Webs of Complete Graphs

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Abstract

Spider web graphs are a recent development related to the study of ×-homotopy of graphs. This paper focuses on spider web graphs between complete graphs. We prove that these graphs are distance degree regular, and give a recursive formula for their distance degree sequences using techniques from group theory and combinatorics.

1 Introduction

In this paper we investigate a novel set of mathematical objects, called *spider web graphs*. These come out of work by Chih and Scull [4] exploring ×-homotopies of graphs as defined in [1,5]. Chih-Scull proved that two graph homomorphisms are ×-homotopic if they are connected by a sequence of *spider moves* which shift one vertex at a time. This leads to the concept of *spider web graphs*, defined by looking at graph morphisms from one graph to another, and creating a graph with these morphisms as vertices and edges connecting morphisms which are related by a spider move. Thus a spider web is a subgraph of the exponential graph.

In this work, we consider the particular case of spider webs between complete graphs. We partially characterize all such spider webs by use of a graph attribute known as distance degree regularity. Distance degree regularity was first introduced in [2] and has been examined to a limited degree as outlined in the survey paper [9]. This paper makes use of basic combinatorial and group theoretic concepts in order to prove that the subset of spider webs in question are distance degree regular, and to provide a method for calculating their distance degree sequence. The question of what additional constraints are required to characterize the graphs up to isomorphism is a difficult open problem [6,10,12].

Our paper begins with a background section, starting with basic definitions and concepts standard in graph theory, followed by a more detailed exposition on spider web graphs, and finishing with a section on distance degree regularity and the associated concept of distance degree sequences. The main section of our paper contains our results on the structure of spider web graphs between complete graphs, including our main theorems, the proof that such graphs are distance degree regular and the computation of the distance degree sequence. We conclude with some consequences of our theorems and a discussion of questions still left unanswered.

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2 Background

2.1 General Graph Theory Definitions

The results here are standard and can be found in many introductory graph theory textbooks, see for example [3].

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Definition 2.1. A **graph** is a pair $G = (V, E)$, where V is a set of *vertices* (singular: vertex), and E is a set of two-sets of vertices which we call *edges*. We denote the vertex set and edge set of a graph G as $V(G)$ and $E(G)$ respectively. We denote the existence of an edge between two vertices $\{v, u\} \in E(G)$ by writing $v \sim u$.

The elements of our edge sets are two-sets and not ordered pairs, so we are considering undirected graphs. In this work, we will always assume that $V(G)$ (and hence $E(G)$) is a finite set. Thus whenever we refer to a graph, we assume it is a finite graph. Since our main interest is the complete graphs defined below, we will also assume that our graphs do not contain loops, although some of these concepts apply more generally to looped graphs.

Of particular interest are the following.

Notation. Let K_n denote the complete graph with n vertices, and edges between every pair of distinct vertices.

Definition 2.2. The **order** of a graph G is the cardinality of its vertex set. So the order of G is $|V(G)|$.

Definition 2.3. A **path** in a graph G is an ordered set of vertices in G such that each vertex (except the last) is connected to its successor by an edge in $E(G)$. That is, if the path consists of the j vertices $(v_1, v_2, ..., v_j)$, then for $1 \leq i < j$ we have $v_i \sim v_{i+1}$.

Definition 2.4. A **graph map** is a function from the vertices of one graph to the vertices of another graph which preserves edges. That is, if G and H are graphs then f is a graph map if $f : V(G) \to V(H)$ such that for $v_1, v_2 \in V(G)$ we have $v_1 \sim v_2$ implies $f(v_1) \sim f(v_2)$. These are often referred to as **graph homomorphisms** in the literature.

Definition 2.5. A **graph isomorphism** is a function $f : V(G) \to V(H)$ which is both a graph map and a bijection on both vertices and edges. If such a function exists, we say that G and H are **isomorphic**, denoted $G \cong H$.

Definition 2.6. An isomorphism from a graph to itself is called an **automorphism**.

Definition 2.7. The **automorphism group** of a graph G is the set of automorphisms on G. This set forms a group under composition. We denote the automorphism group of a graph G by $\mathrm{Aut}(G)$.

Definition 2.8. A graph G is **vertex transitive** if given any two vertices $v, u \in V(G)$, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(v) = u$.

We conclude the section with an example that illustrates the various definitions given.

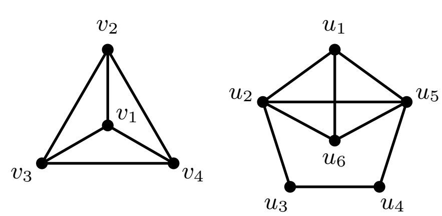
Example 2.9. Let $G = K_4$ be a graph with $V(G) = \{v_1, v_2, v_3, v_4\}$ and

$$E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}$$

Let H be a graph with $V(H) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and

$$E(H) = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_4\}, \{u_4, u_5\}, \{u_5, u_0\}, \{u_2, u_5\}, \{u_0, u_6\}, \{u_1, u_6\}, \{u_5, u_6\}\}.$$

These graphs can be represented by the following diagrams:



Then (v_2, v_3, v_1, v_4) is a path in G and $(u_3, u_2, u_5, u_1, u_6)$ is a path in H.

We can define two graph maps by the functions $f : V(G) \to V(H)$ and $g : V(H) \to V(H)$ given by:

v	$f(v)$
v_1	u_2
v_2	u_1
v_3	u_6
v_4	u_5

u	$g(u)$
u_1	u_1
u_2	u_5
u_3	u_4
u_4	u_3
u_5	u_2
u_6	u_6

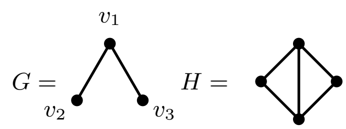
Observe that g constitutes a bijective graph map and so is a graph isomorphism. Therefore we have $g \in \mathrm{Aut}(H)$. While G is vertex transitive, H is not since, for instance, there is no automorphism of H which brings u_3 to u_6.

2.2 Spider Web Graphs

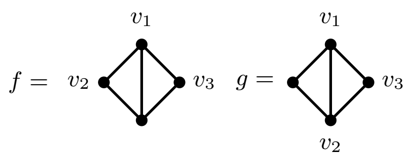
The main objects of study of this paper are graphs called spider webs. Here, we give the definition of these spider web graphs and explain how they relate to a notion of homotopy of graphs from the literature.

Definition 2.10. Let $f, g : G \to H$ be graph maps. We say f and g are a **spider pair** if there exists a single vertex v such that $f(u) = g(u)$ for all $u \neq v$ and $f(v) \neq g(v)$. When we replace f with g we refer to it as a spider move.

Example 2.11. Let G and H be the graphs corresponding to the following diagrams, with the vertices of G labeled as below



Then, the following two diagrams represent functions $f : V(G) \to V(H)$ and $g : V(G) \to V(H)$ which are graph maps forming a spider pair, since $f(v_i) = g(v_i)$ for all i except $i = 2$, and $f(v_2) \neq g(v_2)$.



The motivation for the definition of spider moves comes from the notion of $\times$-homotopy for maps of graphs. The formal definition for $\times$-homotopy can be found in [5], and properties of this definition found in [4, 5]. The connection to spider webs comes from the following result.

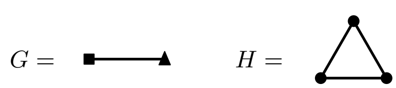
Theorem 2.12 ([4])**.** *If $f, g : G \to H$ and f and g are $\times$-homotopic, then there is a finite sequence of morphisms $f = f_0, f_1, f_2, ..., f_n = g$ such that each successive pair f_k, f_{k+1} is a spider pair.*

Thus information about spider moves between graph maps gives us information about $\times$-homotopies between graph maps. Inspired by this result, we define spider web graphs, our main object of interest in this work. The spider web $W(G, H)$ is a subgraph of the exponential graph H^G, where the vertices are given by graph morphisms and the edges correspond to spider moves.

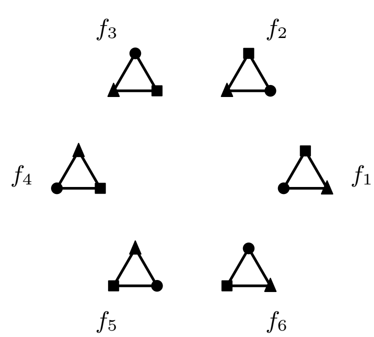
Definition 2.13. Let G, H be graphs. The **spider web** $W(G, H)$ is a graph with a vertex corresponding to each graph map $f : G \to H$. For clarity, we will refer to the vertices of $W(G, H)$ by $\dot{f}$ and denote their associated graph map by f. For any two vertices $\dot{f}$ and $\dot{g}$, then $\dot{f} \sim \dot{g} \in E(W(G, H))$ if and only if f, g are a spider pair per Definition 2.10.

These objects are complex, so we will take a moment to give examples:

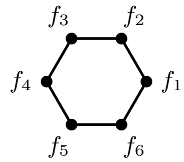
Example 2.14. Let $G = K_2$ and $H = K_3$:



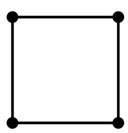
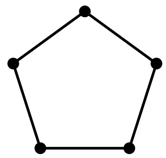
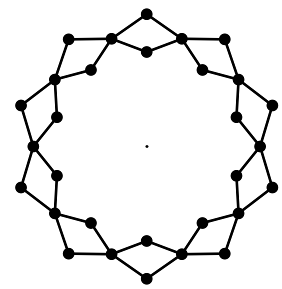
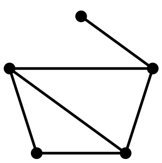
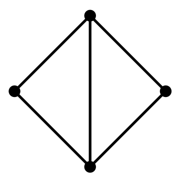
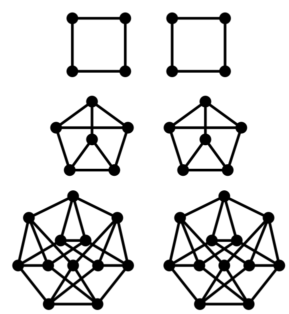
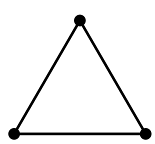
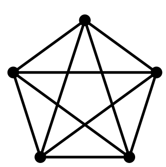
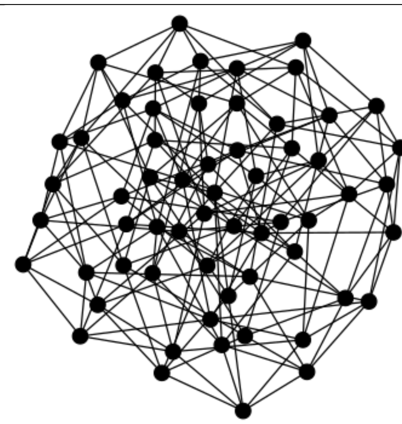
We have labeled the vertices of G by making them square and triangular respectively. The diagram below gives all possible graph maps $G \to H$, labeled as f_1 through f_6 respectively:



Edges in $W(G, H)$ are defined by spider moves between graph maps. In the above diagram, adjacent maps are connected by spider moves around the outside perimeter. Thus, $W(G, H)$ forms a cycle graph on 6 vertices:

$W(G,H) =$ 

Example 2.15.

Domain G	Codomain H	Spider Web $W(G,H)$
		
		
		

Make note in particular of the spider web in the last row of this example. What a mess it appears to be! This is not because it was intentionally laid out in a chaotic manner, it is a genuinely complex object. This is an example of the kind of spider web which we partially characterize in this paper.

2.3 Distance Degree Sequences and Distance Degree Regularity

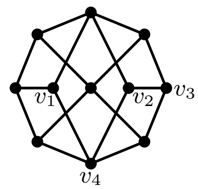
A standard concept in the study of graphs is distance, which follows from Definition 2.3.

Definition 2.16. The **distance** between two vertices, v and u in a graph G is the number of edges in a shortest path between them. We denote the distance by $d(v,u)$.

This concept of distance leads naturally to the following concept.

Definition 2.17. The **diameter** of a graph G is the greatest distance between any pair of vertices in the graph. We denote the diameter by $dia(G)$.

Example 2.18. Consider the graph given by the following diagram with some vertices labeled.



Then we have $d(v_1, v_4) = 1$, $d(v_1, v_2) = 2$ and $d(v_1, v_3) = 3$. In this case the diameter of the graph is 3.

One of many areas of continued study in graph theory is into the properties and patterns associated with distance degree sequences.

Definition 2.19. The **distance degree sequence (dds)** of a vertex v in a graph G is a list of the number of vertices at distance $1, 2, ..., \tau$ from v in G in that order, where τ is the maximum distance from v to any other vertex.

Example 2.20. Consider again the graph from example 2.18. Observe that the distance degree sequence of v_1 is $3, 5, 2$, while the dds of v_4 is $4, 4, 2$. Note that $3 + 5 + 2 + 1 = 4 + 4 + 2 + 1 = 11$, which is the order of our graph.

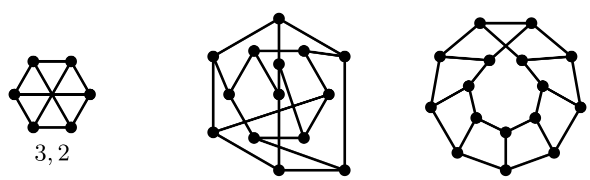
Determining what graph properties are tied to certain distributions of distance degree sequences, which distributions are constructable, and the implications of imposing various conditions on distance degree sequences are all of continuing interest in mathematical research. See [9] for an overview of results in this area. A prime example, and the one which we will be making use of in this work, is the condition that a graph be distance degree regular.

Definition 2.21. A graph G is **distance degree regular (ddr)** if all vertices in the graph have the same distance degree sequence. When this is the case, we can refer to this shared distance degree sequence as the dds of G itself.

In particular we will make use of the following result of [2], given as proposition 5, pp.101.

Theorem 2.22 ([2])**.** *Let G be a graph. If G is vertex transitive, then G is distance degree regular.*

Example 2.23. Three distance degree regular graphs with their respective distance degree sequences.



$3, 2$ $\quad$ $3, 6, 4$ $\quad$ $3, 4, 4, 4$

3 Main Results: Structure of Spider Webs of Complete Graphs

The main results of this paper are on the structure of spider web graphs between complete graphs. We have already seen examples of this, since Example 2.14 is $W(K_2, K_3)$ and $W(K_3, K_5)$ is drawn in Example 2.15. We abbreviate $W(K_n, K_m)$ by $\mathcal{W}_n^m$.

If $n > m$ then $\mathcal{W}_n^m$ is empty, since it is not possible to get a graph homomorphism from a larger K_n to a smaller. When $n = 1$ then $\mathcal{W}_n^m \cong K_m$. In the case that $n = m$ then this problem reduces to the study of $\text{Aut}(K_n)$ which, while worthwhile, has already been done far more thoroughly elsewhere. Thus going forward we will assume that $1 < n < m$. Our goal is to understand the structure of $W(K_n, K_m) = \mathcal{W}_n^m$ in these cases.

We start with one of the first, most obvious questions: "what is the order of $\mathcal{W}_n^m$ in terms of n and m?" To answer this, we begin with the following basic observation.

Observation 3.1. The graph maps from K_n to K_m for $n < m$ correspond to injective functions on vertices: since every pair of distinct vertices is connected, they cannot be mapped to the same vertex, and since all distinct vertices of K_m are connected, any injective vertex function respects connections and gives a map of graphs $K_n \to K_m$.

Theorem 3.2. $\mathcal{W}_n^m$ *has* $\binom{m}{n}n!$ *vertices.*

Proof. Definition 2.13 tells us that the number of vertices in $\mathcal{W}_n^m$ equals the number of graph maps from K_n to K_m, and Observation 3.1 shows that these are all the injective vertex maps. A standard combinatorial argument tells us that number of injective functions $V(K_n) \to V(K_m)$ is given by $\binom{m}{n}n!$ where $\binom{m}{n}$ selects the range and $n!$ determines the placement of the n domain vertices in this range. □

We now examine the structure of the web graph $\mathcal{W}_n^m$ in more detail. We will first establish that $\mathcal{W}_n^m$ is always distance degree regular (ddr), and then provide a complete characterization of its distance degree sequence (dds).

3.1 A Spider Web Between Complete Graphs is ddr

To show that $\mathcal{W}_n^m$ is distance degree regular, we make use of Theorem 2.22 that vertex transitivity implies ddr. This leaves us now the task of showing that $\mathcal{W}_n^m$ is in fact vertex transitive. In order to accomplish this, we will use the following more general result.

Lemma 3.3. *Let G, H be graphs, and let $X = W(G, H)$ be the spider web of graph maps $G \to H$. Let $\varphi \in Aut(H)$ be given. Define $\Psi : V(X) \to V(X)$ such that for $\dot{f} \in V(X)$, we define $\Psi(\dot{f})$ to be the vertex associated to the map φf. Then Ψ is an automorphism of the spider web X.*

Proof. First, we show that Ψ is a bijection. We start by showing it is an injective function. Let $\dot{f}, \dot{g} \in V(X)$ be given such that $\Psi(\dot{f}) = \Psi(\dot{g})$. Then per the definition of Ψ we have $\varphi f = \varphi g$. Since φ is an automorphism, it must be a bijection and therefore has a well defined inverse φ^{-1} and so $\varphi^{-1}\varphi f = \varphi^{-1}\varphi g$ and hence $f = g$. From Definition 2.13 we know that each vertex in X is associated with a unique graph map. Therefore, $f = g$ implies that the vertices $\dot{f} = \dot{g}$ and so Ψ is an injection. Next, Ψ is a function from a finite set to itself, and so its domain and co-domain have the same cardinality. This means that injectivity and surjectivity are equivalent, and showing that Ψ is injective was sufficient to show that it is a bijection.

We now show that Ψ is edge preserving. To that end, let $\dot{f}, \dot{g} \in V(X)$ be such that $\dot{f} \sim \dot{g}$ in X. By definition of spider web, this means that f, g are a spider pair and differ on a single vertex: there is a unique $v \in V(G)$ such that $f(v) \neq g(v)$. To show that $\Psi(\dot{f}) \sim \Psi(\dot{g})$ in X, we must show that φf, φg are also a spider pair. Since φ is injective, $\varphi f(v) \neq \varphi g(v)$. Next, suppose $u \in V(G)$ with $u \neq v$. Then $f(u) = g(u)$ and therefore $\varphi f(u) = \varphi g(u)$. Thus φf, φg are a spider pair, and Ψ is a graph map.

We have shown that Ψ is a bijective graph map and hence an automorphism of X. □

We now use this to show the vertex transitivity of our main spider web of interest $\mathcal{W}_n^m$.

Theorem 3.4. $\mathcal{W}_n^m$ *is vertex transitive.*

Proof. Let distinct $\dot{f}, \dot{g} \in V(\mathcal{W}_n^m)$ be given. We know from Observation 3.1 that the maps $f, g : K_n \to K_m$ are defined by injective functions $f, g : V(K_n) \to V(K_m)$. Define a permutation $\varphi : V(K_m) \to V(K_m)$ by the following: if $x = f(y)$ then $\varphi(x) = g(y)$. This is well-defined since f is injective and it is injective since g is injective. Extend this to a permutation $\varphi : V(K_m) \to V(K_m)$, which defines an automoprhism of the graph K_m; there are many ways to do this, so our permutation is not unique. Then define $\Psi : V(\mathcal{W}_n^m) \to V(\mathcal{W}_n^m)$ by $\Psi(\dot{f}) = \dot{\varphi f}$. This is an automorphism of $\mathcal{W}_n^m$ by Theorem 3.3 and $\Psi(\dot{f}) = \dot{g}$ by construction. Since $\dot{f}$ and $\dot{g}$ were arbitrary vertices of $\mathcal{W}_n^m$, this satisfies Definition 2.8. □

We immediately obtain the desired result from Theorems 2.22 and 3.4.

Theorem 3.5. $\mathcal{W}_n^m$ *is distance degree regular.*

3.2 The dds of A Spider Web Between Complete Graphs

Now that we know that $\mathcal{W}_n^m$ is ddr, it is reasonable to ask if we can determine the dds of $\mathcal{W}_n^m$ given values of n and m. Thus we consider how to find the λth element of the dds of $\mathcal{W}_n^m$ for $1 \leq \lambda \leq \mathrm{dia}(\mathcal{W}_n^m)$. By definition this is equal to the number of vertices which are distance λ away from an arbitrarily chosen vertex in $\mathcal{W}_n^m$. To reference this number, we introduce some notation.

Notation. We denote the number of vertices distance λ away from an arbitrary vertex in $\mathcal{W}_n^m$ by $D_\lambda(n, m)$.

Example 3.6. The spider web from example 2.14 is $\mathcal{W}_2^3$. It is easily observed that $D_1(2, 3) = 2$, $D_2(2, 3) = 2$, and $D_3(2, 3) = 1$. Thus the dds of $\mathcal{W}_2^3$ is $2, 2, 1$.

Finding the dds of $\mathcal{W}_n^m$ is equivalent to finding a formula for $D_\lambda(n,m)$ in terms of λ, n and m. This formula will be combinatorial in nature, but make use of some permutation group results. Thus we will take a moment to ensure everyone is on the same page with the specific group theoretic concepts being referenced and the notation being used to reference them.

Elements of the group of permutations on j elements, S_j, are often written in disjoint cycle form. That is, the element of S_6 which takes the elements $1,2,3,4,5,6$ to the ordered list $4,6,3,1,2,5$ would be written as $(5\ 2\ 6)(1\ 4)$, with an implied cycle (3). It is well established that each element of S_j can be written as a product of disjoint cycles, and this presentation is unique up to the ordering of the cycles. This means that for a given permutation in S_j it makes sense to ask about its cycle structure, that is how many cycles it has of what sizes when written in disjoint cycle form. For example, the above element of S_6 would have cycle structure $\{3,2\}$, with the 1 associated with the trivial cycle omitted. Likewise, the element of S_{20} which takes the elements 1 through 20 to the ordered list $8,11,4,6,10,3,7,1,18,20,2,12,5,15,14,16,9,17,19,13$ can be written as $(20\ 13\ 5\ 10)(18\ 17\ 9)(6\ 3\ 4)(11\ 2)(1\ 8)(14\ 15)$, meaning that it has cycle structure $\{4,3,3,2,2,2\}$, where the multi-set of cycle sizes omits the four implied 1-cycles.

Notation. We will use L to denote a multi-set of cycle sizes. Moreover, L_φ will denote the multi-set of cycle sizes corresponding to the cycle structure of a specific permutation φ.

Notation. Let A be any set or multi-set of integers. Define

$$\mathrm{sum}(A) := \sum_{x\in A} x.$$

We now turn to looking at the distances between vertices $\dot{f},\dot{g} \in V(\mathcal{W}_n^m)$. We will start by considering the case where the images of the corresponding graph maps $f,g : K_n \to K_m$ are the same. We know by Observation 3.1 that both of these maps will be injective on vertices. We can then get the following relationship between these functions.

Observation 3.7. As we saw in the proof of Theorem 3.4 there exists a permutation $\varphi \in S_m$ such that $\varphi f = g$. In the particular case where f,g have the same range, we can choose φ to fix all vertices outside of the range, and we can think of φ as a permutation on n elements (the vertices in the range) rather than the full m elements. Considered as an element of S_n, such a permutation φ is unique: if we restrict the codomain to the range, then f has a well defined inverse, and then the specification $\varphi = gf^{-1}$ uniquely determines the permutation φ. We refer to this as the permutation transforming f to g, and note that this is only well-defined when f,g have the same range.

Our first goal is to count the number of vertices $\dot{g}$ of a given distance from a chosen vertex $\dot{f}$ in the case when the maps f,g have the same range. Leaning on Observation 3.7, we will in fact count the number of permutations which give rise to the given distance. Thus we begin by connecting the cycle structure of the permutation to the distance between the vertices.

Lemma 3.8. *Let distinct $\dot{f},\dot{g} \in V(\mathcal{W}_n^m)$ such that their graph maps f,g have the same range and suppose that the permutation φ transforming f to g consists of a single non-trivial cycle of length j. Then $d(\dot{f},\dot{g}) = j+1$.*

Proof. Per Definitions 2.16 and 2.13 we know that $d(\dot{f},\dot{g})$ will be equal to the number of spider moves required to transform f into g. We know that vertices of $\mathcal{W}_n^m$ correspond to injective functions on the vertices, and so if the ranges are identical we cannot immediately spider move any one vertex into agreement with g. Instead, we first have to spider move a vertex outside of this common range to move it out of the way. Once this is done, the vertex g takes to the newly open target can be spider moved into agreement with g, opening up another target vertex, and continue from there. Thus we can spider move f into agreement with g in exactly $j+1$ spider moves. □

Example 3.9. Consider again the spider web from example 2.14. In particular, observe that f_1 and f_4 have identical ranges in K_3. A permutation consisting of a single two cycle swapping the two vertices in this range would transform f_1 into f_4. Thus, by our lemma we would expect the distance between them to be $2+1$. Observe that, starting from f_1, both the triangle and square are mutually blocking each other from moving into agreement with f_4. We have to spider move one 'out of the way' first, say the square. We do so and end up at the map f_6. Now the triangle can be spider moved into agreement with f_4. We do so and are now at the map f_5. Finally, with the triangle out of the way we can spider move the square into agreement with f_4. That was three spider moves, and so we see that $d(\dot{f}_1,\dot{f}_4) = 3$ in accordance with our expectation.

Corollary 3.10. *Let distinct $\dot{f},\dot{g} \in V(\mathcal{W}_n^m)$ such that their graph maps f,g have the same range, and let φ be the permutation transforming f to g. Then $d(\dot{f},\dot{g})$ is equal to the sum of the sizes of the cycles in φ plus the number of cycles, with fixed-points excluded:*

$$d(\dot{f},\dot{g}) = \mathit{sum}(L_\varphi) + |L_\varphi|$$

Proof. Since the cycles of our permutation are disjoint, moves between them are never necessary and can only add redundant steps to the spider-move transformation. Thus, we can adjust f via spider moves cycle by cycle as per Theorem 3.8: each will take one more than the length of the cycle to spider move into agreement. $\square$

The above result allows us to compute the distance between two vertices whose maps have the same image if we know the cycle structure of the permutation linking them. Thus in order to calculate the number of vertices with the same image of a given distance, we compute the number of cycle structures that give that distance, and then the number of permutations with those cycle structures.

Definition 3.11. We define $\mathfrak{A}_j(\lambda)$ to be the set of cycle structures of permutations on j elements (with fixed points excluded) which result in a distance of exactly λ:

$$\mathfrak{A}_j(\lambda) = \{L : \text{sum}(L) + |L| = \lambda \text{ and sum}(L) \leq j\}$$

Standard group theory then allows us to count the number of permutations with a given cycle structure.

Lemma 3.12. *([11], page 3) Let L be a multi-set representing a possible cycle structure on j elements with fixed-points excluded. For $i \in L$ we denote the multiplicity of i in L with b_i. Then the number of permutations in S_j with cycle structure L is*

$$N(L) = \frac{j!}{(fp)! \prod_{i \in L} (i^{b_i} b_i!)}$$

where fp denotes the number of fixed points the cycle structure L gives rise to as a permutation on j elements: explicitly, $fp = j - sum(L)$.

Thus we can obtain a count of the number of vertices of $\mathcal{W}_n^m$ of a given distance with the same image. Examining the cycle structure of the permutation between vertices also allows us to obtain the maximum distance possible in $\mathcal{W}_n^m$.

Theorem 3.13. *$\mathcal{W}_n^m$ has diameter $\left\lfloor \frac{3n}{2} \right\rfloor$.*

Proof. Let $\dot{f} \in V(\mathcal{W}_n^m)$ be given. We begin by showing that $\text{dia}(\mathcal{W}_n^m) \geq \left\lfloor \frac{3n}{2} \right\rfloor$ by producing vertices which have that distance. If n is even, then we partition the vertices of K_n into $\frac{n}{2}$ 2-cycles whose lengths sum to n. Then lemma 3.10 says that the distance of the vertex $\dot{g}$ where g differs from f by this permutation will have $d(\dot{f}, \dot{g}) = n + \frac{n}{2} = \frac{3n}{2}$. If n is odd, we partition $n-1$ of the vertices of K_n into $\frac{n-1}{2}$ 2-cycles, whose lengths sum to $n-1$, and map the remaining vertex outside of the range of f, adding an extra spider move and increasing the distance from $\dot{f}$ by 1. Then by Lemma 3.10 we have

$$d(\dot{f}, \dot{g}) = (n-1) + \frac{n-1}{2} + 1 = \left\lfloor \frac{3n}{2} \right\rfloor$$

Next, we will show by induction that $\text{dia}(\mathcal{W}_n^m) \leq \left\lfloor \frac{3n}{2} \right\rfloor$. For our base case, let $n = 2$. If $f \neq g$ and f, g differ by a single vertex, then they form a spider pair and $d(\dot{f}, \dot{g}) = 1$. If f and g disagree on both vertices in their domain, then we have two cases: if the ranges are not the same, then at least one vertex can be moved to where it needs to go, followed by the other, giving two spider moves. If the vertices are swapped, it takes three spider moves to realign, as one vertex needs to be moved out of the way and then the two vertices can be moved into agreement in two moves as in the previous case. In all cases, $d(\dot{f}, \dot{g}) \leq 3 = \left\lfloor \frac{3n}{2} \right\rfloor$

For our inductive step, consider $n > 2$ and let $\dot{g} \in \mathcal{W}_n^m$ be given. If g does not have the same range as f then there is some vertex v such that $g(v)$ is not in the range of f. Thus in a single spider move, we can shift f to f' which takes v to $g(v)$ and in all other vertices agrees with f. This leaves us with $n-1$ remaining vertices that need to be considered, and we know by our inductive hypothesis that these can be shifted with a maximum of $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$ spider moves, leaving us with a total distance between f and g less than or equal to $1 + \left\lfloor \frac{3(n-1)}{2} \right\rfloor \leq \left\lfloor \frac{3(n)}{2} \right\rfloor$. If g has the same range as f but breaks down into 2-cycles, then we have seen in the first paragraph that $d(\dot{f}, \dot{g}) = \left\lfloor \frac{3n}{2} \right\rfloor$. Lastly, if g has the same range as f and g differs from f by a permutation that has at least one cycle of length $c > 2$, then if we remove the vertices in this cycle we are left with $n - c$ vertices. We know by induction that it takes $\leq \left\lfloor \frac{3(n-c)}{2} \right\rfloor$ spider moves to move g to agree with f. It takes an additional $c + 1$ moves to realign the vertices of the additional cycle, so the distance $d(\dot{f}, \dot{g}) \leq \frac{3(n-c)}{2} + c + 1 = \frac{3n-c+2}{2}$ But $c > 2$ and so $-c + 2 < 0$ and hence $\frac{3n-c+2}{2} \leq \frac{3n}{2}$. $\square$

We now turn to considering vertices of $\mathcal{W}_n^m$ which have different images. In this case, we will not obtain an explicit formula for the number, but rather develop a recursive formula connecting the number of such vertices to the value of $D_{\lambda'}(n', m')$ for smaller values of λ', n', m'.

Lemma 3.14. *Let $\dot{f} \in V(\mathcal{W}_n^m)$ be given. The number of ways to map exactly r vertices from K_n to vertices which are NOT in the range of f is*

$$\binom{n}{r} \frac{(m-n)!}{(m-n-r)!}$$

Proof. We first have $\binom{n}{r}$ options for which vertices of the domain will be moved outside the range of f. Next we have $\binom{m-n}{r}$ options for which vertices outside the range of f we will be assigning these domain vertices to. Finally we have $r!$ options for how to place the r vertices within the new selected image. By the product principle, our total is then

$$\begin{aligned}\binom{n}{r}\binom{m-n}{r} r! &= \binom{n}{r} \frac{(m-n)!}{(m-n-r)!r!} r! \\ &= \binom{n}{r} \frac{(m-n)!}{(m-n-r)!}\end{aligned}$$

□

Lemma 3.15. *Let $\dot{f} \in V(\mathcal{W}_n^m)$ and let λ be a potential distance, so $1 \leq \lambda \leq dia(\mathcal{W}_n^m)$. Suppose we define a function which takes r of the vertices in K_n to images chosen outside of the range of f. The number of vertices $\dot{g}$ which are distance λ from $\dot{f}$ and associated to maps g with those r vertices mapped as specified, and with all other vertices sent into the range of f, is $D_{\lambda-r}(n-r, n)$.*

Proof. The r vertices already assigned out of the range of f are already set, leaving $n-r$ vertices not previously assigned. These vertices must be taken to vertices which are in the image of f, giving n choices. Thus we are looking at injective vertex maps from $n-r$ vertices to n vertices, or equivalently, maps from K_{n-r} to K_n.

To ensure that we have distance λ, consider that it will take exactly r spider moves to shift the the vertices taken outside of the range of f once all of the $n-r$ other vertices are in their correct place. That means that if our overall distance from $\dot{g}$ to $\dot{f}$ is going to be λ then it must take exactly $\lambda - r$ spider moves to adjust the vertices within the range. Therefore, this question is equivalent to asking how many vertices are distance $\lambda - r$ away from any given vertex in $W(K_{n-r}, K_n)$ which by definition is the value of $D_{\lambda-r}(n-r, n)$. □

Putting all of these results together gives us the following formula for the values in the dds of $\mathcal{W}_n^m$.

Theorem 3.16. *For $1 \leq \lambda \leq dia(\mathcal{W}_n^m)$, the formula for $D_\lambda(n, m)$ is*

$$D_\lambda(n,m) = \sum_{L \in \mathfrak{A}_n(\lambda)} N(L) + \sum_{r=1}^{\ell} \binom{n}{r} \frac{z!}{(z-r)!} D_{\lambda-r}(n-r, n)$$

where

$$N(L) = \frac{n!}{(fp)! \prod_{i \in L} (i^{b_i} b_i!)}$$

and

$$\ell = \min(n, \lambda, m-n) \text{ and } z = m-n$$

in the second term.

Proof. To find the number of vertices in $\mathcal{W}_n^m$ that have distance λ away from $\dot{f}$, we consider two cases: the vertices with the same range, and the vertices with a different range. In the first case where the functions have the same range, then we know that the vertices differ by a permutation. Lemma 3.10 defines the possible cycle structures of these permutations, and Lemma 3.12 that each cycle structure will correspond to $N(L)$ actual permutations. Summing over these possible cases gives us

$$C_1 = \sum_{L \in \mathfrak{A}_n(\lambda)} N(L) = \sum_{L \in \mathfrak{A}_n(\lambda)} \frac{n!}{(fp)! \prod_{i \in L} (i^{b_i} b_i!)}.$$

In the case that the ranges of our functions differ, we consider the possible values for r, the number of differing positions. We know the distance between vertices will be at least r, so $r \leq \lambda$. We know that we only have $m-n$ other places the vertices could possibly go, so $r \leq m-n$. Finally, we only have n vertices to disagree on, so $r \leq n$. Thus we have $r \leq \ell = \min(n, \lambda, m-n)$.

Now suppose we have a value of r in the range $1 < r \leq \ell$. From 3.14 we know that there are $\binom{n}{r}\frac{(m-n)!}{(m-n-r)!}$ ways to place these r vertices. At this point, if $r = \lambda$ then we are done. If $r < \lambda$ then we need to count the number of maps which send these r vertices to their new assignments while sending all other vertices into the range of f creating a distance of λ. By Lemma 3.15 we know that this value is $D_{\lambda-r}(n-r, n)$. Thus, by the product principle we have

$$C_2 = \sum_{r=1}^{\min(n,\lambda,m-n)} \binom{n}{r} \frac{m-n!}{(m-n-r)!} D_{\lambda-r}(n-r, n).$$

The sum of the cases then yields the theorem formula. $\square$

When using this formula to make calculations, it is important that we take $D_\lambda(n,m) = 0$ when $\lfloor \frac{3n}{2} \rfloor < \lambda$ (which follows from Theorem 3.13) and $D_0(n,m) = 1$ (which follows from Definition 2.16). These conditions allow our recursive calculation to terminate.

3.3 Consequences

Observation 3.17. The distance degree sequence of $\mathcal{W}_n^m$ is given by

$$D_1(n,m), D_2(n,m), D_3(n,m), \ldots, D_k(n,m)$$

where $k = dia(\mathcal{W}_n^m) = \lfloor \frac{3n}{2} \rfloor$.

This observation forms the heart of our result, as we have completely characterized the distance degree sequence of the spider webs between complete graphs.

The following demonstrates how our result can be used to directly calculate the dds of the spider web $W(K_3, K_5)$.

Example 3.18. $W(K_3, K_5)$: $n = 3, m = 5, \lambda = 3$:
We have $m-n = 2$, $\min(n, \lambda, m-n) = m-n = 2$ and $\mathfrak{A}(\lambda) = \{2\}$. Therefore

$$\begin{aligned} D_3(3,5) &= \frac{3!}{2^1(1!)} + \binom{3}{1}\frac{2!}{(2-1)!} D_{3-1}(3-1,3) + \binom{3}{2}\frac{2!}{(2-2)!} D_{3-2}(3-2,3) \\ &= \frac{3!}{2(1!)} + \binom{3}{1}\frac{2!}{1!} D_2(2,3) + \binom{3}{2}\frac{2!}{0!} D_1(1,3) \\ &= 3 + 6D_2(2,3) + 6D_1(1,3) \end{aligned}$$

It is easy to obtain that $D_1(1,3) = D_2(2,3) = 2$, Therefore

$$D_3(3,5) = 3 + 6(2+2) = 27$$

This can also be verified by directly checking the spider web itself.

Note that the spider web in this example is the one from the final row in the table from Example 2.15. Despite the chaotic appearance of the object, we now know that every vertex has exactly 27 vertices a distance of 3 away from it!

The next example demonstrates the utility of our result for larger values of n and m. Directly constructing this spider web and then manually finding its distance degree sequence would be computationally prohibitive. However, an algorithm making use of our formula for $D_\lambda(n,m)$ calculates these results in less than a second.

Example 3.19. $W(K_7, K_{11})$: then $n = 7, m = 11$, and we calculate:
order of $\mathcal{W}_n^m = 1663200$.
diameter $= \left\lfloor \frac{3(7)}{2} \right\rfloor = 10$.
dds: $28, 420, 4221, 29890, 146650, 460089, 745752, 249850, 25669, 630$.

The next example deserves special mention, both because of its clear and concise partial characterization of infinitely many spider webs, but also because it served as the jumping off point for the entire paper. This result was discovered by the authors first, and used as a guide for the more general result just demonstrated.

Example 3.20. The order of the graph $W(K_2, K_m)$ is $m^2 - m$ and its dds is

$$2(m-2), (m-1)(m-2), 1$$

We end with the following consequence of our result.

Observation 3.21. If we sum the numbers in the distance degree sequence (including 1 for the original vertex) we obtain the total number of vertices in the graph. Therefore

$$\sum_{\lambda=0}^{\lfloor \frac{3n}{2} \rfloor} D_\lambda(n,m) = \sum_{\lambda=1}^{\lfloor \frac{3n}{2} \rfloor} \left(\sum_{L \in \mathfrak{A}_n(\lambda)} \frac{n!}{(fp)! \prod_{i \in L}(i^{b_i} b_i!)} + \sum_{r=1}^{\ell} \binom{n}{r} \frac{z!}{(z-r)!} D_{\lambda-r}(n-r,n) \right) + 1 = \binom{m}{n} n!$$

This implication of our result is particularly interesting because of the nightmarish complexity of the middle expression and the relative simplicity of the final expression. It seems hard to imagine that any naive observer could see that the middle expression with its double nested triple summation including recursion reduces to one of the most elementary combinatorial expressions.

4 Future Research: Questions Left Open

4.1 With Regard to Distance Degree Regularity

While ddr graphs of diameter 2 are trivially characterized and those of diameter 3 are partially characterized in [8], larger examples are only poorly understood. Open questions include what conditions must be placed on a dds for a ddr graph with that sequence to be constructable, what proportion of k-regular graphs are ddr, and most crucially for our present purposes, what additional conditions need to be placed on two ddr graphs with the same dds in order to ensure they are isomorphic. Plenty of examples of ddr graphs with the same dds which are not isomorphic have been presented, so we know that the condition of sharing a dds alone is not sufficient for isomorphism. If the additional conditions necessary for isomorphism could be identified, it may be possible to characterize our object $\mathcal{W}_n^m$ completely. Despite attempts to find such conditions, the problem appears highly non-trivial [6, 10, 12].

4.2 With Regard to Spider Webs

With the conditions placed on G and H in $\mathcal{W}_n^m = W(G,H)$, we have explored only a tiny portion of possible spider webs in the present work. Further characterization of other kinds of spider webs is a very large project with much work remaining to be done. A promising route forward from the work presented here is to allow either G or H to be more general than a complete graph and try to relate the resulting spider web to those characterized here. Additional work has already been done on entirely different spiderweb graphs, particularly those involving bipartite and and star graphs [7]. The possibility space for the number of constructable spiderwebs is enormous, and the project of systematically characterizing all of them has only just begun.

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