only two basic types of morphisms we need to consider: maps $K[H_1] \to K[H_2]$ for $H_2 \subseteq H_1$, and maps $K_1[H] \to K_2[H]$ for $K_1 \subseteq K_2$. Any other structure map is the composition of maps $K_1[H_1] \to K_1[H_2] \to K_2[H_2]$. This will be useful when defining diagrams in $\mathcal{D}{-}CDGA$ later.

A quasi-isomorphism in the category of $\mathcal{D}{-}CDGAs$ under $\underline{P}$ is a morphism of diagrams which is a quasi-isomorphism at each entry. As with CDGAs, an object $\mathcal{A}$ in the category $\mathcal{D}{-}CDGAs$ under $\underline{P}$ is formal if there is a zig-zag of quasi-isomorphisms $\mathcal{A} \to \mathcal{A}_1 \leftarrow \mathcal{A}_2 \to \cdots \to \underline{H}^*(\mathcal{A})$ between $\mathcal{A}$ and the diagram $\underline{H}^*(\mathcal{A})$ defined by taking the cohomology of each entry $\mathcal{A}(K[H])$. For a space X with the action of a torus group, we define X to be $\mathbb{T}$-equviariantly formal if its model $\underline{\Omega}(X)$ is formal as a $\mathcal{D}{-}CDGA$ under $\underline{P}$. Thus we can interpret equivariant formality as meaning that the equivariant rational homotopy type of the space is determined by its equivariant cohomology, where in this case equivariant cohomology means the diagram of the cohomology of the Borel bundles, which is also the diagram of the Borel cohomology of the fixed sets.

If we consider actions by the circle group $\mathbb{T}^1$, there are more explicit minimal models which can be used, which also have more transparent encoding of geometric information [15]. These can be used to define formality as in [16], and this definition is equivalent to the one just discussed [17]. However, currently these models have only been defined for spaces whose fixed sets are connected as well as simply connected, and so are not well suited for studying Kähler manifolds. Therefore we work here with the more general models described above.

3 Formality of Kähler Manifolds

We now turn to the study of equivariant Kähler manifolds and the proof of the main result. Recall that a Kähler manifold is a complex manifold M admitting a positive definite Hermitian metric

$$H(x,y) = S(x,y) + iA(x,y) \text{ for } x,y,\in T(M)$$

which satisfies the Kähler condition: the imaginary part A of the metric is closed, that is, $dA = 0$. The complex structure is vital to understanding the topology of Kähler manifolds, and so the natural setting for our algebraic investigations is over the complex numbers. Note that a compact simply connected Kähler manifold with a holomorphic action of a torus group will always contain a fixed point to act as the basepoint [11], and so satisfies the conditions for the models discussed in the previous section to exist.

Formality can be defined over any extension field k of $\mathbb{Q}$ by requiring a quasi-isomorphism $\mathcal{M}_X \otimes k \to H^*(X;k)$. In the original work on (nonequivariant) Kähler manifolds, Deligne et al [8] prove formality over $\mathbb{C}$ and $\mathbb{R}$, and then Sullivan's generalization [19] contains (among other things) an argument to descend to formality over $\mathbb{Q}$. In [10] where Triantafillou and Fine consider spaces with actions of finite groups, a similar strategy is used, and their paper proves equivariant formality over $\mathbb{C}$. They give an argument for descent to formality over $\mathbb{Q}$, but this argument does not work when the corrected definition of equivariant minimal model is used; see [15] for a discussion of this issue.

We will work with complex coefficients and do not attempt a descent to $\mathbb{Q}$ here. The main therem of this paper is the following.

Theorem 3.1. *Suppose M is a compact simply connected Kähler manifold with a holomorphic action of a torus group $\mathbb{T}$. Then M is equivariantly formal over $\mathbb{C}$.*

The proof of this theorem consists of two parts. As discussed in the previous section, we usually use a version of the de Rham differential form functor applied to the Borel bundles of the fixed sets to pass from geometry to algebra. In order to make use of the Kähler structure of M, however, we would like a model in which the forms on M appear directly, rather than the forms on the Borel space. Thus we introduce an alternate model based on the Cartan complex. The first part of the proof is to translate from the usual model to this Cartan version, which we do in Proposition 3.2. We then use the Cartan diagram to show formality.

We begin by describing the constuction of the Cartan complex, which is quite general. Let M be a smooth manifold with a smooth action of a connected compact Lie group G. The dual Lie algebra of G is denoted $\mathfrak{g}^*$, and $S(\mathfrak{g}^*)$ is the symmetric algebra generated by $\mathfrak{g}^*$. If the set of vector fields $\{\xi_k\}$ is a basis for $\mathfrak{g}$, then let $\{\xi^k\}$ denote the dual basis for $\mathfrak{g}^*$. Then $\{\xi^k\}$ is also a set of algebra generators for $S(\mathfrak{g}^*)$, and we make $S(\mathfrak{g}^*)$ into a graded algebra by giving the generators degree 2.

We denote the C^∞ de Rham complex of differential forms on M by $\Omega^\infty(M)$. The Cartan complex of equivariant differential forms is defined by

$$A^\bullet_G(M) = ([\Omega^\infty(M) \otimes S(\mathfrak{g}^*)]^G, d_G)$$

The differential d_G is defined to be zero on $S(\mathfrak{g}^*)$, and on $\Omega^\infty(M)$ is given by the formula

$$d_G = d + \Sigma \xi^k i_{\xi_k}$$

where i_{ξ_k} denotes interior multiplication by the generating vector field ξ_k. This complex was first defined by Cartan in ([4], [5]), and its properties are discussed at length in [12]. In the case of the Abelian torus group $\mathbb{T}$, we can get an alternate description of this complex

$$A^\bullet_{\mathbb{T}}(M) = \Omega^\infty_{\mathbb{T}}(M) \otimes S(\mathfrak{t}^*) \simeq \Omega^\infty_{\mathbb{T}}(M)[u_k]$$

where u_k are polynomial generators in degree 2 and $\Omega^\infty_{\mathbb{T}}(M)$ are the (left) $\mathbb{T}$-invariant forms (that is, those for which $\mathcal{L}(\xi_k)(\omega) = 0$ for the generating vector fields ξ_k of the $\mathbb{T}$-action) [1].

Now consider a manifold M with a smooth action of the torus group $\mathbb{T}$. Any fixed set M^K is also a manifold with a smooth action of $\mathbb{T}$, and in fact an action of the quotient torus group $\mathbb{T}/H$ for any connected $H \subseteq K$. Therefore we can define a Cartan object $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ of M in the category of $\mathcal{D}{-}CDGAs$ under $\underline{P}$ as follows. (Since the Cartan model is intrinsically real rather than rational, we need to consider the real version of $\mathcal{D}{-}CDGAs$ under $\underline{P}$, where the CDGAs have underlying real vector spaces and the initial object is $\underline{P}_{\mathbb{R}} = \underline{P} \otimes_{\mathbb{Q}} \mathbb{R}$; we continue to denote this by the same notation.) As with the de Rham model, the idea is that the entry $K[H]$ comes from the fixed set M^K regarded as a $\mathbb{T}/H$-manifold; accordingly, we define the entries $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)(K[H]) = A^\bullet_{\mathbb{T}/H}(M^K)$. We also need to describe the structure maps.

Recall that the homotopy classes of morphisms in the indexing category $\mathcal{D}$ are all compositions of the two basic types of maps: $K_1[H] \to K_2[H]$ for $K_1 \subseteq K_2$, and $K[H_1] \to K[H_2]$ for $H_2 \subseteq H_1$. The inclusion maps between the fixed sets $M^{K_2} \to M^{K_1}$ are smooth and equivariant, so they induce a restriction of differential forms $\phi : \Omega^\infty(M^{K_1}) \to \Omega^\infty(M^{K_2})$ in which invariant forms are preserved. So we get structure maps $A^\bullet_{\mathbb{T}/H}(M^{K_1}) \to A^\bullet_{\mathbb{T}/H}(M^{K_2})$ of the form $\phi \otimes id$ for a fixed H. Structure maps associated to different subgroups $H_2 \subseteq H_1 \subseteq K$ are induced by projections $\mathbb{T}/H_2 \to \mathbb{T}/H_1$, which in turn induce maps $\mathfrak{t}/\mathfrak{h}_1^* \to \mathfrak{t}/\mathfrak{h}_2^*$; so we get maps $\Omega^\infty_{\mathbb{T}/H_1}(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}_1^*) \to \Omega^\infty_{\mathbb{T}/H_2}(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}_2^*)$. Now observe that if $H_i \subseteq K$, any form defined on M^K will be invariant under the action of $\mathbb{T}/H_i$ if and only if it is invariant under the action of $\mathbb{T}/K$; and so these maps will pass to the invariant elements and give the necessary structure maps $A^\bullet_{\mathbb{T}/H_1}(M^K) \to A^\bullet_{\mathbb{T}/H_2}(M^K)$. Taking M to be a one-point space yields a diagram whose entry at $K[H]$ is $S(\mathfrak{t}/\mathfrak{h}^*)$. This is isomorphic to $\underline{P}$, since $S(\mathfrak{t}/\mathfrak{h}^*)$ is isomorphic to $H^*(B\mathbb{T}/H; \mathbb{R})$ and a model for $\Omega^\infty(B\mathbb{T}/H)$ [12]. Therefore the inclusions $S(\mathfrak{t}/\mathfrak{h}^*) \to A^\bullet_{\mathbb{T}/H}(M^K)$ induce the basing map $\underline{P} \to \underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ and we have an object in the category of $\mathcal{D}{-}CDGAs$ under $\underline{P}$.

This Cartan diagram can be used in place of the equivariant model $\underline{\Omega}(M)$ of [14], as shown by the following.

Proposition 3.2. *In the category $\mathcal{D}{-}CDGAs$ under $\underline{P}$, the Cartan diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ is quasi-isomorphic to the model $\underline{\Omega}(M)$.*

The comparison between these diagrams goes through a number of intermediates, and the proof consists of careful checking that each of these comparisons is sufficiently natural to give a quasi-isomorphism of diagrams. We defer this proof to Section 5. In fact, what we need is the following immediate corollary.

Corollary 3.3. *A $\mathbb{T}$-manifold M is equivariantly formal if and only if the Cartan diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ is formal.*

We are now ready to prove the main theorem.

Proof (of Theorem 3.1).

We adapt the argument from [8] to the equivariant setting. We begin with a brief sketch of the original proof, and show how the Kähler structure on a (non-equivariant) manifold M can be used to construct a zig-zag of quasi-isomorphisms between the complex de Rham CDGA of smooth forms $\Omega^\infty(M;\mathbb{C})$ and its cohomology ring $H^*(M;\mathbb{C})$.

The complex structure can be used to refine the grading to a bidgrading on $\Omega^\infty(M;\mathbb{C})$ by splitting the complexified real tangent bundle $T(M)$ into conjugate subbundles $T'(M) \oplus \overline{T'(M)}$, which splits the fundamental vector fields ξ_k into holomorphic and anti-holomorphic parts $\xi_k = Z_k + \overline{Z}_k$. There is an induced splitting on the cotangent bundle $\Lambda^n T(M)^*$ defined by $\Lambda^{p,q}T(M)^* = \Lambda^p T'(M)^* \otimes \Lambda^q \overline{T'(M)}^*$, and each differential form can be written as a unique sum of its (p,q) components; the differential splits as $d = \partial + \overline{\partial}$. The spectral sequences associated to the filtrations in each degree both degenerate at E_1, and the two filtrations are n-oppposite on $H^n(M;\mathbb{C})$, meaning that the two filtrations $'F^p$ and $''F^q$ satisfy $'F^p \oplus'' F^q \simeq H^n(M;\mathbb{C})$ for $p+q-1=n$. This condition can be re-expressed in a number of ways, including:

The $\partial\overline{\partial}$ Lemma ([8], Prop. 5.11) If α is a differential form which is ∂-closed and $\overline{\partial}$-closed, and $\alpha = \partial\beta$, then $\alpha = \partial\overline{\partial}\gamma$ for some γ.

Using the $\partial\overline{\partial}$ lemma, it is straightforward to show that the maps

$$\{\Omega^\infty(M;\mathbb{C}), d\} \xleftarrow{i} \{ker(\overline{\partial}), \partial\} \xrightarrow{\rho} \{H_{\overline{\partial}}(M), \partial\}$$

are quasi-isomorphisms, and also that the differential induced by ∂ on $H_{\overline{\partial}}(M)$ is zero ([8], Section 6). Thus the cohomology of $\{H_{\overline{\partial}}(M), \partial\}$ is $H^*(M;\mathbb{C})$ and we have exhibited a chain of quasi-isomorphisms connecting $\Omega^\infty(M)$ to its cohomology ring.

The ideas of this proof have been adapted to equivariant spaces and Borel cohomology; see Teleman [21] for a sheaf-theoretic approach, and also Lillywhite [13]. Here, we apply the outline of the argument in [21] to the diagram category $\mathcal{D}{-}CDGAs$ under $\underline{P}$. We will invoke Corollary 3.3 to show formality, and so the basic tool will be the Cartan diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ of the Kähler $\mathbb{T}$-manifold M. Note that from here on we will work with complex coefficients, so we use $S(\mathfrak{t}/\mathfrak{h}^*) \otimes_{\mathbb{R}} \mathbb{C}$ which we will continue to denote by $S(\mathfrak{t}/\mathfrak{h}^*)$, and define the Cartan complex by $[\Omega^\infty(M^K;\mathbb{C}) \otimes_{\mathbb{C}} S(\mathfrak{t}/\mathfrak{h}^*)]^{\mathbb{T}/H}$. All cohomology will also have complex coefficients and we will supress this in the notation.

Since the $\mathbb{T}$-action is holomorphic, all of the fixed sets M^K associated to the action are also Kähler. The entries of the diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ consist of $A^\bullet_{\mathbb{T}/H}(M^K)$ for the various subgroups $H \subseteq K$ of $\mathbb{T}$. We use the bigrading described above on $\Omega^\infty(M)$, and extend this bigrading to the Cartan complex by giving $\mathfrak{t}/\mathfrak{h}^*$ bidegree $(1,1)$. Similarly we split the equivariant differential as $d = (\overline{\partial} + \Sigma\xi_k i_{Z_k}) + (\partial + \Sigma\xi_k i_{\overline{Z}_k}) = \overline{\partial}_{\mathbb{T}} + \partial_{\mathbb{T}}$. The result is that each entry of the diagram becomes a first-quadrant double complex.

Consider the filtration defined by the first degree, under the differential $\overline{\partial}_{\mathbb{T}} = (\overline{\partial} + \Sigma\xi_k i_{Z_k})$. We get a spectral sequence

$$E_1^{p,q} = H([\Omega^p(M^K) \otimes S^q(\mathfrak{t}/\mathfrak{h}^*)]^{\mathbb{T}/H}, \overline{\partial}_{\mathbb{T}}) \Rightarrow H^{p+q}(M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H)) \quad \text{(SS1)}$$

To identify this E_1 term, we further refine the p-grading into a bigrading with respect to the two differentials $\overline{\partial}$ and $\Sigma\xi_k i_{Z_k}$ and get a double complex and a spectral sequence

$$\hat{E}_1^{r,s} = H^s(\Omega^r(M^K) \otimes S^q(\mathfrak{t}/\mathfrak{h}^*)]^{\mathbb{T}/H}, \overline{\partial}) \Rightarrow E_1^{p,q} \qquad \text{(SS2)}$$

Now we calculate $\hat{E}_1$:

$$\begin{aligned} H([\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}^*)]^{\mathbb{T}/H}; \overline{\partial}) &\simeq [H(\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}^*)); \overline{\partial}]^{\mathbb{T}/H} \\ &\simeq [H^*_{\overline{\partial}}(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}^*)]^{\mathbb{T}/H} \\ &\simeq H^*_{\overline{\partial}}(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}^*) \\ &\simeq H^*_{\overline{\partial}}(M^K) \otimes H^*(B(\mathbb{T}/H)) \end{aligned}$$

But each M^K is (nonequivariantly) a compact Kähler manifold, and therefore $H^*_{\overline{\partial}}(M^K) \simeq H^*(M^K)$ [8], and

$$\hat{E}_1 \simeq H^*_{\overline{\partial}}(M^K) \otimes H^*(B(\mathbb{T}/H)) \simeq H^*(M^K) \otimes H^*(B(\mathbb{T}/H))$$

We are looking at a holomorphic action of a torus group, and so each component of the space M^K has a $\mathbb{T}$-fixed base point; then a theorem of Blanchard ([2], Chapter XII) now states that the Serre spectral sequence associated to the bundle $M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H) \to B(\mathbb{T}/H)$ collapses. So the target for the original spectral sequence SS1 is $H^*(M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H)) \simeq H^*(M^K) \otimes H^*(B(\mathbb{T}/H))$ and we observe that as graded vector spaces, the $\hat{E}_1$ entries of the second spectral sequence SS2 are already isomorphic to the target vector spaces of SS1. Therefore there can be no non-trivial differentials in either spectral sequence and they both collapse at the first stage.

A similar argument shows that an analogous result also holds for the complex conjugate filtration. Thus we have produced two filtrations of the entries of the diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ associated to $\partial_{\mathbb{T}}$ and $\overline{\partial}_{\mathbb{T}}$ such that the spectral sequence associated to each collapses. Moreover, we recall that these filtrations were induced by n-opposite filtrations on $\Omega^\infty(M^K)$; with the collapse of the associated spectral sequences to $H^*(\Omega^\infty(M^K)) \otimes H^*(S(\mathfrak{t}/\mathfrak{h}^*))$ we see that the two equivariant filtrations on on $A^\bullet_{\mathbb{T}/H}(M^K)$ are also n-opposite. Therefore the same algebraic argument produces an equivariant $\partial_{\mathbb{T}}\overline{\partial}_{\mathbb{T}}$-lemma as described above.

The inclusion maps between the M^K are all holomorphic, and so the splitting $d = \partial_{\mathbb{T}} + \overline{\partial}_{\mathbb{T}}$ is natural with respect to the inclusions of fixed points. On the Cartan complex, the change of group structure maps are induced by inclusions of generators $\mathfrak{t}/\mathfrak{h}_1^* \to \mathfrak{t}/\mathfrak{h}_2^*$. Therefore the arguments of [8], Section 6 can be used to produce quasi-isomorphisms between the diagrams

$$\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M) \hookleftarrow \underline{ker}(\overline{\partial}_{\mathbb{T}}) \to \underline{H}^*(\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M))$$

By Corollary 3.3 this implies that M is equivariantly formal over the complex numbers.

4 Example: Complex Projective Spaces

In this section we discuss a couple of examples coming from linear actions of the circle group on a complex projective space. The circle group $\mathbb{T}^1$ has only two connected subgroups, e and $\mathbb{T}^1$; and so the indexing category $\mathcal{D}$ is relatively simple. Diagrams associated to a $\mathbb{T}^1$-space X have entries at $K[e]$ associated to the space $X^K \times_{\mathbb{T}^1} E\mathbb{T}^1$ for each closed subgroup K of $\mathbb{T}^1$; the only other entry in the diagram is for the subgroup pair $\mathbb{T}^1[\mathbb{T}^1]$, associated to the space $X^{\mathbb{T}^1}$. The base diagram $\underline{P}$ has constant value $\underline{P}(K[e]) = H^*(B\mathbb{T}^1) \simeq \mathbb{C}[c]$ for a polynomial generator c of degree 2, except for the entry $\underline{P}(\mathbb{T}^1[\mathbb{T}^1]) = H^*(*) = \mathbb{C}$. Therefore we can think of a diagram in $\mathcal{D}{-}CDGA$ under $\underline{P}$ as a diagram of $\mathbb{C}[c]$-module CDGAs indexed on subgroups K of $\mathbb{T}^1$, with one extra basing map $\mathbb{T}^1[\mathbb{T}^1] \to \mathbb{T}^1[e]$ which is induced by the projection $X^{\mathbb{T}^1} \times B\mathbb{T}^1 \to X^{\mathbb{T}^1}$. This alternate description is used in [15] for describing equivariant models for actions of the circle group.

A linear action of S^1 on $\mathbb{C}P^n$ is given by $\lambda[z_0 : z_1 : z_2 : \cdots : z_n] = [\lambda^{w_0} z_0 : \lambda^{w_1} z_1 : \lambda^{w_2} z_2 : \cdots : \lambda^{w_n} z_n]$ and is determined by the values of the integer weights $w_0, w_1, \ldots, w_n$. The Borel cohomology of such a space is well understood; as a module over $H^*(B\mathbb{T}) = \mathbb{C}[c]$, it is freely generated by $H^*(\mathbb{C}P^n) = \mathbb{C}[x]/x^{n+1}$ where x is a degree two generator. Moreover, the inclusion of the $\mathbb{T}$ fixed set into the space induces an injection of the Borel cohomology into the Borel cohomology of the fixed space, and the ring structure can be presented as $H^*_{\mathbb{T}}(X) = \mathbb{C}[x,c]/\prod(x - w_i c)$ where w_i are the weights of the action.

We can apply this to a couple of specific examples to demonstrate the calculation of the equivariant models.

Example 4.1. Consider the space $X = \mathbb{C}P^1$ with $\mathbb{T}^1$- action given by $\lambda[z_0 : z_1] = [\lambda z_0 : z_1]$; so the weights are $w_0 = 1$ and $w_1 = 0$. This space is semi-free, with two fixed points $[1 : 0]$ and $[0 : 1]$. So for any nontrivial subgroup $K \subseteq \mathbb{T}^1$, the fixed set $X^K = X^{\mathbb{T}^1}$. Thus in any diagram $\mathcal{B}$ associated to X, we will have that $\mathcal{B}(K[e]) = \mathcal{B}(\mathbb{T}^1[e])$ with the identity map between them for any subgroups $K \neq e$ of $\mathbb{T}^1$. Therefore all of the information in such a system is contained in the portion $\mathcal{B}(e[e]) \to \mathcal{B}(\mathbb{T}[e])$ together with the basing map $\mathcal{B}(\mathbb{T}[\mathbb{T}]) \to \mathcal{B}(\mathbb{T}[e])$.

By Theorem 3.1, the $\mathbb{T}$-equivariant homotopy type of X is determined by the diagram $\mathcal{H}_X$ of Borel cohomologies of the fixed sets X^H. The fixed set consists of two isolated points and so its Borel cohomology is $H^*(B\mathbb{T}) \oplus H^*(B\mathbb{T}) = \mathbb{C}[c] \oplus \mathbb{C}[c]$; the projection $X^{\mathbb{T}^1} \times B\mathbb{T}^1 \to X^{\mathbb{T}^1}$ induces the inclusion $\mathbb{C} \oplus \mathbb{C} \to \mathbb{C}[c] \oplus \mathbb{C}[c]$ as the degree zero piece. The Borel cohomology of the entire space $H^*(X \times_{\mathbb{T}} E\mathbb{T})$ can be calculated from the weights using the formula discussed above, and can be written as $\mathbb{C}[x,c]/x(x-c)$; so the generator x satisfies $x^2 = xc$, and as a module over $\mathbb{C}[c]$ the cohomology is freely generated by the two generators 1 and x.

Therefore the cohomology diagram $\mathcal{H}_X$ of X is given by

$$\begin{array}{ccc} & & \mathcal{H}_X(e[e]) = \mathbb{C}[x,c]/(x^2 = xc) \\ & & \downarrow \\ & & \mathcal{H}_X(K[e]) \\ & & \| \\ \mathcal{H}_X(\mathbb{T}^1[\mathbb{T}^1]) = \mathbb{C} \oplus \mathbb{C} & \longrightarrow & \mathcal{H}_X(\mathbb{T}^1[e]) = \mathbb{C}[c] \oplus \mathbb{C}[c] \end{array}$$

where the non-trivial vertical structure map takes $1 \to (1,1)$ and $x \to (0,c)$. Our theorem states that this is a formal space, and so this is a model for the equivariant complex homotopy type.

Example 4.2. Another simple linear action we consider is the action of $\mathbb{T}^1$ on $Y = \mathbb{C}P^2$ defined by $\lambda[z_0 : z_1 : z_2] = [z_0 : \lambda z_1 : \lambda^2 z_2]$, with weights $0, 1$ and 2. This space has three isolated fixed points $Y^{\mathbb{T}^1} = [1:0:0] \cup [0:1:0] \cup [0:0:1]$. In addition, there is a subspace $[z_0 : 0 : z_2] \cup [0:1:0] \simeq \mathbb{C}P^1 \cup [0:1:0]$ which is fixed by the subgroup $\mathbb{Z}/2 \subseteq \mathbb{T}^1$ generated by $e^{\pi i}$. Therefore to encode the information about Y in a digaram $\mathcal{B}$, we need to consider the entries $\mathcal{B}(e[e]) \to \mathcal{B}(\mathbb{Z}/2[e]) \to \mathcal{B}(\mathbb{T}^1[e])$ together with the basing map $\mathcal{B}(\mathbb{T}^1[\mathbb{T}^1]) \to \mathcal{B}(\mathbb{T}^1[e])$. From this portion of the diagram, we can fill in the rest by observing that for any non-trivial $H \subset \mathbb{T}^1$, $Y^H = Y^{\mathbb{Z}/2}$ if $H = \mathbb{Z}/2$ and $Y^H = Y^{\mathbb{T}^1}$ otherwise; and so $\mathcal{B}(H[e]) = \mathcal{B}(\mathbb{Z}/2[e])$ if $H = \mathbb{Z}/2$, and $\mathcal{B}(H[e]) = \mathcal{B}(\mathbb{T}^1[e])$ otherwise.

The cohomology of the fixed set $Y^{\mathbb{T}^1}$ is given by $\mathbb{C}[c] \oplus \mathbb{C}[c] \oplus \mathbb{C}[c]$, since the fixed set consists of 3 isolated points, with the projection $X^{\mathbb{T}^1} \times B\mathbb{T}^1 \to X^{\mathbb{T}^1}$ inducing the inclusion $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \to \mathbb{C}[c] \oplus \mathbb{C}[c] \oplus \mathbb{C}[c]$ as the degree zero piece. The cohomology of the entire space $Y = Y^e$ can be calculated using the weights, and we get $H^*(Y \times_{\mathbb{T}^1} E\mathbb{T}^1) = \mathbb{C}[y,c]/y(y-c)(y-2c)$; so the generator y satisfies $y^3 = 3y^2c - 2yc^2$, and the module generators of the cohomology are $1, y, y^2$. We can apply the same formula to calculate the Borel cohomology of the fixed space $\mathbb{C}P^1 \subset Y^{\mathbb{Z}/2}$, since the action on this set is also linear, defined by $\lambda[z_0 : 0 : z_2] = [z_0 : 0 : \lambda^2 z_2]$ with weights 0 and 2; so the cohomology can be presented as $H^*(Y^{\mathbb{Z}/2} \times_{\mathbb{T}^1} E\mathbb{T}^1) = \mathbb{C}[x,c]/x(x-2c) \oplus \mathbb{C}[c]$.

Therefore the cohomology diagram $\mathcal{H}_Y$ of Y, and model for the equivariant complex homotopy type of Y, is given by

$$\begin{array}{ccc} & & \mathcal{H}_Y(e[e]) = \mathbb{C}[y,c]/y(y-c)(y-2c) \\ & & \downarrow \alpha \\ & & \mathcal{H}_Y(\mathbb{Z}/2[e]) = \mathbb{C}[x,c]/x(x-2c) \oplus \mathbb{C}[c] \\ & & \downarrow \beta \\ \mathcal{H}_Y(\mathbb{T}^1[\mathbb{T}^1]) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} & \longrightarrow & \mathcal{H}_Y(\mathbb{T}^1[e]) = \mathbb{C}[c] \oplus \mathbb{C}[c] \oplus \mathbb{C}[c] \end{array}$$

The srtructure maps are $\mathbb{C}[c]$-maps induced by the inclusions of the fixed sets, and the map α is defined by $1 \to (1,1)$ and $y \to (x,c)$, while β is defined by $(1,0) \to (1,0,1)$, $(0,1) \to (0,1,0)$ and $(x,0) \to (0,0,2c)$. Note that the map $\mathcal{H}_Y(e[e]) \to \mathcal{H}_Y(\mathbb{T}^1[\mathbb{T}^1])$ is determined by $\beta\alpha$, and is defined by $1 \to (1,1,1)$ and $y \to (0,c,2c)$.

5 Proof of Proposition 3.2

This section contains the proof of the key fact that the Cartan model may be used in place of the standard equivariant model, as we did in the main theorem. The comparison between these diagrams goes through a number of intermediates. In order to work with some of these, it will be convenient to cut down on the size of our diagrams. Therefore we begin with a discussion of how to do this.

For any subset $\mathcal{S}$ of the set of closed subgroups of $\mathbb{T}$, we can restrict the diagram category $\mathcal{D}{-}CDGAs$ under $\underline{P}$ to the entries indexed by the full subcategory on pairs of subgroups from $\mathcal{S}$. Conversely, suppose that $\mathcal{S}$ is closed under arbitrary intersections, and also that for any subgroup H of $\mathcal{S}$, the connected identity component H_0 is also in $\mathcal{S}$. Then we can extend a restricted diagram $\mathcal{N}$ to an object $\widetilde{\mathcal{N}}$ of $\mathcal{D}{-}CDGAs$ under $\underline{P}$. We fill in the missing entries as follows: for any pair $K[H]$, let K' be the intersection $\cap K_i$ of all K_i in $\mathcal{S}$ such that $K \subseteq K_i$, and let H' be the identity component of K'. Then since $K \subseteq K'$ and $H \subseteq K$ with H connected, we also have that $H \subseteq H'$; so we have a map $\underline{P}(K'[H']) \to \underline{P}(K[H])$ induced by the projection $T/H \to T/H'$. Thus we can define $\widetilde{\mathcal{N}}(K[H])$ to be $P(K[H]) \otimes_{P(K'[H']))} \mathcal{N}(K'[H'])$. Then any map of restricted diagrams $\mathcal{N}_1 \to \mathcal{N}_2$ can be extended to a map $\widetilde{\mathcal{N}}_1 \to \widetilde{\mathcal{N}}_2$, and we also have the following result.

Lemma 5.1. *If $\mathcal{N}_1 \to \mathcal{N}_2$ is a quasi-isomorphism of $\mathcal{S}$-diagrams, then the induced map $\widetilde{\mathcal{N}}_1 \to \widetilde{\mathcal{N}}_2$ is a quasi-isomorphism in $\mathcal{D}{-}CDGAs$ under $\underline{P}$.*

Proof. We need to show that we have a quasi-isomorphism between the entries $K[H]$ which are not in $\mathcal{S}$. Since both $\widetilde{\mathcal{N}}_1$ and $\widetilde{\mathcal{N}}_2$ are defined by the process described above, we need to consider the induced map $\widetilde{\mathcal{N}}_1(K[H]) = P \otimes_{P'} \mathcal{N}_1(K'[H']) \to \widetilde{\mathcal{N}}_2(K[H]) = P \otimes_{P'} \mathcal{N}_2(K'[H'])$ where $P = P(K[H])$ and $P' = P(K'[H']))$. We are assuming that the map $\mathcal{N}_1 \to \mathcal{N}_2$ is a quasi-isomorpism of $\mathcal{S}$-diagrams, and therefore $\mathcal{N}_1(K'[H']) \to \mathcal{N}_2(K'[H'])$ is a quasi-isomoprhism, since by constructrion K' and H' belong to $\mathcal{S}$. But in addition, the map $P' \to P$ is given by the map $H^*(B(\mathbb{T}/H')) \to H^*(B(\mathbb{T}/H))$ induced by the projections $\mathbb{T}/H \to \mathbb{T}/H'$ for $H \subseteq H'$, where H, H' are subtori; so this map can be expressed as an inclusion of generators $\mathbb{Q}[c_1, c_2, \ldots, c_i] \to \mathbb{Q}[c_1, c_2, \ldots, c_i, \ldots, c_n]$. Therefore, as a P'-module, P is free, and so the tensor product is exact and preserves quasi-isomorphisms.

Our interest in this construction comes from the fact that for any compact $\mathbb{T}$-manifold, there are only a finite number of isotropy subgroups of the group action. This means that we do not actually need all of the entries of $\underline{\Omega}(M)$ to encode the information contained in a space; we only need those indexed by isotropy groups.

For a given $\mathbb{T}$-manifold M, we can define $\mathcal{I}_M$ to be the collection of subgroups generated by intersections of isotropy subgroups of M. This gives a diagram indexed on a finite number of pairs of subgroups $K[H]$, which still contains all the information needed to describe the structure of M.

Lemma 5.2. *Suppose that M is a compact $\mathbb{T}$-manifold and let $\mathcal{I}_M$ be the set of all closed subgroups of $\mathbb{T}$ which can be produced by taking intersections of isotropy subgroups of M and their connected components. Let $\underline{\Omega}|(M)$ denotes the restriction of $\underline{\Omega}(M)$ to those entries indexed by subgroups in $\mathcal{I}_M$. Then $\widetilde{\underline{\Omega}|(M)}$ is quasi-isomorphic to $\underline{\Omega}(M)$.*

Proof. The entries of $\underline{\Omega}(M)$ are defined by $\Omega(M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H))$, where the $\mathbb{T}/H$ action on M^K is induced by the $\mathbb{T}$ action; since $H \subseteq K$, the space M^K is fixed by H and so the $\mathbb{T}$ action factors through the projection $\mathbb{T} \to \mathbb{T}/H$. Now if H, K are not both in $\mathcal{I}_M$, then $\widetilde{\underline{\Omega}|(M)}$ is defined by $P(K[H]) \otimes_{P(K'[H']))} \Omega(M^{K'} \times_{\mathbb{T}/H'} E(\mathbb{T}/H'))$.

The fixed set M^K consists of all points whose isotropy subgroups contain K, so it is equal to the union $\cup M^{K_i}$ over all isotropy subgroups K_i which contain K. Thus if K' is the intersection of all the isotropy subgroups K_i in $\mathcal{S}$ containing K, then $M^K = M^{K'}$. Moreover, since H' is the connected component of K', then for any $H \subseteq H'$ the $\mathbb{T}/H$-action on $M^{K'}$ is induced by the projection $T/H \to T/H'$. Therefore $M^{K'} \times_{\mathbb{T}/H} E(\mathbb{T}/H)$ is the pullback of the fibration $M^{K'} \times_{\mathbb{T}/H'} E(\mathbb{T}/H') \to B(\mathbb{T}/H')$ over the map $B(\mathbb{T}/H) \to B(\mathbb{T}/H')$. For any fibration of spaces, the CDGA associated to the pullback fibration is quasi-isomorphic to the CDGA pushout of the associated commutative square ([9] Section 15); this pushout is precisely the tensor product $P(K[H]) \otimes_{P(K'[H']))} \Omega(M^{K'} \times_{\mathbb{T}/H'} E(\mathbb{T}/H'))$. Therefore we have that $\Omega(M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H)) = \underline{\Omega}(M)(K[H])$ is quasi-isomorphic to $\widetilde{\underline{\Omega}|(M)}(K[H])$ as claimed.

A quick check of the definitions also shows that a similar result holds for the Cartan diagram.

Lemma 5.3. *If we restrict the Cartan diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ to entries indexed by isotropy subgroups $\mathcal{I}_M$ and denote this restricted diagram by $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}|(M)$, then $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ is quasi-isomorphic to $\widetilde{\underline{\mathbf{A}}^\bullet_{\mathbb{T}}|(M)}$.*

These results on restricted diagrams allow us to make use of the following restricted version of Proposition 3.2.

Proposition 5.4. *If we restrict the to entries indexed by isotropy subgroups $\mathcal{I}_M$, the Cartan diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}|(M)$ is quasi-isomorphic to the model $\underline{\Omega}|(M)$.*

Proof. All diagrams in this proof will be restricted diagrams over $\mathcal{I}_M$; we will drop the bar from the notation for simplicity.

The version of the functor Ω used in the original models of [14] is the Sullivan - de Rham rational polynomial forms applied to the singular simplicial set of the space (see [18]; also defined by Thom, [20]). The first step in the desired comparison is to translate from this model to one using the de Rham algebra of smooth differential forms Ω^∞. For any manifold M, a good functorial way to do this is outlined in [9], Section 11, which we now describe.

In order to make a comparison between smooth and rational polynomial forms, we use as an intermediate the simplicial set given by smooth simplicial forms: for a manifold M, the simplicial complex $S_*^\infty(M)$ is defined using k-simplices which are smooth maps $\triangle^k \to M$ which extend smoothly to some neighbourhood of $\triangle^k$ in $\mathbb{R}^{k+1}$. Then we can define smooth differential forms on $S_*^\infty(M)$ using the smooth structure on each simplex. The complex $S_*^\infty(M)$ is a sub-complex of the singular simplicial complex of M, and the inclusion induces a map $\Omega(S_*(M);\mathbb{R}) = \Omega(M,\mathbb{R}) \to \Omega(S_*^\infty(M),\mathbb{R})$ which is a quasi-isomorphism. Similarly we can use the inclusion of the polynomial forms into the smooth forms on any simplex $\triangle^k$ to get a quasi-isomorphism $\Omega(S_*^\infty(M),\mathbb{R}) \to \Omega^\infty(S_*^\infty(M))$. Lastly, we can restrict any smooth form on M to the image of a smooth simplex $\triangle^k$ and get a quasi-isomorphism $\Omega^\infty(M) \to \Omega^\infty(S_*^\infty(M))$. Thus we obtain an equivalence between $\Omega^\infty(M)$ and $\Omega(M;\mathbb{R})$.

Now we want to apply this to our case of a manifold M with a smooth $\mathbb{T}$ action. The Borel construction $X \times_{\mathbb{T}} E\mathbb{T}$ is of course not a smooth manifold. The usual way to work around this is by describing it as a colimit of compact $\mathbb{T}$-manifolds. The standard description for $E\mathbb{T}$ is $S^\infty \times S^\infty \times \cdots \times S^\infty$, where the number of copies is equal to the rank of the torus $\mathbb{T}$, and the action is given by having each copy of S^1 act on one S^∞ by complex multiplication $\lambda(z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \dots)$. This can be approximated by the compact $\mathbb{T}$-manifolds $(E\mathbb{T})^n = S^{2n+1} \times S^{2n+1} \times \cdots \times S^{2n+1}$, where the inclusions on each factor are given by the complex coordinate maps $(z_1, z_2, \dots, z_n) \to (z_1, z_2, \dots, z_n, 0)$. We can then define smooth forms on $E\mathbb{T}$ to be the limit of the smooth forms on these approximations, and similarly we can approximate $M \times_{\mathbb{T}} E\mathbb{T}$ as a colimit of spaces $M \times_{\mathbb{T}} (E\mathbb{T})^n$, and define smooth forms accordingly.

We now want to define a diagram of smooth differential forms on the Borel spaces $\Omega^\infty(M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H))$ for any pair $K[H]$ of subgroups $H \subseteq K$ from the isotropy subgroups S_M, where H is connected. In order to define the diagram, we need to approximate these Borel spaces by smooth manifolds, and the maps between them by smooth maps. For any closed connected subgroup H of $\mathbb{T}$, the quotient group $\mathbb{T}/H$ is another torus and a smooth approximation of $E(\mathbb{T}/H)$ can be used to define smooth forms on $M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H)$. Inclusions of fixed point sets are smooth maps; for the change of group maps, we also need to ensure that the projection maps $E(\mathbb{T}/H_1) \to E(\mathbb{T}/H_2)$ induced by the projections $\mathbb{T}/H_1 \to \mathbb{T}/H_2$ are smoothly approximated in the models used. To allow for this, we use a variation on the standard models for these universal spaces which have been fattened up to make room for these comparison maps: we define $E\mathbb{T} = \prod_{H \in \mathcal{I}_M} S_H$ where each $S_H = \prod S^\infty$ is a product of $r =$ rank $\mathbb{T}/H$ copies of S^∞ as described above; and $\mathbb{T}$ acts on each S_H through the projection $\mathbb{T} \to \mathbb{T}/H$. Similarly, for any connected H in $\mathcal{I}_M$, we define $E(\mathbb{T}/H) = \prod_{H' \in \mathcal{I}_M | H \subseteq H'} S_{H'}$. Then each of these spaces can be approximated by compact $\mathbb{T}$-manifolds using the same approximations of the spheres S^∞ described above; and the maps between these spaces are given by smoothly approximated projection maps. Thus we can obtain an $\mathcal{I}_M$- diagram $\underline{\Omega}^\infty(M)$ by defining smooth forms as a limit of forms defined on the approximations of each space $M \times_{\mathbb{T}/H} (E\mathbb{T}/H)^n$. Note that it is in this construction where it is particularly useful to be working with diagrams indexed on the finite collection of subgroups in $\mathcal{I}_M$, and not the full $\mathcal{D}$ diagrams which need entries at all subgroups of $\mathbb{T}$.

Once we have described a diagram of spaces $M \times_{\mathbb{T}/H} (E\mathbb{T}/H)$ which are colimits of compact manifolds, with the structure maps a colimit of smooth maps, it is straightforward to apply the comparisons between the smooth and rational polynomial forms, creating the intermediate smooth simplicial forms using the same approximation of the spaces; since all these comparisons were functorial, we see that the $\mathcal{I}_M$-diagrams of smooth and rational polynomial forms are quasi-isomophic.

The next step is to compare $\underline{\Omega}^\infty(M)$ to the Cartan complex we defined. In order to do this we introduce another intermediate complex associated to a $\mathbb{T}$-manifold M known as the Weil complex; again, this is a very general construction [12]. To form the Weil complex, we start with the acyclic algebra generated by $\mathfrak{t}^*$. This is defined by $S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*)$ where generators $\{\theta_k\}$ of $\Lambda(\mathfrak{t}^*)$ are given degree 1 and generators $\{u_k\}$ of $S(\mathfrak{t}^*)$ are given degree 2, and the differential is defined by $d\theta_k = u_k$ and $du_k = 0$. Then the Weil complex $W^\bullet_{\mathbb{T}}(M)$ is defined to be a sub-complex of $\Omega^\infty(M) \otimes S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*)$ consisting of 'basic' elements $[\Omega^\infty(M) \otimes S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*)]_{bas}$. An element is 'basic' if it is both invariant under the group action and horizontal; horizontal means that it is annihilated by i_{ξ_k}, the interior multiplication of all generating vector fields ξ_k of $\mathfrak{t}$; we extend this interior multiplication to $S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*)$ by $i_{\xi_k}(\theta_j) = \delta^j_k$ for the basis $\{\theta_k\}$ dual to the vector fields ξ_k, and $i_{\xi_k}(u_j) = 0$.

The two complexes $W^\bullet_{\mathbb{T}}(M)$ and $A^\bullet_{\mathbb{T}}(M)$ are quasi-isomorphic. The obvious inclusion $\Omega^\infty(M) \otimes S(\mathfrak{t}^*) \to \Omega^\infty(M) \otimes S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*)$ does not induce a quasi-isomorphism from the Cartan complex to the Weil complex, as it does not necessarily land in the basic forms. However there is a map known as the Mathai-Quillen isomorphism $\phi : W^\bullet_{\mathbb{T}}(M) \to A^\bullet_{\mathbb{T}}(M)$ which adjusts a basic form to make it horizontal ([12], Section 4.1). The general form of this isomorphism is somewhat complicated and involves a change of variables to identify $[S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)]_{bas}$ with $S(\mathfrak{g}^*)$ [12]. In the case of a torus group, however, this is unnessessary and it is easy to see that the basic forms $[S(\mathfrak{t}^*) \otimes \Lambda(\mathfrak{t}^*)]_{bas} \simeq S(\mathfrak{t}^*)$, since we have defined $i_{\xi_k}(\theta_k) = 1$ and $i_{\xi_K}(u_j) = 0$. There is also an equivariant de Rham theorem which shows that the Weil complex is a model for the Borel construction $\Omega^\infty(M \times_{\mathbb{T}} E\mathbb{T})$ [1], [12]. Thus the Cartan complex is a model for the Borel bundle.

To adapt this to diagrams, we define a Weil diagram $\underline{W}^\bullet_{\mathbb{T}}(M)$ analogous to the Cartan diagram $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$. Entries are defined by $\underline{W}^\bullet_{\mathbb{T}}(M)(K[H]) = W^\bullet_{\mathbb{T}/H}(M^K)$. For the torus group $\mathbb{T}$, the groups $\mathbb{T}/H$ are also torus groups and so $[S(\mathfrak{t}/\mathfrak{h}^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}^*)]_{bas} \simeq S(\mathfrak{t}/\mathfrak{h}^*)$. So the diagram $\underline{W}^\bullet_{\mathbb{T}}(*)$ associated to the one point space is equivalent to the diagram $\underline{P}$, and the inclusion of $S(\mathfrak{t}/\mathfrak{h}^*)$ in each entry induces a map $\underline{P} \to \underline{W}^\bullet_{\mathbb{T}}(M)$.

As always, we also need to define the structure maps. The structure maps between the entries $K_1[H] \to K_2[H]$ are straightforward, and are of the form $\phi \otimes id$ where ϕ is the induced map $\Omega^\infty(M^{K_1}) \to \Omega^\infty(M^{K_2})$ as with the Cartan diagram. For the change of group structure maps $K[H_1] \to K[H_2]$, we need to define maps $[\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}_1^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}_1^*)]_{bas} \to [\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}_2^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}_2^*)]_{bas}$. The projections $\mathbb{T}/H_2 \to \mathbb{T}/H_1$ induce maps $\mathfrak{t}/\mathfrak{h}_1^* \to \mathfrak{t}/\mathfrak{h}_2^*$; since the subgroups H_i are connected, we can think of this as an inclusion of generators corresponding to the torus H_1/H_2. We need some care in using this map, however, since an element which is basic for $\mathbb{T}/H_1$ may not necessarily be basic for $\mathbb{T}/H_2$; although invariance is automatic, the horizontal condition is not.

We consider the Mathai-Quillen map as inspiration, since this map gives a way of adjusting a differential form to make it horizontal. In order to adjust a form ω so that it vanishes under the interior multiplication i_{ξ_k}, we can replace ω by $\omega + (-1)^{|w|} i_{\xi_k}(\omega)\theta_k$ where θ_k is the dual of the vector field ξ_k; since we have defined $i_{\xi_k}(\theta_j) = \delta_k^j$ the addition of this fudge factor gives a horizontal element. Similarly, if we have a number of generating vector fields $\{\xi_k\}$ of a torus $\mathbb{T}$, we define a fudge factor map $\gamma(w) = (-1)^{|w|} \Sigma_k i_{\xi_k}(w)\theta_k$. Then if we let $\phi = exp\gamma = 1 + \gamma + \frac{1}{2}\gamma^2 + \frac{1}{3!}\gamma^3 + \dots$, the element $\phi(\omega)$ will be horizontal with respect to the vector fields $\{\xi_k\}$, and so land in elements basic with respect to $\mathbb{T}$ as desired.

In our situation, we want to take an element of $[\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}_1^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}_1^*)]_{bas}$ and adjust it so that it will land in $[\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}_2^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}_2^*)]_{bas}$; that is, so that it will be horizontal for the additional generating vector fields ξ_k of the action of the torus group H_1/H_2. The Mathai-Quillen adjustment will do this for us. Here, however, we observe that the structure maps we need defined in the diagram are those for pairs of subgroups $H_2 \subseteq H_1 \subseteq K$; since the forms are defined on M^K, they are automatically fixed by the action any subgroup of K; in particular, by the action of the additional torus H_1/H_2. Therefore in this case the Mathai-Quillen fudge factor is zero, and the inclusion of generators we so optimistically started with turns out to land in the horizontal forms in the cases we are considering; we can take our structure maps $\underline{W}_{\mathbb{T}}^\bullet(M)(K[H_1]) \to \underline{W}_{\mathbb{T}}^\bullet(M)(K[H_2])$ to be the maps induced by the inclusions of generators $\mathfrak{t}/\mathfrak{h}_1^* \to \mathfrak{t}/\mathfrak{h}_2^*$, without needing further adjustments. Moreover, if we denote the Mathai-Quillen adjustment isomorphism on $\mathbb{T}/H$-spaces $W_{\mathbb{T}/H}(M) \to A_{\mathbb{T}/H}(M)$ by ϕ_H, the structure maps give the following commutative diagram

$$\begin{tikzcd}
\underline{W}_{\mathbb{T}}^\bullet(M)(H[K]) \arrow[r, "\phi_H"] \arrow[d] & \underline{\mathbf{A}}_{\mathbb{T}}^\bullet(M)(K[H]) \arrow[d] \\
\underline{W}_{\mathbb{T}}^\bullet(M)(H'[K']) \arrow[r, "\phi_{H'}"] & \underline{\mathbf{A}}_{\mathbb{T}}^\bullet(M)(K'[H'])
\end{tikzcd}$$

showing that the Mathai-Quillen maps fit together to give an isomorphism of diagrams.

Next, we need to check that the comparison maps

$$W_{\mathbb{T}/H}^\bullet(M^K) \to \Omega^\infty(M^K \times_{\mathbb{T}/H} E\mathbb{T}/H)$$

of the equivariant de Rham theorem fit together to give quasi-isomorphisms of diagrams. These comparison maps can be described as follows: choose $\mathbb{T}/H$-invariant forms θ^i in $\Omega^\infty(E\mathbb{T}/H)$ which are everywhere dual to the generating vector fields ξ_i of the $\mathbb{T}/H$-action on $E(\mathbb{T}/H)$; these are called 'connection' forms, and their derivatives $d\theta^i = \mu_i$ are called 'curvature' forms [12]. These forms then induce a map from the Weil algebra $W(\mathfrak{t}/\mathfrak{h}^*) = S(\mathfrak{t}/\mathfrak{h}^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}^*) \to \Omega^\infty(E(\mathbb{T}/H))$ of $\mathbb{T}/H$, and makes $\Omega^\infty(E(\mathbb{T}/H))$ into an algebra over $W(\mathfrak{t}/\mathfrak{h}^*)$. Since both $S(\mathfrak{t}/\mathfrak{h}^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}^*)$ and $\Omega^\infty(E(\mathbb{T}/H))$ are acyclic $W(\mathfrak{t}/\mathfrak{h}^*)$-algebras, $[\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}^*)]_{bas} \to [\Omega^\infty(M^K) \otimes \Omega^\infty(E(\mathbb{T}/H))]_{bas}$ is a quasi-isomorphism. Then a spectral sequence argument shows that

$$[\Omega^\infty(M) \otimes \Omega^\infty(E(\mathbb{T}/H))]_{bas} \to [\Omega^\infty(M \times E(\mathbb{T}/H))]_{bas}$$

is also a quasi-isomorphism, and then the approximation of the universal space $E(\mathbb{T}/H)$ by compact $\mathbb{T}$-manifolds $E(\mathbb{T}/H)^n$ allows us to identify $H^*([\Omega^\infty(M^K \times E(\mathbb{T}/H)]_{bas}) \simeq H^*(M^K \times_{\mathbb{T}/H} E(\mathbb{T}/H))$ [12].

To show that these comparison maps pass to the diagram categories, we need to make sure that these maps are compatible with structure maps and the maps from $\underline{P}$. As usual, the inclusions of fixed sets $M^{K_1} \to M^{K_2}$ present no difficulty, as we can choose our connection forms so that the restriction of the forms on M give the forms on M^K. For the change of group maps $\mathbb{T}/H_2 \to \mathbb{T}/H_1$, we need to show that we can choose the maps from the Weil algebras in a compatible way for each $E(\mathbb{T}/H)$. Recall that our model for $E(\mathbb{T}/H) = \prod_{H' \in \mathcal{I}_M | H \subseteq H'} S_{H'}$, and that each space S_H is a product of infinite dimensional spheres S^∞. For each space S_H, therefore, we can define connection forms dual to the vector fields generating the $\mathbb{T}/H$-action by observing that $\lambda_j \in \mathbb{T}/H$ acts by complex multiplication on the vector v_j of $\mathbf{v} = (v_1, \ldots, v_r) \in S^\infty \times \cdots \times S^\infty$. So if we define $z_{ij}(\mathbf{v}) = i$th coordinate of the vector v_j, let Z be the matrix with entries z_{ij}, then the matrix $\vartheta = Z^t\overline{dZ}$ has components θ^i_H with the desired property [12]. These θ^i_H then can be used to define the required map $W(\mathfrak{t}/\mathfrak{h}^*) \to \Omega^\infty(S_H)$.

Now if $H \subseteq L$ are connected subgroups of $\mathcal{I}_M$, we have an inclusion $W(\mathfrak{t}/\mathfrak{l}^*) \to W(\mathfrak{t}/\mathfrak{h}^*)$ induced on generators by the projection map $\mathbb{T}/H \to \mathbb{T}/L$. We can therefore extend the map defined using the connection forms to a map $W(\mathfrak{t}/\mathfrak{h}^*) \to \Omega^\infty(S_L)$ by zero on the additional generators. Using these maps we can get a map $W(\mathfrak{t}/\mathfrak{h}^*) \to \otimes_{L \supseteq H} \Omega^\infty(S_L) \hookrightarrow \Omega^\infty(\prod_{L \supseteq H} S_L) = \Omega^\infty(E(\mathbb{T}/H))$; and these maps will satisfy the following commutative diagrams:

$$\begin{array}{ccc} W(\mathfrak{t}/\mathfrak{h_1}^*) & \longrightarrow & \Omega^\infty(E(\mathbb{T}/H_1)) \\ \downarrow & & \downarrow \\ W(\mathfrak{t}/\mathfrak{h_2}^*) & \longrightarrow & \Omega^\infty(E(\mathbb{T}/H_2)) \end{array}$$

Therefore the comparison maps respect the structure maps.

Lastly, observe that if we restrict these maps $W(\mathfrak{t}/\mathfrak{h}^*) \to \Omega^\infty(E(\mathbb{T}/H))$ to the basic forms of each complex, we get $S(\mathfrak{t}/\mathfrak{h}^*) \to \Omega^\infty(E(\mathbb{T}/H))_{bas} = \Omega^\infty(B(\mathbb{T}/H))$ acting as a Sullivan minimal model for the space $B(\mathbb{T}/H)$. Therefore these maps act as our basing map $\underline{P} \to \Omega^\infty(B(\mathbb{T}/H))$, and the comparison maps $[\Omega^\infty(M^K) \otimes S(\mathfrak{t}/\mathfrak{h}^*) \otimes \Lambda(\mathfrak{t}/\mathfrak{h}^*)]_{bas} \to [\Omega^\infty(M^K) \otimes \Omega^\infty(E(\mathbb{T}/H))]_{bas}$ also respect the basing maps.

It is now straightforward to check that all the comparison maps in the equivariant de Rham theorem give equivalences of diagrams. Therefore we get quasi-isomorphisms $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M) \leftarrow \underline{W}^\bullet_{\mathbb{T}}(M) \to \underline{\Omega}^\infty(M)$ in the category of $\mathcal{I}_M$-CDGAs under $\underline{P}$.

Proof (of Proposition 3.2). We can use Lemma 5.1 applied to each of the comparison maps used in Proposition 5.4 to show that $\widetilde{\underline{\mathbf{A}}^\bullet_{\mathbb{T}}}|(M)$ is quasi-isomorphic to the model $\widetilde{\underline{\Omega}}|(M)$ in the category of $\mathcal{D}-CDGAs$ under $\underline{P}$. But then Lemmas 5.2 and 5.3 show that this also gives a quasi-isomoprhism between $\underline{\Omega}(M)$ and $\underline{\mathbf{A}}^\bullet_{\mathbb{T}}(M)$ as required.

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