SUPPLY CHAIN MANAGEMENT WITH OVERTIME AND PREMIUM FREIGHT

by

Eric Logan Huggins

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Professor Tava Maryanne Olsen, Chairperson
Professor James C. Bean
Professor Roman Kapuscinski
Professor Mark E. Lewis

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PREFACE

This thesis models a two-stage supply chain where the upstream stage (stage 2) **always** meets demand from the downstream stage (stage 1). We assume that demand is stochastic, and hence shortages will occasionally occur at stage 2. Stage 2 must fill these shortages by expediting, using overtime production and/or shipping parts quickly by air, which we refer to as premium freight shipments. We derive optimal inventory control policies for this supply chain under decentralized, centralized, and coordinated control and perform numerical analysis to compare the results.

In Chapter II, we study the supply chain under decentralized control, where each stage independently minimizes its own costs. Stage 1 ignores stage 2 and follows a simple base-stock policy. Stage 2 also follows a simple base-stock policy, under the assumption that there is no setup cost for regular production at stage 2. When we include a setup cost for regular production at stage 2, two decisions must be made: how much to produce during regular production, and how much to produce during overtime production. We show that the optimal regular production policy is an (s, S) policy and that the optimal overtime production policy depends on the cost parameters.

In Chapter III, we study the supply chain under centralized control, where the two stages work together to minimize system costs. By substituting system variables for stage 2 variables and relaxing some constraints, we show that the optimal inventory control policy at stage 1 has two (or three) order-up-to levels and depends on the available system inventory. We also show that the optimal inventory control policy for the system is a base-stock policy; hence, the optimal inventory control policy for stage 2 is to ensure the system base-stock level is achieved.

We attempt to coordinate the two stages in Chapter IV, by developing two contracts that achieve system optimal (or near-optimal) results. Both contracts depend on a two-tiered (or three-tiered) wholesale cost and a linear transfer payment. Contract A achieves system optimality, but requires the two stages to share cost information. Contract B achieves near-optimality for the system without sharing cost information, and achieves optimality for the average cost case. Under both contracts, an appropriate transfer payment may be negotiated so that both stages improve upon their respective decentralized costs.

In Chapter V, we perform a numerical analysis to compare the supply chain under different forms of control. We show that centralized control can affect significant savings over decentralized control, particularly if the demand variation is high, holding costs are high, or if the fixed cost of expediting is large. We also show that Contract B yields near-optimal results for the system, particularly if the discount factor is close to one.

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CHAPTER I

INTRODUCTION

In 1999, a helicopter crashed near Ann Arbor, Michigan. The helicopter was being used to ship automobile parts across the state. Fortunately, no one was seriously injured, but this accident begs the following question: Does it ever make sense to ship automobile parts by helicopter? Clearly, shipping parts by air is more expensive than shipping by ground transportation, but on the other hand, the shipment will arrive at its destination very quickly. Finding an answer to this question was the original motivation for the research that follows. In this thesis, we study a two-stage supply chain where the upstream facility **always** meets demand from the downstream facility. We assume that demand is stochastic, and hence, shortages will occasionally occur at the upstream facility. In order to fill these shortages, the upstream facility must employ either overtime production or air shipments, which we refer to '*premium freight*.' We study this supply chain under decentralized control, where the two facilities operate independently; under centralized control, where a single controller makes all decisions for both facilities; and, under coordinated control, where two independent facilities follow a contract which leads to system optimal performance.

1.1 Overview

In traditional supply chain situations, downstream facilities make decisions about their order quantities without regard to the actual inventory available upstream. If the upstream facilities do not have enough inventory on hand to fill the orders, it is often assumed that the downstream facility will take what it can get and backorder the rest. In order to avoid these shortages, the upstream facilities have traditionally set their inventory levels high enough so that the likelihood of not meeting downstream demand is low. However, the shift towards lean inventory has caused a reduction in inventories, possibly increasing the likelihood of these shortages. Moreover, the cost of backorders (already high, though hard to estimate) is certainly not decreasing in today's competitive markets. Therefore, many facilities use various forms of expediting to meet supply requests when shortages occur.

We consider a problem with stochastic demand where the downstream facility's supply requests are always met by the upstream facility, and the upstream facility has at least one of two methods of expediting available: overtime production and premium freight. Overtime production occurs at the end of a period (typically a day) after demand has been realized. The parts produced during overtime production cost more to build, but do not incur additional shipping expenses. Premium freight consists of building parts at the beginning of the same period they are required downstream and shipping them in an expedited fashion (e.g., by airplane or helicopter). We assume that these parts arrive downstream in time to be used during the same period in which they are shipped. Parts shipped by premium freight cost more to ship, but do not incur additional production costs. In our work, both methods of expediting may be expensive, incurring fixed and per unit costs. The two methods of expediting are mathematically equivalent, and our analysis will often consider the case where just one method is available; however, we extend all of our results to the case where both overtime production and premium freight may be used.

We have modelled our problem after the actual supply chain issues faced by the Ford Motor Corporation and Visteon Automotive Systems, specifically the Visteon plant in Ypsilanti, Michigan. Visteon produces mostly engine parts used in vehicle assembly at Ford. Inventory levels are relatively low, about “half a day’s worth” according to our contacts at the Ypsilanti plant, and the company is trying to “be lean.” According to Vasilash [60], at Visteon “there is a codified set of ‘Lean Design Rules’, there are ‘Lean Assessment Tools’ for benchmarking, and all manufacturing engineers have been trained in lean methods.” However, they follow a policy of meeting all supply requests, frequently using overtime production and/or premium freight when shortages occur. Backordering is not considered an option because the parts they send downstream are essential in keeping the assembly lines moving, and the cost of shutting down the assembly lines at Ford is extremely high. We have heard a wide range of estimates for this cost, but all have been in tens of thousands of dollars per hour! Therefore, overtime production and premium freight shipments are commonly used in order keep the assembly lines moving. Again, according to [60], “Expediting . . . is commonly created as a result of a schedule change made by the customer . . . and it necessitates such things as overtime and having to pay for faster shipping."

In our three models, we have attempted to capture the essence of the interaction between Visteon and Ford. Originally, Visteon was a wholly owned subsidiary of Ford. The decisions made and the costs incurred at Visteon were simply a part of the larger process at Ford. This situation is represented by our centralized model in Chapter III. In the past few years, Visteon has begun the process of breaking off from Ford and becoming an independent firm. This situation is represented by our decentralized model in Chapter II. Lastly, as Visteon becomes an independent firm, it may be worthwhile to consider possible contracts that will benefit both firms. This situation is represented by the coordinated models in Chapter IV.

We feel that our model and results may apply elsewhere in the automobile industry and in other industries, and that our results show that supply requests can always be met, but at a cost. For example, shipping automobile parts by air is not uncommon. According to an article in The Detroit News [53], Willow Run Airport outside of Detroit has recently become the nation's third largest cargo airport due to shipment of automobile parts. The article states that "hardly a car or truck is made anywhere in the United States that doesn't include parts that have traveled through Willow Run Airport" and that "increasingly, Detroit's automakers are flying parts from city to city and from continent to continent." We feel that our model may apply to the computer and electronics industries as well, where many manufacturers have reduced or even eliminated their requirements for warehousing and receive parts in just-in-time fashion. Finally, we feel that our results yield new insight into a common assumption made in the inventory literature. In most single location inventory models, it is assumed that supply requests upstream are always met, without considering how and at what cost. Our results show that supply requests can always be met upstream using some form of expediting, but that it may be much less expensive for the system if the downstream facility is sensitive to the inventory situation upstream and adjusts supply requests accordingly.

The rest of this chapter consists of a literature review and a description of our basic model and assumptions. The literature review in Section 1.2 covers three major areas that correspond to our three supply chain models. The first part reviews single location inventory problems, since our decentralized model in Chapter II breaks down into two, separate, single location problems. First we discuss the seminal papers in the field and then we discuss papers that consider either expediting or fixed costs for shortages, which relate specifically to our decentralized model. The second part of our literature review covers centrally controlled supply chain models, corresponding to our centralized model in Chapter III. Again, we discuss the seminal work in the field and then discuss papers that are closely related to our model. This discussion has significant overlap with the third part of the literature review, which focuses on supply chain coordination issues, as we do in Chapter IV. Finally, we conclude the literature review with a brief discussion on logconcavity, an assumption we impose on our demand distribution.

In Section 1.3, we outline the two-stage supply chain model that we use throughout this thesis. We define the various inventory and decision variables for both stages and the associated costs. We list the assumptions that will hold throughout the thesis, although additional assumptions are made in later sections. We describe a time line that indicates when shipments occur, when decisions are made, when costs are charged, etc. We also discuss some general notation about one period costs and total objective function costs that will be standard throughout this document. Finally, our model as defined in this section contains options for both overtime production and premium freight; however, throughout this thesis, our initial analysis in each chapter will consider overtime production as the only method of expediting. (Similarly, we could have chosen only premium freight, as both methods of expediting are mathematically equivalent.) This assumption clarifies the exposition for the reader. At the end of each chapter, we consider the original problem with both methods of expediting, and show that similar, although more complicated, results hold.

In Chapter II, we consider the situation where the two stages in the supply chain are completely independent firms and seek to minimize their own respective costs. The decisions made at each firm depend only on the costs and information available at that particular firm. In Section 2.1, we show that the optimal inventory control policy at stage 1 is a simple base-stock policy. Importantly, we also show that stage 1 passes the exact demand it experiences back to stage 2. In Section 2.2, we show that the optimal inventory control policy at stage 2 is also a base-stock policy. This proof requires additional work to show that the expected one period costs at stage 2 are quasiconvex, under the assumption that our demand distribution is logconcave. In Section 2.3, we again consider stage 2, but we include the premium freight option and we include a setup cost for regular production at stage 2. Under these conditions, we show that the optimal overtime production policy depends on the problem data, and that the optimal regular production policy is an (s, S) policy. The proof involves first defining a relationship between overtime and regular production, then deriving the overtime results, and lastly deriving the regular production results.

In Chapter III, we consider the situation where the two stages are actually part of a single firm and are controlled by a single manager. That manager has complete information from both stages, makes all decisions, and attempts to minimize the total system costs. In Section 3.1, we develop the relaxed version of the centralized problem. In Section 3.2, we show that under relaxed conditions, the optimal inventory control policy at stage 1 is an interesting, three-tiered policy that depends on the system inventory. The proof involves relaxing two constraints and substituting system inventory variables for stage 2 variables. In Section 3.3, again under relaxed conditions, we show that the optimal inventory control policy for the entire system is a base-stock policy, and hence the optimal policy for stage 2 depends on the base-stock level for the system and the inventory position chosen for stage 1. In Section 3.4, we prove that the optimal policies found for the relaxed problem are also optimal for the original, fully constrained problem. At the end of the chapter, we show that similar results hold when we include the premium freight option. In this case, the stage 1 policy becomes four-tiered and the system policy remains base-stock, but the proof requires additional steps concerning when to employ overtime production and when to employ premium freight.

In Chapter IV, we consider the situation where the two stages are independent firms, but are willing to work together under contract to achieve system optimal performance, assuming both firms benefit from the contract. We consider two different contracts. In Section 4.1, we describe the wholesale cost and linear transfer payment that we use in both contracts. In Section 4.2, we show that Contract A achieves system optimality, but requires the firms to share cost and inventory information in order to be implemented. In Section 4.3, we show that under Contract B, stage 1 follows the optimal centralized policy, a base-stock policy is optimal for the system, but that the base-stock level may be too high. However, numerical analysis provided in the next chapter shows that Contract B performs near-optimally from a system perspective, and under Contract B the two firms do not have to share cost information. In Section 4.4, we discuss appropriate values of the linear transfer payment and in Section 4.5, we consider the average cost case of some of our previous models and show that under the average cost criterion, Contract B performs optimally. In Section 4.6, we reconsider the wholesale cost when both methods of expediting are available.

In Chapter V, we compare costs and determine managerial insights using numerical analyses. In Section 5.1, we compare the costs and inventories under the centralized and decentralized models. We show that the centralized model can affect significant savings, particularly when the demand variance is high, holding costs are high, or the setup cost for overtime production is large. In Section 5.2, we compare the system costs under Contract B to those under centralized control. We show that in general, Contract B will produce near-optimal system performance. Lastly, in Section 5.3, we study two example problems to gain insight as to how centralized control saves costs over decentralized control.

We conclude the thesis in Chapter VI. We discuss our main results from each preceding chapter and comment on the techniques used to prove them. We also discuss extensions to our problem worthy of further study.

1.2 Literature Review

In this thesis, we study an inventory control problem with expediting between two firms. The literature in the area of inventory control/supply chain management is vast, beginning with articles in the late 1950's and continuing to flourish through today. It is a very safe bet that the current copy of *Management Science* has at least one article on some supply chain issue, and it is likely that the copy of *Management Science* ten years from now will also contain some article discussing inventory control/supply chain management, although the inventory related catch-phrase may have changed by then. The year 1958 may be considered the inception of stochastic inventory control, with the publication of *Studies in the Mathematical Theory of Inventory and Production* [5]. Nearly all current articles in the field can trace their ideas back to this excellent text. A more current update on the status of stochastic inventory theory by Porteus [49] may be found in Chapter 12 of the text edited by Heyman and Sobel [29].

Single Location Inventory Control

The seminal journal articles in the field of optimal control of single location inventory problems are due to Scarf [52] and Veinott [62]. Scarf proved that in general, (s, S) policies are optimal for inventory control problems with setup costs for production. Scarf assumes that the expected costs per period are convex and proves his results using the notion of "K-convexity." Under the more general assumption that the expected costs per period are quasiconvex, Veinott proved the same result. Both authors considered inventory problems over a finite horizon. In this thesis, we consider problems over an infinite horizon, and thus we rely heavily on the results from Zheng [66]. In Zheng's paper, he generalizes the results of Scarf and Veinott over the infinite horizon in a novel way. He shows that, given that the expected one period costs are quasiconvex, (s, S) policies are optimal for both the discounted and average cost cases. One area where our assumptions differ from Zheng's are that he assumes that backorders are allowed; we have to modify this assumption in Section 2.3.

In Chapter II, we study single location problems with expediting. In Section 2.1, we study stage 1 of the supply chain as a single location problem. In this case, stage 1 benefits from expediting, but does not pay for it, and the analysis is straightforward. In Section 2.2, we study stage 2 of the supply chain as a single location problem under the assumption that only one method of expediting is available. We assume that the expedited shipment has lead time 0, and again, the analysis is straightforward. In Section 2.3, we include both methods of expediting and a setup cost for regular production at stage 2, and the analysis gets interesting. In this section, we basically cover the material presented in our first paper [30]. In that paper, we study a single location inventory control problem where demand must always be met, two methods of expediting are available (both with 0 lead time), and both methods of expediting incur fixed plus linear costs.

The literature on single location problems with expediting began with studies by Daniel [19] and Fukuda [24]. Daniel derives optimal policies where there are two supply modes, expedited and regular, with lead times of 0 or 1 periods, respectively. Fukuda extends these results to the case where the lead times are k or $k+1$ periods. Whittmore and Saunders [64] discuss conditions where only one supply mode is optimal when the difference in lead times is more than one period. A more recent study where lead times differ by one period over an infinite horizon may be found in Zhang [65]. When the lead times differ by more than one period, the analysis of the problem becomes complicated, if not intractable. Authors have tried various approaches to handle this complexity.

In two different papers ([16] and [17]), Chiang and Gutierrez consider a situation where the supply lead times are shorter than the review periods. In the first paper, they include a fixed cost for emergency orders and assume a base-stock policy is optimal; they then show that for a particular base-stock level, either only regular shipments are used or there exists an inventory threshold under which expedited shipments are used. In the second paper, they do not include fixed costs for emergency orders and they derive how to calculate the optimal base-stock level. In three recent papers by Tagaras and Vlachos [55], Vlachos and Tagaras [63], and Teunter and Vlachos [57], an inventory system with regular and emergency shipments is considered. Due to the complexity of the analysis, all three papers consider approximate cost functions and near-optimal policies for both kinds of shipments are discussed. The first paper discusses at heuristic that quickly calculates near-optimal policies. The second paper takes capacity restriction into account. The third paper considers two sets of approximate optimality conditions, one which has a complicated form but is quite accurate, another which has a simple form but may be far from optimal.

Another approach to this problem is under the continuous review setting. Moinzadeh has been prolific in this area. In two papers with Nahmias ([45] and [46]), the authors first consider (Q, R) policies for both regular and expedited orders; they discuss procedures for calculating these parameters and use simulation to check their results. In the second paper, the authors consider a fixed contract with emergency delivery adjustments; they prove that finding the optimal policy is intractable, but discuss procedures for finding approximate solutions. In [43], Moinzadeh and Schmidt assume a "one-for-one" $(S-1, S)$ inventory policy and consider an expediting policy that depends on the age of the inventory. Mohebbi and Posner [41] consider a continuous review system with emergency orders and lost sales. Arslan et al. [4] study a make-to-order system under both continuous review and periodic review. They assign a fixed and linear cost for expediting, and show that expediting should be used when the number of backorders surpasses a threshold. Duenyas et al. [20] consider a supplier that faces a quota where overtime production may be employed to meet the quota if necessary, incurring both fixed and linear costs. Under the assumption that the supplier uses an order-up-to policy for regular production, they show that the overtime production policy will be an (s, S) policy. Other recent papers related to inventory control with some form of expediting are Hill [27], Johansen and Thorstenson [35], and Lovejoy and Sethuraman [40].

Other papers that relate to our model in Chapter II are those where either backorders are not allowed or where shortages incur fixed costs. Smith [54] considers an $(S-1, S)$ system without backorders where a per unit penalty L is assessed for each unit of unmet demand. The results of this paper basically mirror the results of an inventory problem with lost sales. Moinzadeh considers an $(S-1, S)$ inventory system with partial backorders in his 1989 paper [42] and describes how the system operates under steady state. In the model of Çentinkaya and Parlar [10], backorders are allowed which incur fixed and per unit costs, but there is no setup cost for production. The authors show that a myopic base-stock policy is optimal over the infinite horizon under certain conditions. Aneja and Noori [3] consider a problem where unmet demand is met by “some external arrangement” with both per unit and fixed costs. They assume that if a shortage occurs, the inventory level will be brought up to 0 and they show that (s, S) policies are optimal over the finite horizon. Ishigaki and Sawaki [34] give a condition based on the problem parameters for (s, S) policies to be optimal for a finite horizon model with both fixed and per unit holding and lost sales costs.

Centralized Supply Chain Management

The seminal paper in the area of optimal control of multi-echelon inventory models is by Clark and Scarf in 1960 [18]. In this paper, the authors show that the multi-echelon problem can be decomposed into separate, single location problems. Here, setup costs are not allowed except at the location furthest upstream, and base-stock policies are shown to be optimal at all echelons except the location furthest upstream, where an (s, S) policy is optimal. Except for the location furthest downstream, all locations face an induced penalty for potentially causing shortages downstream which effectively increases their inventories. In Federgruen and Zipkin [21], the authors extend the results from Clark and Scarf to the infinite horizon, for both the discounted and average cost cases. They also show that calculating the various order-up-to levels is much easier in the infinite horizon case. In Chapter III, we study our problem over the infinite horizon and compare our results with those of Federgruen and Zipkin.

We cover the material from our second paper [31] in our chapter on the centralized model. In the first three sections, we assume that only one method of expediting is available. We show that the optimal policies at both stage 1 and stage 2 depend on the system inventory and the optimal policy for the entire system is a base-stock policy. We repeat the analysis in the fourth section, including both methods of expediting. In our analysis, we assume that the costs of expediting (both fixed and linear) are shared by the two stages, as the costs charged depend on the inventory available at both stages. This assumption prevents us from decomposing the problem into two separate, single location problems as is frequently done in the literature. However, by a novel substitution of system variables for stage 2 variables, we are able to solve for the optimal policies. Our solution shows that since stage 1 shares the expediting costs with stage 2, it is often optimal for stage 1 to order everything that stage 2 has available, but no more, in order to avoid excessive expediting costs.

The literature on centralized supply chains (and the comparison to decentralized and coordinated supply chains) is extensive. A recent, thorough overview of this literature may be found in the text edited by Tayur, Ganeshan and Magazine [56]. Of particular interest to us are papers that model two-stage supply chains where either the demand from stage 1 is always met by stage 2, or some form of expediting is used. The papers by Gavirneni et al. [26] and Lee et al. [37] are examples of the former, and the paper by Lawson and Porteus [36] is an example of the latter. In Gavirneni et al., the authors study a capacitated two-stage supply chain under different levels of information and show that order-up-to policies are optimal, then discuss the value of the information to the supplier (stage 2). In their model, they assume that if the supplier faces a shortage, the retailer (stage 1) "acquires the missing part of the order elsewhere." In Lee et al., the authors attempt to quantify the value of information in a two-stage supply chain with correlated demand. They assume that when the manufacturer (stage 2) faces a shortage, the manufacturer "obtains units from an alternative source" which they resupply later. Although the manufacturer always meets demand from the retailer (stage 1), they effectively pay a backorder penalty. In both models, stage 1 does not suffer the consequences of expediting, unlike in our centralized model. In fact, in a footnote to [37], the authors point out that:

> In the current paper, we make the assumption that the expedite cost is borne solely by the manufacturer so as to isolate the benefits of information sharing to the manufacturer. A similar assumption was made by most other researchers ... If this assumption is relaxed, then information sharing could bring benefits to both the manufacturer and the retailer, but this requires much more complex modeling of the contractual relationship between the manufacturer and the retailer.

We model the shared costs in Chapter III and discuss the contractual relationship in Chapter IV. Examples of other papers that discuss the value of information in the supply chain are Bourland et al. [8], Chen [11] and [12], and Cachon and Fisher [9].

The literature on multi-echelon inventory problems that directly addresses expediting is limited, with the paper by Lawson and Porteus being one notable exception. In this paper, the authors model an $m+1$-stage supply chain without setup costs where at each stage the option exists to ship by regular means, expedite, or hold inventory. The authors show that a "top-down base stock" policy is optimal, where the modified order-up-to decisions are made, in order, from the upstream stage to the downstream stage, and the regular shipment decisions are made before the expedited shipment decisions at each stage. Again, the expediting costs at each stage are not shared with other stages, and the authors use decomposition to solve the problem. Other papers that address a multi-echelon setting with expediting are due to Aggarwal and Moinzadeh [1], [44]. In both papers, the authors consider continuous review systems with a warehouse that follows an $(S-1, S)$ policy and several retailers. In the first paper, they describe methods for approximating the optimal ordering policies for the system; in the second paper, they discuss more accurate method for approximating the optimal ordering policies that depend on the inventory levels and the remaining lead times of outstanding orders.

In Chapter III, we show that the optimal policy at stage 1 has two distinct order-up-to levels, as well as a range of values where stage 1 uses up the available system inventory, but no more. Two-tiered policies have appeared in previous literature. Federgruen and Zipkin [22] study a capacitated inventory model which results in the following policy: order up to a base-stock level if capacity is sufficient, otherwise use up all the capacity. Here, the two order-up-to levels are the base-stock level and the production capacity. Henig et al. [28] model a supply and transportation contract and obtain an optimal policy where if the inventory is low, the order-up-to level is high; if the inventory is high, the order-up-to level is low; otherwise, it is optimal to order the amount specified in the contract. Hwang and Singh [32] model a production flow system with uncertain capacity. They derive a policy where below a low threshold, it is optimal not to produce, and above the low threshold, it is optimal to produce as much as possible up to a high threshold. Lastly, Fry et al. [23] study a vendor managed inventory system where the supplier can only produce once every T periods. They show that a "replenish-up-to" policy is optimal, where the replenish levels are constant at the beginning of the production cycle, but the replenish levels decrease as the end of the production cycle approaches and outsourcing is required.

As noted before, the literature on centralized supply chain management is vast and growing. Some researchers are interested in very general results whereas others investigate the traditional multi-echelon inventory problem under more specific conditions. An example of the former is Gallego and Zipkin [25]. In this paper, the authors address common questions to most supply chains, such as how do different holding costs at different stages affect inventory, and they define the "fundamental equation of supply chain theory," which is the recursion used to determine optimal base-stock policies. An example of a more specific paper is by Chen [14]. He considers the traditional multi-echelon inventory control problem, but with batch ordering, and shows that an (R, nQ) policy is optimal.

Coordinated Supply Chain Management

Research on coordinating supply chains is currently very popular. An excellent review of supply chain contract coordination may be found in Tsay et al. [58]. This review, published in 1998, lists eighty-eight different references to articles in the field. This area of research is important because it enables decentralized stages of the supply chain to work together to achieve the cost savings of the centrally controlled supply chain. In Chapter IV, we discuss two contracts that attempt to coordinate our decentralized model. The first contract does achieve system optimality, but may only be practical if the two stages are in the same firm. The second contract achieves near-optimal results for the system, but is more realistic for two, independent firms. Tsay et al. list eight different types of contract clauses; our contracts in Chapter

IV contain two of these types: specification of decision rights and pricing. In our contracts, stage 1 is given some control over the inventory decisions at stage 2, but shares the cost of expediting with stage 2 through a two-tiered wholesale cost plus a linear transfer payment. In other supply chain models, various other contracts have been shown to induce system optimality, such as quantity flexibility contracts and buyback policies (e. g., see [59] and [47]).

In our coordinated model, we try to follow the "good" contract properties of Lee and Whang [38], with partial success. We discuss these properties in Chapter IV. In our model, we assume that stage 2 has a 100% fill rate, and so stage 1 always receives its supply requests. In traditional coordinated supply chains, upstream stages face an induced penalty that increases their inventory levels, which in turn increases their fill rate. Recent articles have discusses novel ways to achieve this induced penalty function. Chen and Zheng [15] study a supply chain with fixed costs at each stage. They establish lower bounds on system costs by using a "cost-allocation scheme" where they assign echelon costs at each stage and then consider each stage individually. Porteus [50] uses "responsibility tokens", where if an upstream stage can not meet downstream demand, they pass responsibility tokens downstream instead of actual parts and the shortage is accounted for using these tokens. Under this framework, each stage can be "brilliantly self-serving" while together the stages achieve system optimality.

Logconcavity

Finally, in all our models we assume that the demand distribution is logconcave, which we define in the next section. For a discussion of the mathematical properties of logconcave functions, see Ibragrimov [33]. Porteus [48] uses the assumption of logconcavity (which he refers to as Polya Frequency 2) to show the optimality of (s, S) policies for inventory problems with concave costs. Rosling [51] provides a broad overview of logconcavity which he discusses in the context of quasiconvex cost functions. The paper by An [2] covers logconcavity for discrete probability distributions.

1.3 Supply Chain Model

We consider a periodic review, two-stage supply chain where an upstream supplier (stage 2) must meet supply requests from the downstream assembler (stage 1). Assume for convenience that each period is a day. During the current day, stage 1 and stage 2 each produce up to chosen inventory levels. At the end of the day, stage 1 experiences exogenous demand and then a decision is made about how much to produce at stage 1 for the next day; stage 2 experiences demand from stage 1 which is equal to the desired production quantity at stage 1 for the next day. If stage 2 cannot meet this demand from its current inventory, then there exists a shortage which must be filled using either overtime production or premium freight. Overtime production occurs at the end of the current day, is shipped overnight by regular shipment, and is available at stage 1 at the beginning of the next day. Products shipped by premium freight are actually built at stage 2 early the day they are required downstream and shipped very quickly, arriving in time for same day production at stage 1. Define the following variables:

D_t = the exogenous demand during period t

$x_{1,t}$ = the stage 1 inventory position at the start of period t

$z_{1,t}$ = the stage 1 production quantity during period t

$y_{1,t}$ = the stage 1 inventory level after production during period t

$x_{2,t}$ = the stage 2 inventory position at the start of period t

$z_{2,t}$ = the stage 2 regular production quantity during period t

$y_{2,t}$ = the stage 2 inventory level after regular production during period t

$\tilde{x}_{2,t}$ = the stage 2 inventory position at the start of overtime,
after receiving demand from stage 1 during period t

$\tilde{z}_{2,t}$ = the stage 2 overtime production quantity during period t

$\tilde{y}_{2,t}$ = the stage 2 inventory level after overtime production during period t

During the production processes at each stage, various costs are incurred. At stage 1, linear costs are assessed for production (c_1), holding (h_1) and backordering (b_1). Note that we allow backordering at stage 1, but not at stage 2. At stage 2, linear costs are assessed for production (c_2) and holding (h_2), overtime production incurs linear (c_o) plus fixed (K_o) costs, and premium freight shipments incur linear (c_p) plus fixed (K_p) costs as well. These costs are assumed to be discounted every period by a factor of α, with $0 < \alpha < 1$, although we consider $\alpha = 1$ in Section 4.5. Throughout this thesis, we utilize the following notation: $x^+ = x$ if $x > 0$, and 0 otherwise; $x^- = |x|$ if $x < 0$, and 0 otherwise; and, $\delta(x) = 1$ if $x > 0$, and 0 otherwise. To better understand the sequence of actions at both stages (and the associated costs in parentheses), consider the time line in Table 1.1. Note that, in the time line, the decisions are made separately by the two stages, as in our decentralized and coordinated models. Under the assumptions of the centralized model, all decisions about $y_{1,t+1}$, $\tilde{y}_{2,t}$ and $y_{2,t+1}$ are made at the same time, where stage 1 makes its decision in the time line.

In all three of our models, we make the following assumptions. First, as mentioned above, we assume a discount factor α, with $0 < \alpha < 1$. Second, we assume that demand is discrete, non-negative, stationary, and from a logconcave probability distribution. We define p_d as the probability that demand equals d and $F(d)$ as the probability that demand is less than or equal to d, and we assume that the expected value of demand, μ, is positive and finite. Third, we assume that the per unit cost of overtime production at stage 2 is greater than the per unit cost of regular production at stage 2. Fourth, we assume that the cost of backordering at stage 1 is not so small that it is cheaper to always backorder than to produce. All of these assumptions are fairly standard, except for the assumption of logconcavity (also referred to as Polya Frequency 2 or strongly unimodal), which merits some discussion.

A function $F(x)$ is said to be logconcave in x if $log(F(x))$ is concave in x. This assumption means that we require our demand distribution to have a smooth shape with at most one mode; however, the assumption is not as restrictive as it may sound, as most commonly used distribution are in fact logconcave. The Exponential, Normal, Uniform, Beta and Gamma distributions are all continuous logconcave distributions; the Poisson, Discrete Uniform, and Binomial distributions are all discrete

Stage 1 Timing	**Stage 2 Timing**
Starts period with inventory $x_{1,t}$.	Starts period with inventory $x_{2,t}$.
	If $x_{2,t} < 0$, produce $\lvert x_{2,t} \rvert$ units, and ship by premium freight. $(K_p \delta(x_{2,t}^-) + (c_p + c_2)x_{2,t}^-)$
Receives supply request from stage 2.	
Produces up to $y_{1,t} = x_{1,t} + z_{1,t}$. $(c_1(y_{1,t} - x_{1,t}))$	Produces up to $y_{2,t} = x_{2,t} + z_{2,t}$. $(c_2(y_{2,t} - x_{2,t}^+))$
Experiences demand D_t.	
Inventory next period: $x_{1,t+1} = y_{1,t} - D_t$	
Pays holding and backorder costs. $(h_1(y_{1,t} - D_t)^+ + b_1(y_{1,t} - D_t)^-)$	
Decides $y_{1,t+1} = x_{1,t+1} + z_{1,t+1}$.	
Places order of $z_{1,t+1}$ to stage 2.	Experiences demand $z_{1,t+1}$ from stage 1.
	Overtime inventory is $\tilde{x}_{2,t} = y_{2,t} - z_{1,t+1}$.
	Decides $x_{2,t+1} = \tilde{y}_{2,t} = \tilde{x}_{2,t} + \tilde{z}_{2,t}$.
	Produces $\tilde{z}_{2,t}$ with overtime production. $(K_o \delta(\tilde{z}_{2,t}) + c_o \tilde{z}_{2,t})$
	Ships $\min\{z_{1,t+1}, y_{2,t} + \tilde{z}_{2,t}\}$ units to stage 1.
	Pays holding costs. $(h_2 \tilde{y}_{2,t}^+)$
	Decides $y_{2,t+1} = x_{2,t+1} + z_{2,t+1} \geq 0$

Table 1.1: Decision Time Line for Both Stages

logconcave distributions. For the discrete distributions that we consider, we use two nice properties of logconcave functions: the fraction $\frac{F(x+1)-F(x)}{F(x)}$ is nonincreasing in x and the convolution of a quasiconvex function with a logconcave demand distribution remains quasiconvex. For later reference, we label our assumptions as follows:

(A1) $0 < \alpha < 1$.

(A2) Demand is discrete, stationary, non-negative, and logconcave.

(A3) For all t, $0 < \mu < \infty$.

(A4) $c_2 < c_o$.

(A5) $b_1 > (1 - \alpha)c_1$.

Finally, throughout this thesis, we assume that the ultimate goal is to minimize expected, total, discounted costs over an infinite horizon, except in Section 4.5 where we consider the average cost case. We originally considered our problem under a finite horizon, but found that due to end-of-horizon effects, finite horizon results were more difficult to prove. For all of our models, we follow the notational conventions of Bertsekas [7]; here is a brief review. For each minimization problem, we start with a cost per period g(period "k" variables), which consists of all the costs incurred during period "k". Note that the period "k" may not start and end at the same points in time as listed in the time line and that some of the variables and costs may occur in different time periods. We then may make changes to the cost per period, moving terms and possibly relaxing constraints. To represent the expected cost per period, we use the notation G(period "k" variables) $= E_D[g$(period "k" variables)$]$. For a given policy π, the expected total discounted cost over the infinite horizon is

$f_\pi(x_0)$, where x_0 is the initial inventory information:

$$f_\pi(x_0) \equiv \lim_{N\to\infty} E\left[\sum_{k=0}^{N-1} \alpha^k g(\text{period ``}k\text{'' variables})\right].$$

We are interested in finding the optimal policy π out of all possible admissible policies Π and hence the optimal expected total discounted cost over the infinite horizon, $f^*(x_0)$. This function will exist according to Proposition 1.1 on page 137 of Bertsekas [7] if g(period "k" variables)≥ 0. We seek to solve the optimal cost function $f^*(x_0) \equiv \min_{\pi\in\Pi} f_\pi(x_0)$ in order to determine the optimal inventory control policies.

CHAPTER II

DECENTRALIZED MODEL

In this chapter, we study the decentralized model. We assume that the two stages of the supply chain are independent firms that each seek to minimize their own costs. In Section 2.1, we prove the optimal inventory control policy for stage 1 is a base-stock policy. In Section 2.2, we justify that the optimal inventory control policy for stage 2 is a base-stock policy under the assumption that overtime production is the only method of expediting available. The stage 2 proof is slightly more complicated than the stage 1 proof because we must show that the stage 2 expected one period costs are quasiconvex. In Section 2.3, we again consider stage 2 but include premium freight as an option and we include an additional setup cost for regular production at stage 2. We proceed to show that the optimal inventory control policy for overtime production depends on stage 2 costs and we show that the optimal inventory control policy for regular production is an (s, S) policy.

Note that in Section 2.1, we are not proving anything new. (Actually, most of Section 2.1 is a homework problem in [7] taken from [29]!) However, we derive the optimal policy for stage 1 step-by-step for two reasons. First, the more complicated derivations later in the thesis tend to follow the same steps, and we feel that the proof in Section 2.1 is a good introduction to this methodology. Second, one of the steps we will frequently use is to "move" a term back to the previous period; we justify this step rigorously in Section 2.1 and then refer to it later in the thesis when necessary.

2.1 Stage 1 Optimal Policy Under Decentralized Control

Under decentralized control, stage 1 is an independent firm. The manager at stage 1 makes decisions based on the initial inventory available, x_1, and the potential costs incurred including the production cost c_1, and the holding and backordering costs, h_1 and b_1, respectively. The manager's decision is particularly straightforward because stage 2 always meets its supply requests, ensuring that stage 1 can reach its production goals. All the variables discussed in this section occur during the same period t, so we drop the subscript t for notational convenience. The one period costs experienced by stage 1 are

$$g_{1,dec}(x_1, y_1, D) = c_1(y_1 - x_1) + h_1(y_1 - D)^+ + b_1(y_1 - D)^-$$

with $y_1 \geq x_1$. Clearly, $g_{1,dec}(\cdot) \geq 0$ and hence by [7], the optimal cost function $f^*_{1,dec}(x_1)$ satisfies

$$f^*_{1,dec}(x_1) = \min_{y_1 \geq x_1} E_D[g_{1,dec}(x_1, y_1, D) + \alpha f^*_{1,dec}(y_1 - D)].$$

The argument that minimizes this equation is the optimal inventory control policy which we seek. In order to determine this policy, we now move the $-c_1 x_1$ term back to the previous period as $-\alpha c_1(y_1 - D)$ (using a technique similar to Veinott [61] which we justify below) and define our moved one period costs as

$$\begin{aligned} g_{1,dec,m}(x_1, y_1, D) &= c_1 y_1 + h_1(y_1 - D)^+ + b_1(y_1 - D)^- - \alpha c_1(y_1 - D) \\ &= (1-\alpha)c_1 y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^- \end{aligned}$$

with $y_1 \geq x_1$.

Lemma 1 *The optimal policy which solves $f^*_{1,dec}(x_{1,0})$ also solves $f^*_{1,dec,m}(x_{1,0})$ and*

$$f^*_{1,dec}(x_{1,0}) = -c_1 x_{1,0} + f^*_{1,dec,m}(x_{1,0}).$$

Proof:

$$\begin{aligned}
& f^*_{1,dec}(x_{1,0}) \\
&= \min_{\pi \in \Pi} f_{1,dec,\pi}(x_{1,0}) \\
&= \min_{\pi \in \Pi} \lim_{N \to \infty} E_D \left[\sum_{k=0}^{N-1} \alpha^k g_{1,dec}(x_{1,k}, y_{1,k}, D_k) \right] \\
&= \min_{\pi \in \Pi} \lim_{N \to \infty} E_D \left[\sum_{k=0}^{N-1} \alpha^k (c_1(y_{1,k} - x_{1,k}) + h_1(y_{1,k} - D_k)^+ + b_1(y_{1,k} - D_k)^-) \right] \\
&= -c_1 x_{1,0} + \\
& \quad \min_{\pi \in \Pi} \lim_{N \to \infty} E_D \left[\begin{array}{l} \sum_{k=0}^{N-2} \alpha^k \left(\begin{array}{l} c_1 y_{1,k} + h_1(y_{1,k} - D_k)^+ + b_1(y_{1,k} - D_k)^- \\ -\alpha c_1(y_{1,k} - D_k) \end{array} \right) + \\ \alpha^{N-1}(c_1 y_{1,N-1} + h_1(y_{1,N-1} - D_{N-1})^+ + b_1(y_{1,N-1} - D_{N-1})^-) \end{array} \right] \\
&= -c_1 x_{1,0} + \min_{\pi \in \Pi} \left\{ \begin{array}{l} \lim_{N \to \infty} E_D \left[\sum_{k=0}^{N-2} \alpha^k (g_{1,dec,m}(x_{1,k}, y_{1,k}, D_k)) \right] + \\ \lim_{N \to \infty} E_D \left[\begin{array}{l} \alpha^{N-1}(c_1 y_{1,N-1} + h_1(y_{1,N-1} - D_{N-1})^+ \\ +b_1(y_{1,N-1} - D_{N-1})^-) \end{array} \right] \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= -c_1 x_{1,0} + \min_{\pi \in \Pi} \lim_{N \to \infty} E_D \left[\sum_{k=0}^{N-1} \alpha^k (g_{1,dec,m}(x_{1,k}, y_{1,k}, D_k)) \right] + 0 \\
&= -c_1 x_{1,0} + f^*_{1,dec,m}(x_{1,0})
\end{aligned}
$$

The first, second, third, and seventh equalities are by definition. The fourth equality holds since $x_{1,0}$ is constant and $y_{1,k} - D_k = x_{1,k+1}$ for $k = 0 \ldots N-2$. The fifth equality holds since the last term is positive and the sixth equality holds since the last term is finite. □

Now consider the optimal cost function $f^*_{1,dec,m}(x_1)$:

$$
\begin{aligned}
f^*_{1,dec,m}(x_1) &= \min_{y_1 \geq x_1} E_D[g_{1,dec,m}(x_1, y_1, D) + \alpha f^*_{1,dec,m}(y_1 - D)] \\
&= \min_{y_1 \geq x_1} \left\{ G_{1,dec,m}(y_1) + \alpha E_D[f^*_{1,dec,m}(y_1 - D)] \right\}
\end{aligned}
$$

where $G_{1,dec,m}(y_1) = E_D[g_{1,dec,m}(x_1, y_1, D)]$. Clearly, $G_{1,dec,m}(y_1)$ is convex and so $-G_{1,dec,m}(y_1)$ is unimodal. To apply the results from Zheng's paper [66], we need that $-G_{1,dec,m}(y_1)$ is unimodal and that $G_{1,dec,m}(y_1) \to \infty$ as $|y_1| \to \infty$. For $y_1 < 0$, the slope (in the discrete sense) of $G_{1,dec,m}(y_1)$ is $-b_1 + (1-\alpha)c_1 < 0$ by assumption (A5); thus, as $y_1 \to -\infty$, $G_{1,dec,m}(y_1) \to \infty$. As $y_1 \to +\infty$, the slope of $G_{1,dec,m}(y_1)$ becomes $(1-\alpha)c_1 + h_1 > 0$, and thus $G_{1,dec,m}(y_1) \to \infty$. Hence, we have from [66] that the optimal inventory control policy at stage 1 is a base-stock policy. Define the optimal base-stock level as $S^*_{1,dec}$. Under the assumption that the initial inventory is not more than this value, $x_1 \leq S^*_{1,dec}$, we can calculate $f^*_{1,dec}(x_1)$.

$$
\begin{aligned}
&f^*_{1,dec}(x_1) \\
&\quad = -c_1 x_1 + f^*_{1,dec,m}(x_1) \\
&\quad = -c_1 x_1 + \min_{y_1 \geq x_1} \left\{ G_{1,dec,m}(y_1) + \alpha E_D[f^*_{1,dec,m}(y_1 - D)] \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= -c_1 x_1 + G_{1,dec,m}(S^*_{1,dec}) + \alpha E_D[f^*_{1,dec,m}(S^*_{1,dec} - D)] \\
&= -c_1 x_1 + G_{1,dec,m}(S^*_{1,dec}) + \alpha G_{1,dec,m}(S^*_{1,dec}) + \alpha^2 E_D[f^*_{1,dec,m}(S^*_{1,dec} - D)] \\
&= -c_1 x_1 + G_{1,dec,m}(S^*_{1,dec})(1 + \alpha + \alpha^2 + \ldots) \\
&= -c_1 x_1 + \frac{G_{1,dec,m}(S^*_{1,dec})}{1 - \alpha}
\end{aligned}
$$

We will use this relationship to compare costs in Chapter IV.

Hence, under decentralized control, the optimal policy at stage 1 is to order up to $S^*_{1,dec}$ every period. It is important to note for the next section that, due to the base-stock policy, stage 1 will pass the exact demand it experiences back to stage 2. So, stage 2 will face the same, logconcave demand that stage 1 faces.

2.2 Stage 2 Optimal Policy Under Decentralized Control

Under decentralized control, stage 2 is also an independent firm. The manager at stage 2 makes decisions based on the initial inventory available, x_2, and the potential costs incurred including the production cost c_2, holding cost h_2, and the overtime production costs c_o and K_0. Stage 2 faces the same demand distribution as stage 1. In this section, we assume that overtime production is the only method of filling shortages. Thus, the overtime decision is straightforward: if there is a shortage, run overtime production to fill it. Since the per unit cost of overtime production is more than that of regular production, it will never be cost-effective to produce more than the shortage with overtime production.

Mathematically, during period t, stage 1 will order $z_{1,t+1} = D_t$ from stage 2. At the beginning of overtime, the overtime inventory level is $\tilde{x}_{2,t} = y_{2,t} - D_t$. If this quantity is negative, overtime production must be employed. The overtime production quantity is $(y_{2,t} - D_t)^-$. Again, all the variables that follow in this section occur during the same time period t, so we drop the subscript. Given that stage 2 starts with initial inventory x_2, the one period costs experienced by stage 2 are

$$g_{2,dec}(x_2, y_2, D) = c_2(y_2 - x_2^+) + K_o\delta((y_2 - D)^-) + c_o(y_2 - D)^- + h_2(y_2 - D)^+$$

with $y_2 \geq x_2^+$. Clearly, $g_{2,dec}(\cdot) \geq 0$ and hence by [7], the optimal cost function $f^*_{2,dec}(x_2)$ satisfies

$$f^*_{2,dec}(x_2) = \min_{y_2 \geq x_2^+} E_D[g_{2,dec}(x_2, y_2, D) + \alpha f^*_{2,dec}(y_2 - D)].$$

The argument that minimizes this equation is the optimal inventory control policy which we seek. In order to determine this policy, we now move the $-c_2 x_2^+$ term back to the previous period as $-\alpha c_2(y_2 - D)^+$ as we did in the previous section. Define our moved one period costs as

$$g_{2,dec,m}(x_2, y_2, D) \quad = \quad c_2 y_2 + K_o\delta((y_2 - D)^-) + c_o(y_2 - D)^- + (h_2 - \alpha c_2)(y_2 - D)^+$$

with $y_2 \geq x_2^+$. Note that the optimal policy which solves $f^*_{2,dec}(x_2)$ also solves $f^*_{2,dec,m}(x_2)$ and

$$f^*_{2,dec}(x_2) = -c_2 x_2^+ + f^*_{2,dec,m}(x_2).$$

Consider the optimal cost function $f^*_{2,dec,m}(x_1)$:

$$\begin{aligned}
f^*_{2,dec,m}(x_1) &= \min_{y_2 \geq x_2^+} E_D[g_{2,dec,m}(x_2, y_2, D) + \alpha f^*_{2,dec,m}(y_2 - D)] \\
&= \min_{y_2 \geq x_2^+} \left\{G_{2,dec,m}(y_2) + \alpha E_D[f^*_{2,dec,m}(y_2 - D)]\right\}
\end{aligned}$$

where $G_{2,dec,m}(y_2) = E_D[g_{2,dec,m}(x_2, y_2, D)]$. To apply the result from Zheng [66] as we did before, we need to show $G_{2,dec,m}(y_2) \to \infty$ as $|y_2| \to \infty$ and that $-G_{2,dec,m}(y_2)$ is unimodal or that $G_{2,dec,m}(y_2)$ is quasiconvex. For $y_2 < 0$, the slope of $G_{2,dec,m}(y_2)$ is $c_2 - c_o < 0$ by assumption (A4); thus, as $y_2 \to -\infty$, $G_{2,dec,m}(y_2) \to \infty$. As $y_2 \to +\infty$, the slope of $G_{2,dec,m}(y_2)$ becomes $h_2 + (1-\alpha)c_2 > 0$, and thus $G_{2,dec,m}(y_2) \to \infty$. We now show that $G_{2,dec,m}(y_2)$ is quasiconvex in the following lemma. In fact, we prove the lemma two different ways: the first proof is brief, using results from Porteus ([49]) and An ([2]) and the assumption that demand is logconcave; the second proof is longer, but more instructive, showing exactly where the logconcavity assumption is required.

Lemma 2 *The expected one period cost $G_{2,dec,m}(y_2)$ is quasiconvex.*

Proof 1: Consider the one period cost function

$$\begin{aligned} g_{2,dec,m}(x_2, y_2, d) &= c_2 y_2 + K_o \delta((y_2 - d)^-) + c_o (y_2 - d)^- + (h_2 - \alpha c_2)(y_2 - d)^+ \\ &= \begin{cases} K_o - (c_o - c_2) y_2 + c_o d & \text{if } y_2 < d \\ (h_2 + (1-\alpha) c_2) y_2 - (h_2 - \alpha c_2) d & \text{if } y_2 \geq d \end{cases} \end{aligned}$$

for a given value of d. This function is quasiconvex in y_2. To the left of d, it decreases with slope $c_2 - c_o < 0$ by (A4). From point d to the right, it increases with slope $(h_2 + (1-\alpha)c_2) > 0$. Now, $G_{2,dec,m}(y_2) = E_D[g_{2,dec,m}(x_2, y_2, D)]$, so $G_{2,dec,m}(y_2)$ is the convolution of a quasiconvex function and a logconcave demand distribution. According to Porteus ([49], page 619) and An ([2], Proposition 10), the convolution of a quasiconvex function with a logconcave demand distribution remains quasiconvex; therefore, $G_{2,dec,m}(y_2)$ is also quasiconvex.

Proof 2: We calculate the differential of $G_{2,dec,m}(y_2)$ and show that it goes from negative to positive, changing signs exactly once. Define

$$\begin{aligned}
\Delta G_{2,dec,m}(y_2) &\equiv G_{2,dec,m}(y_2+1) - G_{2,dec,m}(y_2) \\
&= c_2(y_2+1) + K_o \sum_{d=y_2+2}^{\infty} p_d + c_o \sum_{d=y_2+2}^{\infty} (d-(y_2+1))p_d + \\
&\quad (h_2 - \alpha c_2)\sum_{d=0}^{y_2+1}((y_2+1)-d)p_d - c_2 y_2 - K_o \sum_{d=y_2+1}^{\infty} p_d \\
&\quad -c_o \sum_{d=y_2+1}^{\infty}(d-y_2)p_d - (h_2-\alpha c_2)\sum_{d=0}^{y_2}(y_2-d)p_d \\
&= c_2 - K_o p_{y_2+1} - c_o(1-F(y_2)) + (h_2 - \alpha c_2)F(y_2) \\
&= c_2 - c_o - K_o p_{y_2+1} + (h_2 + c_o - \alpha c_2)F(y_2) \\
&= F(y_2)\left[(c_2 - c_o)\frac{1}{F(y_2)} - K_o \frac{p_{y_2+1}}{F(y_2)} + (h_2 + c_o - \alpha c_2)\right]
\end{aligned}$$

for $F(y_2) > 0$. Assume $d_0 \geq 0$ is the smallest d such that $p_d > 0$. For $y_2 < d_0 - 1$, $\Delta G_{2,dec,m}(y_2) = c_2 - c_0 < 0$. At $y_2 = d_0 - 1$, $\Delta G_{2,dec,m}(y_2) = c_2 - c_0 - K_o p_{d_0} < 0$. So, the differential is negative from the left up to d_0. From d_0 to the right, the differential is the product of $F(y_2)$, a positive, nondecreasing function, and the term in the brackets. The terms in the brackets are also nondecreasing, which we will explain below, and hence the product of the two terms can change signs at most once. And, as $y_2 \to +\infty$, $\Delta G_{2,dec,m}(y_2) \to h_2 + (1-\alpha)c_2 > 0$. Thus, the differential changes from negative to positive exactly once, and hence $G_{2,dec,m}(y_2)$ is quasiconvex.

To understand why the terms in the brackets nondecreasing, note that the first term is the product of a negative value and a nonincreasing function, and is hence nondecreasing. The third term is constant, and hence nondecreasing. The second term is the product of a negative value and $\frac{p_{y_2+1}}{F(y_2)}$. According to Rosling [51], discrete logconcave distributions have the property that the fraction $\frac{p_{d+1}}{F(d)}$ is nonincreasing. Hence, the second term in nondecreasing and so the sum of all three terms is nondecreasing. Without this property, it is possible that the second term could vary enough to cause more than one minimum for $G_{2,dec,m}(y_2)$ and a base-stock policy may not be optimal for stage 2. $\square$

Using the result from [66], we have that the optimal inventory control policy at stage 2 is a base-stock policy. Define the optimal base-stock level as $S^*_{2,dec}$. Under the assumption that the initial inventory is not more than this value, $x_2 \leq S^*_{2,dec}$, we can further calculate $f^*_{2,dec}(x_2)$.

$$\begin{aligned}
&f^*_{2,dec}(x_2) \\
&\quad= -c_2 x_2^+ + f^*_{2,dec,m}(x_2) \\
&\quad= -c_2 x_2^+ + \min_{y_2 \geq x_2^+} \left\{ G_{2,dec,m}(y_2) + \alpha E_D[f^*_{2,dec,m}(y_2 - D)] \right\} \\
&\quad= -c_2 x_2^+ + G_{2,dec,m}(S^*_{2,dec}) + \alpha E_D[f^*_{2,dec,m}(S^*_{2,dec} - D)] \\
&\quad= -c_2 x_2^+ + G_{2,dec,m}(S^*_{2,dec}) + \alpha G_{2,dec,m}(S^*_{2,dec}) + \alpha^2 E_D[f^*_{2,dec,m}(S^*_{2,dec} - D)] \\
&\quad= -c_2 x_2^+ + G_{2,dec,m}(S^*_{2,dec})(1 + \alpha + \alpha^2 + \ldots) \\
&\quad= -c_2 x_2^+ + \frac{G_{2,dec,m}(S^*_{2,dec})}{1 - \alpha}
\end{aligned}$$

Again, we will use this relationship to compare costs in Chapter IV. Under decentralized control with overtime production as the only expediting option, the optimal policy at stage 2 is to order up to $S^*_{2,dec}$ every period.

2.3 Stage 2 Optimal Policy with Premium Freight and a Setup Cost for Regular Production

In this section, we reconsider stage 2 under decentralized control. First, we include premium freight as an option for expediting. Now, the manager has a choice of how to fill shortages, either using overtime production, premium freight, or some combination of the two. Unlike the previous section, the overtime production quantity is now a more challenging decision. Second, we include an additional setup cost, K_2, for regular production. This inclusion adds realism to the problem, but also adds additional mathematical challenges. In this section only, we have three distinct fixed costs: for regular production at stage 2, for overtime production at stage 2, and for premium freight shipments. We originally tried to include a setup cost K_1 at stage 1 as well, but this assumption caused stage 2 to experience non-logconcave demand by creating a bullwhip effect (see [39]), and the problem became intractable. For a discussion of a two-stage supply chain with setup costs at both stages, see the paper by Chen [13].

Our goal in this section is to determine the optimal overtime production strategy and the optimal regular production strategy. In that order, we first show that the overtime production policy depends on the problem data. For example, if overtime production costs are relatively inexpensive and premium freight shipments are costly, it will be always be optimal to fill shortages with overtime production. However, if the two expediting costs are similar, whether or not to use overtime production depends on the actual shortage. Second, we show that the optimal regular production policy is an (s, S) policy, which is a typical result for a single location inventory problems with a setup cost for production.

To prove our results in this section, we require two additional assumptions about the costs involved. First, we assume that the setup cost for regular production is not more than the setup cost for overtime production, $K_2 \leq K_o$. Second, we assume that the per unit cost of regular production is less than the discounted per unit cost of premium freight and regular production, $c_2 < \alpha(c_2 + c_p)$. This assumption is similar to the assumption that $c_2 < c_o$. We label these assumptions as follows:

(A6) $K_2 \leq K_o$

(A7) $c_2 < \alpha(c_2 + c_p)$

2.3.1 Relationship Between Overtime and Regular Production

In this section, we will consider two types of time periods which we refer to as the regular-period and the overtime-period. We consider these two periods so that we may analyze various costs starting at different instances in the production cycle. Referring to the time line in Table 1.1, the regular-period begins where stage 2 decides its regular production quantity for the upcoming period. During a regular-period, all actions and costs occur during the same time period t. The overtime-period begins where stage 2 decides its overtime production quantity for the current period. During an overtime-period, some actions and costs occur during the current time period t, while others actually occur during the following time period $t + 1$. To keep track of what is happening when, we do not drop the time subscripts at the beginning of this section as we did in the previous two sections. However, to prevent subscripts from overwhelming us, we point out here that all discussion for the rest of this section focuses on the stage 2, decentralized problem; hence, we drop the subscript $_2$ and the subscript $_{dec}$ from our functions and variables. For both regular- and overtime-periods, this problem has functions representing the cost per stage, the expected, total, discounted cost, and the optimal, expected, total, discounted cost, using the same notation as before. For the regular-period, we will let g represent the cost per period, f_π represent the total cost, and f^* represent the optimal cost. We distinguish the overtime-period functions with a $\widetilde{\text{tilde}}$; we will let $\tilde{g}$ represent the one period cost, $\tilde{f}_\pi$ represent the total cost, and $\tilde{f}^*$ represent the optimal cost of an overtime-period. We now formally define these functions.

The one period costs experienced during regular production are

$$\begin{aligned} g(x_t, y_t, \tilde{x}_t, \tilde{y}_t) &= K_p\delta(x_t^-) + c_p x_t^- + K_2\delta(y_t - x_t) + c_2(y_t - x_t) \\ &\quad + K_o\delta(\tilde{y}_t - \tilde{x}_t) + c_o(\tilde{y}_t - \tilde{x}_t) + h_2\tilde{y}_t^+ \end{aligned}$$

with $y_t \geq x_t^+$ and $\tilde{y}_t \geq \tilde{x}_t$. The first two costs are for premium freight, the next two costs are for regular production, the next two costs are for overtime production, and the last cost is the holding cost. Note that three of these costs, $K_p\delta(x_t^-)$, $c_p x_t^-$, and $-c_2 x_t$ are predetermined from overtime decision of the previous period. We move these three costs back to the previous regular-period and define:

$$\begin{aligned} g_m(x_t, y_t, \tilde{x}_t, \tilde{y}_t) &\equiv K_2\delta(y_t - x_t) + c_2 y_t + K_o\delta(\tilde{y}_t - \tilde{x}_t) \\ &\quad + c_o(\tilde{y}_t - \tilde{x}_t) + h_2\tilde{y}_t^+ + \alpha[K_p\delta(\tilde{y}_t^-) + c_p\tilde{y}_t^- - c_2\tilde{y}_t] \end{aligned}$$

with $y_t \geq x_t^+$ and $\tilde{y}_t \geq \tilde{x}_t$. Note that $\tilde{x}_t = y_t - D_t$. Ideally, we would like to show that $g_m(x_t, y_t, \tilde{x}_t, \tilde{y}_t) \geq 0$ in order to apply the result from Bertsekas [7] as before, but it may be negative when $0 < \tilde{x}_t < \tilde{y}_t$; however, we will show this case is never optimal in the overtime section.

We now define our overtime-period cost per stage as

$$\begin{aligned}\tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1}) &= K_o\delta(\tilde{y}_t - \tilde{x}_t) + c_o(\tilde{y}_t - \tilde{x}_t) + h_2\tilde{y}_t^+ \\ &\quad +\alpha[K_p\delta(\tilde{y}_t^-) + c_p\tilde{y}_t^- + K_2\delta(y_{t+1} - \tilde{y}_t) + c_2(y_{t+1} - \tilde{y}_t)]\end{aligned}$$

with $\tilde{y}_t \geq \tilde{x}_t$ and $y_{t+1} \geq \tilde{y}_t^+$. Note that $\tilde{y}_t = x_{t+1}$ and

$$\begin{aligned}\tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1}) &\geq c_o(\tilde{y}_t - \tilde{x}_t) + h_2\tilde{y}_t^+ + \alpha c_p\tilde{y}_t^- - \alpha c_2\tilde{y}_t + \alpha c_2 y_{t+1} \\ &\geq -\alpha c_2\tilde{y}_t + \alpha c_2 y_{t+1} = \alpha c_2(y_{t+1} - \tilde{y}_t) = \alpha c_2(y_{t+1} - x_{t+1}) \\ &\geq 0\end{aligned}$$

where the first inequality is true because all setup costs are non-negative, the second inequality is true because the three terms dropped are non-negative, and the third inequality is true because $y_{t+1} \geq x_{t+1}$.

Before defining our total cost functions, consider that π is an admissible policy if $y_t \geq x_t^+$ and $\tilde{y}_t \geq \tilde{x}_t$ for all t and both y_t and $\tilde{y}_t$ are chosen in a non-anticipatory fashion. In other words, y_t may only depend on x_t and $(x_i, y_i, D_i, \tilde{x}_i, \tilde{y}_i)$ where $i < t$; $\tilde{y}_t$ may only depend on $\tilde{x}_t$ and $(\tilde{x}_{i-1}, \tilde{y}_{i-1}, x_i, y_i, D_i)$ where $i \leq t$. Let Π be the set of all such policies. For the regular-period, let

$$f_{m,\pi}(x_0) \equiv \limsup_{N\to\infty} E_{\mathcal{D}_0}\left[\sum_{t=0}^{N-1} \alpha^t g_m(x_t, y_t, \tilde{x}_t, \tilde{y}_t)\right] \tag{2.1}$$

where $\mathcal{D}_0 = \{D_0, D_1, D_2, \ldots\}$. Note that $f_\pi(x_0) = K_p\delta(x_0^-) + c_p x_0^- - c_2 x_0 + f_{m,\pi}(x_0)$. For the overtime-period, let

$$\tilde{f}_\pi(\tilde{x}_0) \equiv \lim_{N\to\infty} E_{\mathcal{D}_1}\left[\sum_{t=0}^{N-1} \alpha^t \tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1})\right] \tag{2.2}$$

where $\mathcal{D}_1 = \{D_1, D_2, D_3, \ldots\}$. Note that the limit is known to exist in this case since $\tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1}) \geq 0$. There is a strong relationship between these two functions.

$$\begin{aligned}
f_{m,\pi}(x_0) &= \limsup_{N\to\infty} E_{\mathcal{D}_0}\left[\sum_{t=0}^{N-1} \alpha^t g_m(x_t, y_t, \tilde{x}_t, \tilde{y}_t)\right] \\
&= \limsup_{N\to\infty} E_{\mathcal{D}_0}\left[K_2\delta(y_0 - x_0) + c_2 y_0 + \left[\sum_{t=0}^{N-1} \alpha^t \tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1})\right]\right] \\
&= K_2\delta(y_0 - x_0) + c_2 y_0 + \limsup_{N\to\infty} E_{\mathcal{D}_0}\left[\sum_{t=0}^{N-1} \alpha^t \tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1})\right] \\
&= K_2\delta(y_0 - x_0) + c_2 y_0 + \lim_{N\to\infty} E_{\mathcal{D}_0}\left[\sum_{t=0}^{N-1} \alpha^t \tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1})\right] \\
&= K_2\delta(y_0 - x_0) + c_2 y_0 + \lim_{N\to\infty} E_{D_0}\left[E_{\mathcal{D}_1}\left[\sum_{t=0}^{N-1} \alpha^t \tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1})\middle| D_0\right]\right] \\
&= K_2\delta(y_0 - x_0) + c_2 y_0 + E_{D_0}\left[\lim_{N\to\infty} E_{\mathcal{D}_1}\left[\sum_{t=0}^{N-1} \alpha^t \tilde{g}(\tilde{x}_t, \tilde{y}_t, y_{t+1})\middle| D_0\right]\right] \\
&= K_2\delta(y_0 - x_0) + c_2 y_0 + E_{D_0}\left[\tilde{f}_\pi(\tilde{x}_0)\right] \\
&= K_2\delta(y_0 - x_0) + c_2 y_0 + E_{D_0}\left[\tilde{f}_\pi(y_0 - D_0)\right]
\end{aligned}$$

where $y_0 \geq x_0^+$. The first equality is true by definition of $f_{m,\pi}$, the second equality is true by the definitions of g_m and $\tilde{g}$, the fourth equality is true since $\tilde{g} \geq 0$, the sixth equality is true by the Monotone Convergence Theorem since $\tilde{g} \geq 0$, and the seventh equality is true by the definition of $\tilde{f}_\pi$ and because the system is Markovian. Thus, under the restriction that $y_0 \geq x_0^+$, we have

$$f_{m,\pi}(x_0) = K_2\delta(y_0 - x_0) + c_2 y_0 + E_{D_0}\left[\tilde{f}_\pi(y_0 - D_0)\right]. \tag{2.3}$$

Recall that in equation (2.1), we used $\limsup$ to define $f_{m,\pi}(x_0)$ rather than the limit. Now, by equations (2.2) and (2.3) and since $y_0 \geq x_0^+$, we see that the limit for $f_{m,\pi}(x_0)$ exists. Similarly, we can write $\tilde{f}$ in terms of f_m and under the restriction that $\tilde{y}_0 \geq \tilde{x}_0$, we have

$$\tilde{f}_\pi(\tilde{x}_0) = K_o\delta(\tilde{y}_0-\tilde{x}_0)+c_o(\tilde{y}_0-\tilde{x}_0)+h_2\tilde{y}_0^+ +\alpha[K_p\delta(\tilde{y}_0^-)+c_p\tilde{y}_0^- -c_2\tilde{y}_0]+\alpha f_{m,\pi}(\tilde{y}_0). \quad (2.4)$$

Finally, we define the optimal expected, total, discounted cost functions starting in regular time and overtime, respectively, as

$$f_m^*(x) \equiv \min_{\pi\in\Pi} f_{m,\pi}(x)$$

and

$$\tilde{f}^*(\tilde{x}) = \min_{\pi\in\Pi} \tilde{f}_\pi(\tilde{x})$$

where $x \in \mathcal{I}$, $\tilde{x} \in \mathcal{I}$, and $\mathcal{I}$ is the set of all integers. From equations (2.3) and (2.4) we can write our optimal cost functions as combinations of each other and we get

$$f_m^*(x) \equiv \min_{y\geq x^+} \left\{K_2\delta(y-x) + c_2 y + E_D[\tilde{f}^*(y-D)]\right\} \quad (2.5)$$

and

$$\tilde{f}^*(\tilde{x}) \equiv \min_{\tilde{y}\geq\tilde{x}} \left\{K_o\delta(\tilde{y}-\tilde{x}) + c_o(\tilde{y}-\tilde{x}) + h_2\tilde{y}^+ + \alpha[K_p\delta(\tilde{y}^-) + c_p\tilde{y}^- - c_2\tilde{y}] + \alpha f_m^*(\tilde{y})\right\}. \quad (2.6)$$

Given the definitions above, the following lemma shows that the optimal cost functions are finite.

Lemma 3 *The optimal cost functions $\tilde{f}^*(\tilde{x})$ and $f_m^*(x)$ are finite for all $\tilde{x}, x \in \mathcal{I}$.*

Proof: Observe that $\tilde{f}_\pi(\tilde{x})$ and $f_{m,\pi}(x)$ are non-negative for all policies $\pi \in \Pi$ and for all $\tilde{x}, x \in \mathcal{I}$. (Note: $\tilde{f}_\pi$ is non-negative since it is the sum of non-negative $\tilde{g}$'s and $f_{m,\pi}$ is non-negative by equation (2.3)). It suffices to show that there exists a policy γ such that $\tilde{f}_\gamma(\tilde{x}) < \infty$ and $f_{m,\gamma}(x) < \infty$ for all $\tilde{x}, x \in \mathcal{I}$. Let this γ be such that whenever the inventory level is negative produce up to 0; otherwise, do nothing. Note that this policy applies to both overtime and regular production and that this policy is stationary. Consider the cost per stage during an overtime-period under this policy:

$$\tilde{g}_\gamma(\tilde{x}, \tilde{y}, y) = K_o\delta(\tilde{y} - \tilde{x}) + c_o(\tilde{y} - \tilde{x}) + h_2\tilde{y}^+ = \begin{cases} h_2\tilde{x} & \text{if } \tilde{x} \geq 0 \\ K_o - c_o\tilde{x} & \text{if } \tilde{x} < 0 \end{cases}$$

since $\tilde{y} = \tilde{x}^+$. Thus, since γ is stationary and $\tilde{g}_\gamma(\tilde{x}, \tilde{y}, y) \geq 0$, by Corollary 1.1.1 of Bertsekas ([7], page 139) we have that

$$\tilde{f}_\gamma(\tilde{x}) = \begin{cases} h_2\tilde{x} + \alpha E_D[\tilde{f}_\gamma(\tilde{x} - D)] & \text{if } \tilde{x} \geq 0 \\ K_o - c_o\tilde{x} + \alpha E_D[\tilde{f}_\gamma(0 - D)] & \text{if } \tilde{x} < 0. \end{cases}$$

Now note that

$$\begin{aligned} E_D[\tilde{f}_\gamma(0 - D)] &= \sum_{d=0}^{\infty} \Big(K_o + c_o d + \alpha E_D[\tilde{f}_\gamma(0 - D)]\Big)\, p_d - K_o p_0 \\ &= K_o(1 - p_0) + c_o\mu + \alpha E_D[\tilde{f}_\gamma(0 - D)] \end{aligned}$$

which implies that $E_D[\tilde{f}_\gamma(0 - D)] = \frac{K_o(1-p_0)+c_o\mu}{1-\alpha} < \infty$ by assumption (A3). Thus,

$$\tilde{f}_\gamma(\tilde{x}) = \begin{cases} h_2\tilde{x} + \alpha E_D[\tilde{f}_\gamma(\tilde{x} - D)] & \text{if } \tilde{x} \geq 0 \\ K_o - c_o\tilde{x} + \alpha\frac{K_o(1-p_0)+c_o\mu}{1-\alpha} & \text{if } \tilde{x} < 0 \end{cases}$$

and $\tilde{f}_\gamma(\tilde{x}) < \infty$ for $-\infty < \tilde{x} < 0$. Now, when $\tilde{x} \geq 0$, the inventory level will remain non-negative for some time T and will then eventually become negative at time $T + 1$. Note, $T < \infty$ almost surely (a.s.) since $\mu > 0$ by assumption (A3). While the inventory level is non-negative, there will be a holding cost of at most $h\tilde{x}$ for T

discounted periods. When the inventory goes negative, to some value N say, there will be an α^{T+1} discounted cost of $K_o - c_o N + \alpha\frac{K_o(1-p_0)+c_o\mu}{1-\alpha}$ where $-\infty < N < 0$ a.s. and $-E_D[N] \leq \mu$. So, for $\tilde{x} \geq 0$,

$$\begin{aligned}\tilde{f}_\gamma(\tilde{x}) &\leq E_D\left[\sum_{t=0}^{T} \alpha^t h_2 \tilde{x} + \alpha^{T+1}\left(K_o - c_o N + \alpha\frac{K_o(1-p_0)-c_o\mu}{1-\alpha}\right)\right] \\ &\leq \frac{h_2\tilde{x}}{1-\alpha} + K_o + c_o\mu + \alpha\frac{K_o(1-p_0)+c_o\mu}{1-\alpha} < \infty.\end{aligned}$$

Thus $\tilde{f}_\gamma(\tilde{x}) < \infty$ for $0 \leq \tilde{x} < \infty$. So, $\tilde{f}^*(\tilde{x}) \leq \tilde{f}_\gamma(\tilde{x}) < \infty$ for all $\tilde{x} \in \mathcal{I}$. Now, $f_m^*(x)$ is also finite since

$$f_{m,\gamma}(x) = \left\{\begin{array}{ll} K_2 + E_D[\tilde{f}_\gamma(0-D)] & \text{if } x < 0 \\ E_D[\tilde{f}_\gamma(x-D)] & \text{if } x \geq 0 \end{array}\right\} < \infty.$$

Thus, $f_m^*(x) \leq f_{m,\gamma}(x) < \infty$ for all $x \in \mathcal{I}$. The optimal cost functions are both finite. $\square$

To review what we have covered in this subsection, we have shown that optimal, expected, total, discounted costs exist and are finite for both regular- and overtime-periods. The argument that minimizes $f_m^*(x)$ is the optimal inventory control policy which we seek, and it is the same solution as that of $f^*(x)$. Also, the solution to $\tilde{f}^*(\tilde{x})$ is nearly the same inventory control policy as the solution to $f_m^*(x)$, but the optimal inventory control policy for the overtime-period is simply lacking the first decision y_0.

2.3.2 Optimal Overtime Production Policies

In this subsection, we characterize the structure of the optimal overtime policies. At the beginning of overtime, the inventory level is known and we must choose how much to produce during overtime, or in effect, what the inventory level will be at the start of regular time during the next regular-period. We show that if the inventory level is non-negative during overtime, it is best not to run overtime production. If the inventory is negative, we show that there are three choices: (a) produce and ship the entire shortage as premium freight at the beginning of the next period, (b) produce enough during overtime to fill the shortage (bringing the inventory level up to 0), or (c) produce more than enough to fill the shortage during overtime (bringing the inventory level up to some positive quantity), in order to avoid the fixed cost for regular production. Next, we show that four different optimal policies are possible, depending on the problem parameters when the overtime inventory level is negative. These policies are to do premium freight when the shortage is small and overtime when the shortage is large, to only do overtime, to only use premium freight, and do overtime when the shortage is small and premium freight when the shortage is large. The first three policies are $(\tilde{s}, \tilde{S})$ policies for overtime production, but the fourth is not. (The tildes indicate we are considering overtime production.)

At this point, we drop the time subscripts for notational convenience, keeping in mind that some of the overtime-period variables actually occur during the next time period. Consider the cost per stage for an overtime-period

$$\tilde{g}(\tilde{x}, \tilde{y}, y) = K_o\delta(\tilde{y}-\tilde{x})+c_o(\tilde{y}-\tilde{x})+h_2\tilde{y}^+ +\alpha[K_p\delta(\tilde{y}^-)+c_p\tilde{y}^- +K_2\delta(y-\tilde{y})+c_2(y-\tilde{y})]$$

where $\tilde{y} \geq \tilde{x}$ and $y \geq \tilde{y}^+$. Since $\tilde{g}(\tilde{x}, \tilde{y}, y) \geq 0$, we refer to [7] and the optimal cost function $\tilde{f}^*$ satisfies

$$\tilde{f}^*(\tilde{x}) = \min_{\tilde{y}\geq\tilde{x}, y\geq\tilde{y}^+} E_D\left[\tilde{g}(\tilde{x}, \tilde{y}, y) + \alpha\tilde{f}^*(y-D)\right]. \tag{2.7}$$

Note that $\tilde{f}^*(\tilde{x})$ is finite for all $\tilde{x} \in \mathcal{I}$ by Lemma 3. Now, $\tilde{g}(\tilde{x}, \tilde{y}, y)$ is piecewise linear in $\tilde{y}$ and $\tilde{g}(\tilde{x}, \tilde{y}, y) \to \infty$ as $|\tilde{y}| \to \infty$. So, the minimum over $\tilde{y}$ occurs at either $\tilde{y} = \tilde{x}$, $\tilde{y} = 0$, or $\tilde{y} = y$. We now consider what happens for different values of $\tilde{x}$. When $\tilde{x} \geq 0$, then, as we show below, we do not run overtime production, the optimal $\tilde{y} = \tilde{x}$, and

$$\tilde{g}(\tilde{x}, \tilde{x}, y) = (h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y - \tilde{x}) + \alpha c_2 y.$$

The intuition behind this result is that it is cheaper to produce next period with regular production than to produce now with overtime production. Overtime production has higher setup and per unit costs plus a holding cost, while the costs next period are discounted by α. Mathematically, (since $g(\tilde{x}, 0, y)$ is inadmissable),

$$\begin{aligned}
\tilde{g}(\tilde{x}, \tilde{x}, y) - \tilde{g}(\tilde{x}, y, y) &= (h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y - \tilde{x}) + \alpha c_2 y \\
&\quad - (K_o \delta(y - \tilde{x}) + (h_2 + c_o) y - c_o \tilde{x}) \\
&\leq (h_2 - \alpha c_2)\tilde{x} + \alpha c_2 y - ((h_2 + c_o) y - c_o \tilde{x}) \\
&= (h_2 + c_o - \alpha c_2)(\tilde{x} - y) \\
&\leq 0
\end{aligned}$$

since $y \geq \tilde{x}$. Thus, for $\tilde{x} \geq 0$, equation (2.7) becomes

$$\begin{aligned}
\tilde{f}^*(\tilde{x}) &= \min_{y \geq \tilde{x}} E_D \left[\tilde{g}(\tilde{x}, \tilde{x}, y) + \alpha \tilde{f}^*(y - D) \right] \\
&= \min_{y \geq \tilde{x}} E_D \left[(h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y - \tilde{x}) + \alpha c_2 y + \alpha \tilde{f}^*(y - D) \right] \\
&= (h_2 - \alpha c_2)\tilde{x} + \min_{y \geq \tilde{x}} \left\{ \alpha K_2 \delta(y - \tilde{x}) + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)] \right\} \\
&= (h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y^*_{\tilde{x}} - \tilde{x}) + \alpha c_2 y^*_{\tilde{x}} + \alpha E_D[\tilde{f}^*(y^*_{\tilde{x}} - D)] \qquad (2.8)
\end{aligned}$$

where

$$y^*_{\tilde{x}} = \arg\min_{y \geq \tilde{x}} \left\{ \alpha K_2 \delta(y) + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y-D)] \right\}.$$

Note that $y^*_{\tilde{x}}$ exists because we consider only integer values of y, the linear term goes to infinity as y goes to infinity, and the expectation term is non-negative.

Now when $\tilde{x} < 0$, we must decide whether to utilize overtime production and/or premium freight. Given the three options above ($\tilde{y} = \tilde{x}$, $\tilde{y} = 0$, or $\tilde{y} = y$), we get the following costs per stage where the first case is the premium freight case, the second case is where we produce up to 0 in overtime, and the third case is where we produce up to a positive quantity in overtime:

$$\tilde{g}(\tilde{x}, \tilde{y}, y) = \begin{cases} \alpha K_p - \alpha(c_p + c_2)\tilde{x} + \alpha K_2 + \alpha c_2 y & \text{if } \tilde{y} = \tilde{x} \\ K_o - c_o \tilde{x} + \alpha K_2 \delta(y) + \alpha c_2 y & \text{if } \tilde{y} = 0 \\ K_o + c_o(y - \tilde{x}) + h_2 y & \text{if } \tilde{y} = y. \end{cases}$$

So, when $\tilde{x} < 0$,

$$\begin{aligned}
\tilde{f}^*(\tilde{x}) &= \min_{y \geq 0, \tilde{y} = \tilde{x}, 0, y} E_D \left\{ \tilde{g}(\tilde{x}, \tilde{y}, y) + \alpha \tilde{f}^*(y - D) \right\} \\
&= \min \left\{ \begin{array}{l} \min_{y \geq 0} E_D \left\{ \tilde{g}(\tilde{x}, \tilde{x}, y) + \alpha \tilde{f}^*(y-D) \right\} \\ \min_{y \geq 0} E_D \left\{ \tilde{g}(\tilde{x}, 0, y) + \alpha \tilde{f}^*(y-D) \right\} \\ \min_{y \geq 0} E_D \left\{ \tilde{g}(\tilde{x}, y, y) + \alpha \tilde{f}^*(y-D) \right\} \end{array} \right\} \\
&= \min \left\{ \begin{array}{l} \alpha K_p - \alpha(c_p + c_2)\tilde{x} + \alpha K_2 + \min_{y \geq 0} \left\{ \alpha c_2 y + \alpha E_D[\tilde{f}^*(y-D)] \right\} \\ K_o - c_o \tilde{x} + \min_{y \geq 0} \left\{ \alpha K_2 \delta(y) + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y-D)] \right\} \\ K_o - c_o \tilde{x} + \min_{y \geq 0} \left\{ (c_o + h_2) y + \alpha E_D[\tilde{f}^*(y-D)] \right\} \end{array} \right\}
\end{aligned}$$

$$= \min \left\{ \begin{array}{l} \alpha K_p - \alpha(c_p + c_2)\tilde{x} + \alpha K_2 + \alpha c_2 y_p^* + \alpha E_D[\tilde{f}^*(y_p^* - D)] \\ \\ K_o - c_o\tilde{x} + \alpha K_2 \delta(y_0^*) + \alpha c_2 y_0^* + \alpha E_D[\tilde{f}^*(y_0^* - D)] \\ \\ K_o - c_o\tilde{x} + (c_o + h_2)\tilde{y}_+^* + \alpha E_D[\tilde{f}^*(\tilde{y}_+^* - D)] \end{array} \right\} \qquad (2.9)$$

where

$$\begin{aligned} y_p^* &= \arg\min_{y \geq 0} \left\{ \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)] \right\}, \\ y_0^* &= \arg\min_{y \geq 0} \left\{ \alpha K_2 \delta(y) + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)] \right\}, \text{ and} \\ \tilde{y}_+^* &= \arg\min_{y \geq 0} \left\{ (c_o + h_2) y + \alpha E_D[\tilde{f}^*(y - D)] \right\}. \end{aligned}$$

As before, y_p^*, y_0^*, and $\tilde{y}_+^*$ exist because we consider only integer values of y, the linear term goes to infinity as y goes to infinity, and the expectation term is non-negative.

So, when the overtime inventory is negative ($\tilde{x} < 0$), we have three choices. The first choice is to produce nothing during overtime ($\tilde{y} = \tilde{x}$), fill the shortage with regular production next period and ship by premium freight, and continue regular production up to $y_p^* \geq 0$. The second choice is to produce up to 0 with overtime production ($\tilde{y} = 0$) and then produce up to $y_0^* \geq 0$ next period during regular production. Note that when $y_0^* > 0$, $y_p^* = y_0^*$; when $y_0^* = 0$, the two values may be different. Finally, the third choice is to produce up to $\tilde{y}_+^* > 0$ during overtime ($\tilde{y} = \tilde{y}_+^*$) and to do no regular production next period ($y = \tilde{y}_+^*$).

Now, for $\tilde{x} < 0$, we can rewrite

$$\tilde{f}^*(\tilde{x}) = \min \left\{ C_{PF}^* - \alpha(c_p + c_2)\tilde{x}, C_0^* - c_o\tilde{x}, C_+^* - c_o\tilde{x} \right\}$$

where

$$C_{PF}^* = \alpha K_p + \alpha K_2 + \alpha c_2 y_p^* + \alpha E_D[\tilde{f}^*(y_p^* - D)],$$

$$\begin{aligned} C_0^* &= K_o + \alpha K_2 \delta(y_0^*) + \alpha c_2 y_0^* + \alpha E_D[\tilde{f}^*(y_0^* - D)], \text{ and} \\ C_+^* &= K_o + (c_o + h_2)\tilde{y}_+^* + \alpha E_D[\tilde{f}^*(\tilde{y}_+^* - D)]. \end{aligned}$$

At this point it should be noted that the three constants above are, in fact, just constants. The y^* terms are numbers found by minimization and the $E_D[\tilde{f}^*(y^* - D)]$ are just numbers found by taking the expectation over D. Also note that for all values of $\tilde{x}$, either overtime up to 0 or overtime up to $\tilde{y}_+^*$ is better depending on the values of C_0^* and C_+^*. Define $C_{OT}^+ = \min\{C_0^*, C_+^*\}$, define

$$\tilde{S} \equiv \begin{cases} 0 & \text{if } C_0^* < C_+^* \\ \tilde{y}_+^* & \text{otherwise} \end{cases}$$

and if $c_o \neq \alpha(c_p + c_2)$, define

$$\tilde{s} \equiv \left\lfloor \frac{C_{OT}^* - C_{PF}^*}{c_o - \alpha(c_p + c_2)} \right\rfloor .$$

Here, we have that for $\tilde{x} < 0$, $\tilde{f}^*(\tilde{x}) = \min\left\{C_{PF}^* - \alpha(c_p + c_2)\tilde{x}, C_{OT}^* - c_o\tilde{x}\right\}$ and we must determine when the premium freight option is better than the overtime option. Premium freight is cheaper than overtime when $C_{PF}^* - \alpha(c_p + c_2)\tilde{x} < C_{OT}^* - c_o\tilde{x}$, or when $(c_o - \alpha(c_p + c_2))\tilde{x} < C_{OT}^* - C_{PF}^*$. This inequality depends on the relative values of C_{OT}^* and C_{PF}^* and the relative values of c_o and $\alpha(c_p + c_2)$. For example, if $C_{OT}^* > C_{PF}^*$ and $c_o < \alpha(c_p + c_2)$, premium freight is better for $\tilde{s} < \tilde{x} < 0$ and overtime is better for $\tilde{x} \leq \tilde{s}$. If $C_{OT}^* > C_{PF}^*$ and $c_o > \alpha(c_p + c_2)$, premium freight is better for all $\tilde{x} < 0$ and overtime is never used. The results are listed in Table 2.1

Now we have four interesting cases to study. Let case 1 be where premium freight is better for $\tilde{s} < \tilde{x} < 0$ and overtime is better for $\tilde{x} \leq \tilde{s}$. Let case 2 be where overtime

	$C^*_{OT} > C^*_{PF}$	$C^*_{OT} = C^*_{PF}$	$C^*_{OT} < C^*_{PF}$
$c_o < \alpha(c_p + c_2)$	PF for $\tilde{s} < \tilde{x} < 0$ OT for $\tilde{x} \leq \tilde{s}$	OT for all $\tilde{x} < 0$	OT for all $\tilde{x} < 0$
$c_o = \alpha(c_p + c_2)$	PF for all $\tilde{x} < 0$	Any	OT for all $\tilde{x} < 0$
$c_o > \alpha(c_p + c_2)$	PF for all $\tilde{x} < 0$	PF for all $\tilde{x} < 0$	OT for $\tilde{s} < \tilde{x} < 0$ PF for $\tilde{x} \leq \tilde{s}$

Table 2.1: Premium Freight Versus Overtime

is always better than premium freight and case 3 where premium freight is always better than overtime when $\tilde{x} < 0$. Finally, let case 4 be where overtime is better for $\tilde{s} < \tilde{x} < 0$ and premium freight is better for $\tilde{x} \leq \tilde{s}$. We discuss case 1 here in detail; the analyses for the other three cases are very similar, and are included in the appendix at the end of this thesis.

Case 1

In this case, we have the following stationary policy μ_1, which happens to be an $(\tilde{s}, \tilde{S})$ policy for overtime production:

$$\mu_1 = \begin{cases} \tilde{x} \geq 0 & \rightarrow \quad \tilde{y} = \tilde{x}, y = y^*_{\tilde{x}} \\ \tilde{s} < \tilde{x} < 0 & \rightarrow \quad \tilde{y} = \tilde{x}, y = y^*_p \\ \tilde{x} < \tilde{s} & \rightarrow \quad \tilde{y} = \tilde{S}, y = \begin{cases} y^*_0 & \text{if } \tilde{S} = 0 \\ \tilde{y}^*_+ & \text{if } \tilde{S} = \tilde{y}^*_+. \end{cases} \end{cases}$$

From equations (2.8) and (2.9), we have also that $\tilde{f}^*(\tilde{x}) =$

$$\begin{cases} (h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y^*_{\tilde{x}} - \tilde{x}) + \alpha c_2 y^*_{\tilde{x}} + \alpha E_D[\tilde{f}^*(y^*_{\tilde{x}} - D)] & \text{if } \tilde{x} \geq 0 \\ \alpha K_p - \alpha(c_p + c_2)\tilde{x} + \alpha K_2 + \alpha c_2 y^*_p + \alpha E_D[\tilde{f}^*(y^*_p - D)] & \text{if } \tilde{s} < \tilde{x} < 0 \\ \begin{cases} K_o - c_o\tilde{x} + \alpha K_2 \delta(y^*_0) + \alpha c_2 y^*_0 + \alpha E_D[\tilde{f}^*(y^*_0 - D)] & \text{if } \tilde{S} = 0 \\ K_o - c_o\tilde{x} + (c_o + h_2)\tilde{y}^*_+ + \alpha E_D[\tilde{f}^*(\tilde{y}^*_+ - D)] & \text{if } \tilde{S} = \tilde{y}^*_+ \end{cases} & \text{if } \tilde{x} \leq \tilde{s}. \end{cases} \tag{2.10}$$

Also, plugging μ_1 into equation (2.6) we get:

$$\tilde{f}^*(\tilde{x}) = \begin{cases} (h_2 - \alpha c_2)\tilde{x} + \alpha f^*(\tilde{x}) & \text{if } \tilde{x} \geq 0 \\ \alpha K_p - \alpha(c_p + c_2)\tilde{x} + \alpha f^*(\tilde{x}) & \text{if } \tilde{s} < \tilde{x} < 0 \\ K_o - c_o\tilde{x} + (c_o + h_2 - \alpha c)\tilde{S} + \alpha f^*(\tilde{S}) & \text{if } \tilde{x} \leq \tilde{s}. \end{cases} \tag{2.11}$$

It turns out that in any case, we have:

Theorem 1 *An optimal, stationary overtime production policy exists and has structure as in Table 2.1.*

Proof: Since $\tilde{g}(\tilde{x}, \tilde{y}, y) \geq 0$, then according to Proposition 1.3 of Bertsekas ([7], page 143), a stationary policy μ is optimal if and only if

$$\min_{\tilde{y} \geq \tilde{x}, y \geq \tilde{y}^+} E_D\left[\tilde{g}(\tilde{x}, \tilde{y}, y) + \alpha \tilde{f}^*(y - D)\right] = E_D\left[\tilde{g}(\tilde{x}, \tilde{y}_\mu, y_\mu) + \alpha \tilde{f}^*(y_\mu - D)\right]$$

where $\tilde{y}_\mu$ and y_μ are overtime inventory position and regular inventory position, respectively, under policy μ. This is exactly what we have just shown for case 1 using policy μ_1 in equation (2.10). The same result holds for cases 2, 3, and 4 (as can be found in the appendix) and the proof is complete. $\square$

2.3.3 Optimal Regular Production Policy

In this section, we will show that the optimal regular production policy at stage 2 is an (s, S) policy. To do so, we will show that for each case from the previous theorem, $f_m^*(x) = \min_{y \geq x^+} \{K_2\delta(y-x) + G(y) + \alpha E_D[f_m^*(y-D)]\}$ has an appropriate $G(y)$ function that fits the model (or a slight modification thereof) from Zheng [66]. Before we begin the proof, we first prove several lemmas. In Lemma 4, we derive logconcavity results required in our proof of quasiconvexity. In Lemma 5, we derive relationships for the regular- and overtime-period optimal cost functions. In Lemma 6, we calculate the total cost of doing overtime production. In Lemma 7, we derive results required since our problem is discrete and in Lemma 8, we show that our problem generally meets the requirements of Zheng. We also modify Lemma 1 from Zheng by eliminating the possibility of backorders.

Lemma 4 *Given $F(x)$ is logconcave, $m \geq 0$ and $n > 0$,*

(i) $\frac{p_{x+n}}{F(x)}$ is a non-increasing function that tends down to 0 as $x \to \infty$ and

(ii) $\frac{F(x+m)}{F(x)}$ is a non-increasing function that tends down to 1 as $x \to \infty$.

Proof: Given $F(x)$ is logconcave, Rosling ([51], page 4) yields that both $\frac{p_{x+1}}{p_x}$ and $\frac{p_{x+1}}{F(x)}$ are non-increasing in x. Note that for any $n > 0$ with $p_{x+n-1} > 0$, $p_{x+n-2} > 0$, $\ldots$, $p_{x+1} > 0$,

$$\frac{p_{x+n}}{F(x)} = \frac{p_{x+n}}{p_{x+n-1}} \frac{p_{x+n-1}}{p_{x+n-2}} \cdots \frac{p_{x+2}}{p_{x+1}} \frac{p_{x+1}}{F(x)}.$$

This is the product of non-negative, non-increasing functions and hence is non-increasing. Also, $p_{x+n} \to 0$ as $n \to \infty$. Note that if $p_{x+n} = 0$, then $p_{x+n+1} = 0$,

$p_{x+n+2} = 0, \ldots$ by logconcavity. Thus, *(i)* is true. To prove *(ii)*, consider that

$$\frac{F(x+m)}{F(x)} = \frac{F(x)}{F(x)} + \frac{p_{x+1}}{F(x)} + \cdots + \frac{p_{x+m-1}}{F(x)} + \frac{p_{x+m}}{F(x)}$$

is the sum of 1 and m non-negative, non-increasing functions that tend down to 0 as $x \to \infty$. Clearly, this is a non-increasing function which tends to 1 as $x \to \infty$. $\square$

Lemma 5 *Define* $f_m^*(-) \equiv f_m^*(x)$ *for any* $x < 0$. *Then,*

$$f_m^*(-) = K_2 + c_2 y_p^* + E_D[\tilde{f}^*(y_p^* - D)] \tag{2.12}$$

$$f_m^*(0) = K_2\delta(y_0^*) + c_2 y_0^* + E_D[\tilde{f}^*(y_0^* - D)] \tag{2.13}$$

$$f_m^*(\tilde{y}_+^*) = c_2\tilde{y}_+^* + E_D[\tilde{f}^*(\tilde{y}_+^* - D)] \tag{2.14}$$

Proof: Recall from equation (2.5) that $f_m^*(x) = \min_{y \geq x^+}\{K_2\delta(y-x) + c_2 y + E_D[\tilde{f}^*(y-D)]\}$. To prove the first equation, note that the definition of $f_m^*(-)$ is consistent as we are minimizing for $y \geq x^+$ and the only x in the minimization occurs in the form of $\delta(y-x)$. Then $f_m^*(-) = \min_{y \geq 0}\{K_2 + c_2 y + E_D[\tilde{f}^*(y-D)]\} = K_2 + c_2 y_p^* + E_D[\tilde{f}^*(y_p^* - D)]$, by definition of y_p^*. To prove the second equation, consider $f_m^*(0) = \min_{y\geq 0}\{K_2\delta(y) + c_2 y + E_D[\tilde{f}^*(y-D)]\} = K_2\delta(y_0^*) + c_2 y_0^* + E_D[\tilde{f}^*(y_0^* - D)]$, by definition of y_0^*. Finally, to prove the last equation, consider that when $\tilde{S} = \tilde{y}_+^*$, then $C_{OT}^* = C_+^* < C_0^*$. So,

$$\begin{aligned} C_+^* &= K_o + (c_o + h_2)\tilde{y}_+^* + \alpha E_D[\tilde{f}^*(\tilde{y}_+^* - D)] \qquad (2.15) \\ &< C_0^* \\ &= K_o + \alpha K_2\delta(y_0^*) + \alpha c_2 y_0^* + \alpha E_D[\tilde{f}^*(y_0^* - D)] \\ &= K_o + \min_{y\geq 0}\{\alpha K_2\delta(y) + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y-D)]\} \end{aligned}$$

$$
\begin{aligned}
&\leq K_o + \min_{y \geq 0}\{\alpha K_2 + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)]\} \\
&\leq K_o + \min_{y \geq \tilde{y}_+^*}\{\alpha K_2 + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)]\}. \qquad (2.16)
\end{aligned}
$$

Hence from the first and last equalities above, $(c_o + h_2)\tilde{y}_+^* + \alpha E_D[\tilde{f}^*(\tilde{y}_+^* - D)] < \min_{y \geq \tilde{y}_+^*}\{\alpha K_2 + \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)]\}$. Since $c_2 < \frac{c_o+h_2}{\alpha}$ by assumptions (A1) and (A4),

$$
c_2\tilde{y}_+^* + E_D[\tilde{f}^*(\tilde{y}_+^* - D)] < \frac{c_o + h_2}{\alpha}\tilde{y}_+^* + E_D[\tilde{f}^*(\tilde{y}_+^* - D)] < \min_{y \geq \tilde{y}_+^*}\{K_2 + c_2 y + E_D[\tilde{f}^*(y - D)]\}. \tag{2.17}
$$

Thus,

$$
\begin{aligned}
f^*(\tilde{y}_+^*) &= \min_{y \geq \tilde{y}_+^*}\left\{K_2\delta(y - x) + c_2 y + E_D[\tilde{f}^*(y - D)]\right\} \\
&= \min\left\{\begin{array}{l} c_2\tilde{y}_+^* + E_D[\tilde{f}^*(\tilde{y}_+^* - D)] \\ \min_{y > \tilde{y}_+^*}\{K_2 + c_2 y + E_D[\tilde{f}^*(y - D)]\} \end{array}\right\} \\
&= c_2\tilde{y}_+^* + E_D[\tilde{f}^*(\tilde{y}_+^* - D)].
\end{aligned}
$$

□

Before we prove the next lemma, note that we can rewrite $C_{PF}^* = \alpha K_p + \alpha f_m^*(-)$, $C_0^* = K_o + \alpha f_m^*(0)$, and $C_+^* = K_o + (c_o + h_2 - \alpha c_2)\tilde{y}_+^* + \alpha f_m^*(\tilde{y}_+^*)$ when $\tilde{S} = \tilde{y}_+^*$.

Lemma 6 *Define* $K_{all} \equiv K_o + (c_o + h_2 - \alpha c_2)\tilde{S} + \alpha f_m^*(\tilde{S}) - \alpha f_m^*(-)$. *Then,*

$$
K_{all} = C_{OT}^* - C_{PF}^* + \alpha K_p \geq 0
$$

Proof: The proof depends on whether $\tilde{S} = 0$ or $\tilde{S} = \tilde{y}_+^*$. If $\tilde{S} = 0$, then $C_{OT}^* = C_0^* \leq C_+^*$. From the definition above, $K_{all} = K_o + \alpha f_m^*(0) - \alpha f_m^*(-) = C_0^* - C_{PF}^* +$

$\alpha K_p = C^*_{OT} - C^*_{PF} + \alpha K_p$, proving the necessary equality. To prove the necessary inequality,

$$
\begin{aligned}
K_{all} &= K_o + \alpha f^*_m(0) - \alpha f^*_m(-) \\
&= K_o + \alpha \left(K_2\delta(y^*_0) + c_2 y^*_0 + E_D[\tilde{f}^*(y^*_0 - D)] \right) - \\
&\quad \alpha \left(K_2 + c_2 y^*_p + E_D[\tilde{f}^*(y^*_p - D)] \right) \\
&\geq \alpha \left(\min_{y \geq 0} \left\{ K_2\delta(y) + c_2 y + E_D[\tilde{f}^*(y - D)] \right\} - \min_{y \geq 0} \left\{ c_2 y + E_D[\tilde{f}^*(y - D)] \right\} \right) \\
&\geq 0
\end{aligned}
$$

where the first inequality is true since $K_o > \alpha K_2$ by assumptions (A1) and (A6) and the second inequality is true since $K_2\delta(y) \geq 0$.

Now, if $\tilde{S} = \tilde{y}^*_+$, then $C^*_{OT} = C^*_+ < C^*_0$. From the definition above, $K_{all} = K_o + (c_o + h_2 - \alpha c_2)\tilde{y}^*_+ + \alpha f^*_m(\tilde{y}^*_+) - \alpha f^*_m(-) = K_o + (c_o + h_2)\tilde{y}^*_+ + \alpha(f^*_m(\tilde{y}^*_+) - c_2\tilde{y}^*_+) - \alpha f^*_m(-) = K_o + (c_o + h_2)\tilde{y}^*_+ + \alpha E_D[\tilde{f}^*(\tilde{y}^*_+ - D)] - \alpha f^*_m(-) = C^*_+ - C^*_{PF} + \alpha K_p = C^*_{OT} - C^*_{PF} + \alpha K_p$ by equation (2.12), proving the necessary equality for the lemma. To prove the necessary inequality,

$$
\begin{aligned}
K_{all} &= K_o + (c_o + h_2 - \alpha c_2)\tilde{y}^*_+ + \alpha f^*_m(\tilde{y}^*_+) - \alpha f^*_m(-) \\
&= K_o + (c_o + h_2 - \alpha c_2)\tilde{y}^*_+ + \alpha \left(c_2\tilde{y}^*_+ + E_D[\tilde{f}^*(\tilde{y}^*_+ - D)] \right) \\
&\quad -\alpha \left(K_2 + c_2 y^*_p + E_D[\tilde{f}^*(y^*_p - D)] \right) \\
&\geq \left((c_o + h_2)\tilde{y}^*_+ + \alpha E_D[\tilde{f}^*(\tilde{y}^*_+ - D)] \right) - \alpha \left(c_2 y^*_p + E_D[\tilde{f}^*(y^*_p - D)] \right) \\
&= \min_{y \geq 0} \left\{ (c_o + h_2)y + \alpha E_D[\tilde{f}^*(y - D)] \right\} - \min_{y \geq 0} \left\{ \alpha c_2 y + \alpha E_D[\tilde{f}^*(y - D)] \right\} \\
&\geq 0
\end{aligned}
$$

where the first inequality follows from $K_o > \alpha K_2$ and the second inequality is true since $c_o > \alpha c_2$. $\square$

Lemma 7 *For cases 1 and 4, there exists β with $0 \leq \beta < 1$ such that*

$$C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s} = \beta(c_o - \alpha(c_p + c_2))$$

Proof: In cases 1 and 4, $\tilde{s}$ is the floor of the following negative fraction so

$$\tilde{s} + 1 > \frac{C^*_{OT} - C^*_{PF}}{c_o - \alpha(c_p + c_2)} \geq \tilde{s}.$$

For case 1, the denominator of the fraction is negative so we get that

$$(c_o - \alpha(c_p + c_2))(\tilde{s} + 1) < C^*_{OT} - C^*_{PF} \leq (c_o - \alpha(c_p + c_2))\tilde{s}$$

which implies that

$$0 \geq C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s} > (c_o - \alpha(c_p + c_2)).$$

Hence, for case 1,

$$C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s} = \beta(c_o - \alpha(c_p + c_2))$$

for some $0 \leq \beta < 1$. For case 4, the denominator of the fraction is positive so we get that

$$(c_o - \alpha(c_p + c_2))(\tilde{s} + 1) > C^*_{OT} - C^*_{PF} \geq (c_o - \alpha(c_p + c_2))\tilde{s}$$

which implies that

$$0 \leq C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s} < (c_o - \alpha(c_p + c_2)).$$

Hence, for case 4,

$$C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s} = \beta(c_o - \alpha(c_p + c_2))$$

for some $0 \leq \beta < 1$. $\square$

We now proceed to prove for case 1 that an (s, S) policy is optimal for regular production; the proofs for cases 2, 3, and 4 are very similar and are in the appendix. From Lemma 3, we know that f^*_m and $\tilde{f}^*$ are finite. Thus, from equation (2.5) and then equation (2.11),

$$\begin{aligned} f^*_m(x) &= \min_{y \geq x^+} \left\{ K_2\delta(y-x) + c_2 y + E_D[\tilde{f}^*(y-D)] \right\} \\ &= \min_{y \geq x^+} \left\{ \begin{array}{l} K_2\delta(y-x) + c_2 y + \\ E_D \left[\begin{array}{l} \left\{ (h_2 - \alpha c_2)(y-D) + \alpha f^*_m(y-D) \right\} 1(y - D \geq 0) + \\ \left\{ \alpha K_p - \alpha(c_p + c_2)(y-D) \right. \\ \left. + \alpha f^*_m(y-D) \right\} 1(\tilde{s} < y - D < 0) + \\ \left\{ K_o - c_o(y-D) + \right. \\ \left. (c_o + h_2 - \alpha c_2)\tilde{S} + \alpha f^*_m(\tilde{S}) \right\} 1(y - D \leq \tilde{s}) \end{array} \right] \end{array} \right\} \end{aligned}$$

where 1 is the indicator function with 1(true) = 1 and 1(false) = 0. We now have that

$$f^*_m(x) = \min_{y \geq x^+} \left\{ \begin{array}{l} K_2\delta(y-x) + c_2 y + \\ E_D \left[\begin{array}{l} \left\{ (h_2 - \alpha c_2)(y-D) + \alpha f^*_m(y-D) \right\} 1(y - D \geq 0) + \\ \left\{ \alpha K_p - \alpha(c_p + c_2)(y-D) \right. \\ \left. + \alpha f^*_m(y-D) \right\} 1(\tilde{s} < y - D < 0) + \\ \left\{ K_o - c_o(y-D) + (c_o + h_2 - \alpha c_2)\tilde{S} \right. \\ \left. + \alpha f^*_m(\tilde{S}) - \alpha f^*_m(-) + \alpha f^*_m(y-D) \right\} 1(y - D \leq \tilde{s}) \end{array} \right] \end{array} \right\}$$

$$
\begin{aligned}
&= \min_{y \geq x^+} \left\{ \begin{array}{l} K_2\delta(y-x) + c_2 y + \alpha E_D[f_m^*(y-D)] + \\ E_D \left[\begin{array}{l} (h_2 - \alpha c_2)(y-D)1(y-D \geq 0) + \\ \Big\{\alpha K_p - \alpha(c_p + c_2)(y-D)\Big\} 1(\tilde{s} < y - D < 0) + \\ \Big\{K_o - c_o(y-D) + (c_o + h_2 - \alpha c_2)\tilde{S} \\ + \alpha f_m^*(\tilde{S}) - \alpha f_m^*(-)\Big\} 1(y - D \leq \tilde{s}) \end{array} \right] \end{array} \right\} \\
&= \min_{y \geq x^+} \left\{ K_2\delta(y-x) + G_1(y) + \alpha E_D[f_m^*(y-D)] \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(y) \equiv\; & c_2 y + \sum_{d=0}^{y} (h_2 - \alpha c_2)(y-d)p_d + \sum_{d=y+1}^{y-\tilde{s}-1} \left(\alpha K_p - \alpha(c_p + c_2)(y-d)\right) p_d \\
& + \sum_{d=y-\tilde{s}}^{\infty} \left(K_{all} - c_o(y-d)\right) p_d.
\end{aligned}
$$

Lemma 8 *Under the conditions of case 1 ($c_o < \alpha(c_p + c_2)$ and $C^*_{OT} > C^*_{PF}$), $-G_1(y)$ is unimodal, $G_1(y) \to \infty$ as $|y| \to \infty$, and the minimum point $y^0 \geq 0$.*

Proof: We may prove this lemma using the results from [49] and [2], but we choose to actually calculate the differential to again show where the logconcavity assumption is used. Define $\Delta G_1(y) \equiv G_1(y+1) - G_1(y)$. This represents the rate of change as y increases by 1.

$$
\begin{aligned}
\Delta G_1(y) =\; & c_2(y+1) - c_2 y + \sum_{d=0}^{y+1} (h_2 - \alpha c_2)(y+1-d)p_d - \sum_{d=0}^{y} (h_2 - \alpha c_2)(y-d)p_d \\
& + \sum_{d=y+2}^{y-\tilde{s}} \left(\alpha K_p - \alpha(c_p + c_2)(y+1-d)\right) p_d - \\
& \sum_{d=y+1}^{y-\tilde{s}-1} \left(\alpha K_p - \alpha(c_p + c_2)(y-d)\right) p_d + \\
& \sum_{d=y-\tilde{s}+1}^{\infty} \left(K_{all} - c_o(y+1-d)\right) p_d - \sum_{d=y-\tilde{s}}^{\infty} \left(K_{all} - c_o(y-d)\right) p_d
\end{aligned}
$$

$$\begin{aligned}
&= c_2 + (h_2 - \alpha c_2)\sum_{d=0}^{y} p_d - (\alpha K_p + \alpha(c_p + c_2))p_{y+1} \\
&\quad +(\alpha K_p - \alpha(c_p + c_2)(\tilde{s}+1))p_{y-\tilde{s}} \\
&\quad -\alpha(c_p + c_2)\sum_{d=y+2}^{y-\tilde{s}-1} p_d - (K_{all} - c_o\tilde{s})p_{y-\tilde{s}} - c_o \sum_{d=y-\tilde{s}+1}^{\infty} p_d \\
&= c_2 + (h_2 - \alpha c_2)F(y) - \alpha(c_p + c_2)(F(y-\tilde{s}) - F(y)) - c_o(1 - F(y-\tilde{s})) \\
&\quad -\alpha K_p p_{y+1} + ((c_o - \alpha(c_p + c_2))\tilde{s} - (K_{all} - \alpha K_p))\, p_{y-\tilde{s}}
\end{aligned}$$

By Lemma 6,

$$\begin{aligned}
\Delta G_1(y) &= c_2 - c_o + (h_2 + \alpha c_p)F(y) + (c_o - \alpha(c_p + c_2))F(y-\tilde{s}) - \\
&\quad (C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s})\, p_{y-\tilde{s}} - \alpha K_p p_{y+1} \\
&= c_2 - c_o + (h_2 + \alpha c_p)F(y) + (c_o - \alpha(c_p + c_2))F(y-\tilde{s}-1) + \\
&\quad (1-\beta)(c_o - \alpha(c_p + c_2))p_{y-\tilde{s}} - \alpha K_p p_{y+1}
\end{aligned}$$

by Lemma 7 where $0 \leq \beta < 1$. At this point, note that as $y \to -\infty$, $\Delta G_1(y) \to c_2 - c_o < 0$ by assumption (A4) and as $y \to \infty$, $\Delta G_1(y) \to c_2 - c_o + (h_2 + \alpha c_p) + (c_o - \alpha(c_p + c_2)) = h_2 + (1-\alpha)c_2 > 0$. Thus, as $y \to -\infty$, $G_1(y) \to \infty$ and as $y \to \infty$, $G_1(y) \to \infty$. Also note that for $y < 0$,

$$\begin{aligned}
\Delta G_1(y) &= c_2 - c_o + (c_o - \alpha(c_p + c_2))F(y-\tilde{s}-1) \\
&\quad +(1-\beta)(c_o - \alpha(c_p + c_2))p_{y-\tilde{s}} - \alpha K_p p_{y+1} \leq c_2 - c_o < 0
\end{aligned}$$

and so any minimum point of $G_1(y)$, y^0, will be non-negative. Finally, it remains to show that $-G_1(y)$ is unimodal. We will do so by showing that $\Delta G_1(y)$ changes sign exactly once. Let d_0 be the smallest demand such that $p_{d_0} > 0$ (d_0 exists by assumption (A3)); note that for $y < d_0$, $\Delta G_1(y) < 0$. For $y \geq d_0$, rewrite $\Delta G_1(y)$ as

$$\Delta G_1(y) \;=\; c_2 - c_o + F(y)F_1(y)$$

where

$$\begin{aligned} F_1(y) \;=\;& (h_2 + \alpha c_p) + (c_o - \alpha(c_p + c_2))\frac{F(y-\tilde{s}-1)}{F(y)} \\ &+(1-\beta)(c_o - \alpha(c_p + c_2))\frac{p_{y-\tilde{s}}}{F(y)} - \alpha K_p \frac{p_{y+1}}{F(y)}. \end{aligned}$$

Observe that $F(y)$ is increasing up to 1 and by Lemma , (i) $\frac{F(y-\tilde{s}-1)}{F(y)}$ is non-increasing down to 1 and (ii) $\frac{p_{y-\tilde{s}}}{F(y)}$ and $\frac{p_{y+1}}{F(y)}$ are nonincreasing down to 0. Also observe that $F_1(y)$ is nondecreasing. The first term is a constant, the second term is product of a negative term (since this is case 1) and a nonincreasing function (by (i)), and the third and fourth terms are products of negative terms and nonincreasing functions (by (ii)). Thus, $F(y)F_1(y)$ is the product of two nondecreasing functions. Since $F(y) > 0$, the product can only change signs at most once. Since $\Delta G_1(y)$ is negative on the left and eventually positive on the right, the product must change signs exactly once. Thus, $-G_1(y)$ is unimodal. □

Zheng [66] shows that an (s, S) policy is optimal for a "standard discrete-time inventory model" where the one period cost function $G_\alpha(y)$ has the properties we have just shown exist for our $G_1(y)$. So, we see that our constrained optimal cost function$f^*_m(x) = \min_{y \geq x^+} \{K_2\delta(y-x) + G_1(y) + \alpha E_D[f^*_m(y-D)]\}$ matches the unconstrained optimal cost function f_α defined by Zheng on page 807 except that we are minimizing over $y \geq x^+$, rather than $y \geq x$. Zheng shows that an (s^*, S^*) policy is optimal for his unconstrained problem. So, consider our problem with the relaxed restriction that $y \geq x$ rather than $y \geq x^+$. According to Zheng, an (s^*, S^*) policy is optimal for this problem. Now, if $s^* \geq -1$, it is always better to order during regular time when inventory is negative and there will be no backorders. In this case, the (s^*, S^*) policy defined by Zheng is also optimal for our constrained problem and we have shown that the optimal regular time policy is an (s, S) policy with $-1 \leq s < S$.

Now, if $s^* < -1$, this policy is not admissible for our problem. So, reconsider Zheng's Lemma 1 ([66], page 806) with the restriction that the reorder point must be at least -1. Call this Lemma 1^+.

Lemma 1^+. *For given $\alpha (0 < \alpha \leq 1)$, let y^0 be a minimum point of $G_1(y)$. There exist s^+ and S^+ that satisfy*

(I) $c_\alpha^+ \equiv c_\alpha(s^+, S^+) = \min_{-1 \leq s < S} c_\alpha(s, S)$;

(II) $s^+ < y^0 \leq S^+$;

(III) $G_1(s^+) < c_\alpha^+$

(IV) $c_\alpha^+ \leq G_1(S^+)$.

Proof Part (I) is true by the same argument as Zheng for part (i) with an extra bound. Part (II) is true by the same argument as Zheng for part (ii), noting that $y^0 \geq 0$ by Lemma 8 so that s^+ is always feasible. Part (IV) is true by the same argument for Zheng's part (iv). Finally, since $s^* < -1$, (iii) from Zheng yields that $G_1(s^*) \geq c_\alpha^* > G_1(s^* + 1) \geq G_1(s^+)$ and clearly $c_\alpha^+ \geq c_\alpha^*$. Combining these two inequalities gives the inequality of part (III). □

Theorem 2 *The optimal regular production policy at stage 2 is an (s, S) policy with*

$-1 \le s < S$.

Proof: If $s^* \ge -1$, Zheng's result holds and we are done. If $s^* < -1$, we must show that $s^+ = -1$. Using (11) from Zheng (dropping the subscript α's for notational convenience):

$$c(-1, S^+) = \frac{K_2 + \sum_{j=0}^{S^+} G_1(S^+ - j)m(j)}{\sum_{j=0}^{S^+} m(j)}$$

$$\Rightarrow \qquad c(-1, S^+) = \frac{K_2 + \sum_{j=0}^{S^+-s^+-1} G_1(S^+-j)m(j) + \sum_{j=S^+-s^+}^{S^+} G_1(S^+-j)m(j)}{\sum_{j=0}^{S^+-s^+-1} m(j) + \sum_{j=S^+-s^+}^{S^+} m(j)}$$

$$\Rightarrow \quad c(-1, S^+)\left(\sum_{j=0}^{S^+-s^+-1} m(j) + \sum_{j=S^+-s^+}^{S^+} m(j)\right) - \sum_{j=S^+-s^+}^{S^+} G_1(S^+ - j)m(j)$$

$$= K_2 + \sum_{j=0}^{S^+-s^+-1} G_1(S^+ - j)m(j)$$

$$\Rightarrow \qquad c(-1, S^+)\sum_{j=0}^{S^+-s^+-1} m(j) + \sum_{j=S^+-s^+}^{S^+} m(j)\left(c(-1, S^+) - G_1(S^+ - j)\right)$$

$$= K_2 + \sum_{j=0}^{S^+-s^+-1} G_1(S^+ - j)m(j).$$

At this point, notice that the $G_1(S^+ - j)$ in the second sum ranges from $G(0)$ to $G_1(s^+)$. By (III) and the unimodality of $G_1(y)$, each of these terms is less than c^+. Thus,

$$c(-1, S^+)\sum_{j=0}^{S^+-s^+-1} m(j) + \sum_{j=S^+-s^+}^{S^+} m(j)\left(c(-1, S^+) - c^+\right) \le K_2 + \sum_{j=0}^{S^+-s^+-1} G_1(S^+ - j)m(j)$$

$$\Rightarrow c(-1, S^+)\sum_{j=0}^{S^+-s^+-1} m(j) \le K_2 + \sum_{j=0}^{S^+-s^+-1} G_1(S^+ - j)m(j)$$

$$\Rightarrow c(-1, S^+) \le \frac{K_2 + \sum_{j=0}^{S^+-s^+-1} G_1(S^+ - j)m(j)}{\sum_{j=0}^{S^+-s^+-1} m(j)} = c^+.$$

The second inequality follows from $c(-1, S^+) \ge c^+$ and the third inequality follows from the definition of c^+. Since $c(-1, S^+) \le c^+$, we can choose to let $s^+ = -1$. Now we must prove that a $(-1, S^+)$ policy is optimal. Following the proof from Zheng in Section 3, we see that (13) on page 807 may not be true. However, it doesn't matter because for all $i \leq -1$, we must order to avoid backorders. So, we order for every inventory level less than or equal to s^+. The proof of Zheng's (14) is exactly the same with a superscript $^+$ replacing every *. Hence, if $s^* < -1$, an (s, S) policy is optimal for our constrained problem with $-1 = s < S$.

Thus, the optimal regular production policy at stage 2 is an (s, S) policy with $-1 \leq s < S$. □

Given the results of Theorems 1 and 2, we can now relate several of our parameters to each other. We have an (s, S) policy with $s \geq -1$. If $s \geq 0$, then it is optimal to order regular production up to S when our inventory level is 0. In this case, $y_p^* = y_0^* = S$; the first equality holds by the definitions y_p^* and y_0^* and the second equality holds by the definition of an (s, S) policy. If $s = -1$, then $y_p^* = S \geq y_0^* = 0$. Another relationship between our parameters is that $\tilde{S} \leq S$. We know that $\tilde{S} = 0$ or $\tilde{S} = \tilde{y}_+^* > 0$. The first case is trivial since $S \geq 0$. In the second case, compare the definitions of y_p^* and $\tilde{y}_+^*$ in Section 2.3.2. Since $\alpha c_2 < c_o + h_2$, we have that $y_p^* \geq \tilde{y}_+^*$. Hence, $\tilde{S} = \tilde{y}_+^* \leq y_p^* = S$.

2.4 Conclusion and Insights

In this chapter, we have considered our two-stage supply chain under decentralized control. Each stage of the supply chain behaves independently and determines its optimal inventory control policy based only on its own costs and inventory available. In Section 2.1, we showed by a straightforward argument that a base-stock policy is optimal at stage 1. In Section 2.2, we showed that a base-stock policy is optimal at stage 2 given that only one method of expediting is available. In Section 2.3, we considered stage 2 under the more interesting scenario that both methods of expediting are available and there exists a setup cost for regular production. Under these conditions, we showed that two production decisions must be made: how much to produce during regular production and how much to produce during overtime production. We first showed that these two decisions are related and then proved that the optimal overtime decision depends on the various costs involved. Lastly, we proved that whatever the optimal overtime policy is, an (s, S) policy is optimal for regular production.

We believe that our theoretical results yield important managerial insights for firms that operate in a decentralized fashion. First, our results from the first section show that stage 1 clearly ignores the inventory situation at stage 2. Despite the fact that a large order from stage 1 may force stage 2 to incur possibly high setup costs for expediting, stage 1 will request large orders whenever the demand is large, due to the base-stock policy. In the next chapter, we will show that when the two stages share these setup costs, stage 1 is not so quick to make large orders. Our results in the second section show that stage 2 basically reacts to the same demand distribution as stage 1. Because stage 1 does not temper the demand process, stage 2 must account for the cost of overtime production with additional inventory.

Our results from the third section yield several managerial insights. First, although it may be obvious, the production decision during overtime directly affects the regular production decision the next day. Clearly, if there is no shortage during overtime, overtime production is not appropriate since it costs more. If there is a shortage, the manager must decide how to fill it. We show that it is never optimal to use both methods of expediting, since such a decision would incur both setup costs. In fact, there exists a threshold such that for some shortages, overtime production is optimal, and for other shortages, premium freight is the way to go. This threshold depends on how much the two methods of expediting cost, and it may turn out that one method is always better. By comparing relative costs, a manager should be able to get a sense of whether one option is always better, or whether the shortage will determine which option is cost-effective.

Another interesting result is that in some situations when a shortage is faced, it is optimal to run overtime production not only to fill the shortage, but to bring the inventory level up to a positive quantity and to avoid regular production the next day. This result may seem counter-intuitive at first since we assume that overtime production costs are greater than regular production costs. However, assume that the per unit costs are relatively close and a manager faces a shortage; if the manager must run overtime production to at least fill the shortage, the overtime setup cost becomes a sunk cost and it makes sense to continue production beyond filling the shortage to avoid paying the setup cost for regular production the next day. Whether or not this policy is practical considering labor relations is not clear, but we feel that managers should be aware that this policy may be optimal if the per unit cost of overtime production is not too expensive.

Lastly, the best policy for regular production is what most inventory managers would expect for a production problem with a setup cost. An (s, S) policy for regular production is optimal for all four of the possible overtime production policies. One final observation for managers is that the overtime production level will never be greater than the regular production level; that is, $\tilde{S} \leq S$. This result makes intuitive sense given the relative costs of the two types of production.

CHAPTER III

CENTRALIZED MODEL

In this chapter, we study the centralized model. We assume that the two stages of the supply chain are part of a single firm and are controlled by a manager whose goal is to minimize system costs. This manager knows the inventory levels at both stages and makes three different decisions concerning production at stage 1, regular production at stage 2, and overtime production at stage 2. In the first four sections, we assume as before that overtime production is the only method of expediting. In the first section, we develop the relaxed version of our problem. In Section 3.2, we derive the optimal inventory control policy for stage 1 under the relaxed conditions; we show that stage 1 will occasionally "underorder" to avoid expediting at stage 2, but will force expediting at stage 2 if the system inventory is very low. In Section 3.3, again under the relaxed conditions, we show that the optimal inventory control policy for the entire system is base-stock. We prove that the optimal relaxed policies are actually the optimal policies for the original, fully-constrained problem in Section 3.4 and we list the optimal policies for both stages and the system. In Section 3.5, we include premium freight as an expediting option and in a more complicated proof, show that similar results hold. Finally, in Section 3.6, we conclude the chapter and discuss managerial insights for the centralized model.

To prove the results of this chapter, we require two further assumptions about the holding costs at both stages and the backordering cost at stage 1. Basically, we assume that the holding cost at stage 2 is less than the holding cost at stage 1 and we strengthen (A5) that the backordering cost at stage 1 is not too small. These assumptions ensure that the stage 1 policy is "reasonable" (discussed more formally later in the chapter) and are fairly standard. We also require that the initial inventory at stage 1 is less than or equal to the optimal base-stock level at stage 1 under decentralized control. We feel that this is a reasonable assumption if we are considering switching from a decentralized system to a centralized system. We list these assumptions as

(A8) $b_1 \geq (1-\alpha)c_1 + (\frac{c_o}{\alpha} - c_2)$

(A9) $h_2 \leq \alpha(h_1 + (1-\alpha)c_1)$

(A10) $x_1 \leq S^*_{1,dec}$

3.1 Relaxed Problem

The timing of the centralized problem is somewhat different than that of the previous chapter. In this problem, the centralized manager makes three decisions at once about production at the the two stages. The manager first decides the production level for stage 1 for the next period; this decision directly affects the overtime decision for the current period at stage 2, and hence affects the regular production decision for the next period at stage 2 as well. Referring to the time line in Table 1.1, all decisions occur where the stage 1 decision occurs on the time line, after stage 1 experiences demand. Assume we are in the middle of time period t. The centralized manager currently knows the inventory position for next period at stage 1, $x_{1,t+1}$, and the current stage 2 inventory level, $y_{2,t}$. The manager must now decide $y_{1,t+1}$, $\tilde{y}_{2,t}$, and $y_{2,t+1}$. Note that for the first four sections of this chapter, we assume overtime production is the only expediting option. Under these conditions, the overtime decision is straightforward. If stage 1 orders more than stage 2 has on hand, stage 2 must fill the shortage with overtime production. In other words, if $z_{1,t+1} = y_{1,t+1} - x_{1,t+1} > y_{2,t}$, stage 2 must produce $(y_{1,t+1} - x_{1,t+1} - y_{2,t})^+$ parts during overtime.

The decision period we will consider for the centralized model occurs over two time periods. The overtime production variables and costs and the stage 2 holding costs occur during one time period, and the rest of the variables and costs occur during the next time period. For example, during time period t, we determine the overtime production, $(y_{1,t+1} - x_{1,t+1} - y_{2,t})^+$, and potentially pay overtime costs K_o and c_o or holding costs h_2. During time period $t+1$, we use the variables $y_{1,t+1}$ and $y_{2,t+1}$, demand D_{t+1} occurs and all costs incurred are discounted by α. At this point, we will drop the time subscripts for notational convenience, keeping in mind that the variables occur in different time periods. This creates one problem in that we have two terms for the inventory level at stage 2, which were originally $y_{2,t}$ and $y_{2,t+1}$. To alleviate this problem, we define

$\bar{x}_{2,t} \equiv y_{2,t} =$ the stage 2 inventory level after regular production during period t

and after dropping the time subscript, $\bar{x}_2$ represents the stage 2 inventory level when the centralized manager makes inventory decisions. So, the one period costs experienced by the entire system are

$$\begin{aligned} &g_{cen}(x_1, y_1, \bar{x}_2, y_2, D) \equiv \\ &\quad K_o\delta((y_1 - x_1) - \bar{x}_2) + c_o((y_1 - x_1) - \bar{x}_2)^+ + h_2(\bar{x}_2 - (y_1 - x_1))^+ + \\ &\quad \alpha\left(c_1(y_1 - x_1) + c_2(y_2 - (\bar{x}_2 - (y_1 - x_1))^+) + h_1(y_1 - D)^+ + b_1(y_1 - D)^-\right) \end{aligned}$$

with $y_1 \geq x_1$ and $y_2 \geq (\bar{x}_2 - (y_1 - x_1))^+$. The first two terms are overtime production costs, the third term is the holding cost at stage 2, the fourth term is the production cost at stage 1, the fifth term is the production cost at stage 2, and the last two terms are holding and backordering costs at stage 1. Note that we assume there is no fixed cost for regular production at either stage for the remainder of the thesis. Clearly, $g_{cen}(\cdot) \geq 0$ and hence by [7], the optimal cost function $f^*_{cen}(x_1, \bar{x}_2)$ satisfies

$$f^*_{cen}(x_1, \bar{x}_2) = \min_{\substack{y_1 \geq x_1 \\ y_2 \geq (\bar{x}_2 - (y_1 - x_1))^+}} E_D[g_{cen}(x_1, y_1, \bar{x}_2, y_2, D) + \alpha f^*_{cen}(y_1 - D, y_2)].$$

The argument that minimizes this equation is the optimal inventory control policy which we seek. Moving the $-\alpha c_1 x_1$ back to the previous period as $-\alpha^2 c_1(y_1 - D)$ (as in the previous chapter) and rearranging terms, we get:

$$\begin{aligned} &g_{cen,m}(x_1, y_1, \bar{x}_2, y_2, D) \equiv \\ &\quad \alpha(1-\alpha)c_1 y_1 + \alpha^2 c_1 D + K_o\delta((y_1 - x_1) - \bar{x}_2) + c_o((y_1 - x_1) - \bar{x}_2)^+ + \\ &\quad (h_2 - \alpha c_2)(\bar{x}_2 - (y_1 - x_1))^+ + \alpha c_2 y_2 + \alpha(h_1(y_1 - D)^+ + b_1(y_1 - D)^-) \end{aligned}$$

under the same restrictions. Note that $f^*_{cen}(x_1, \bar{x}_2) = -\alpha c_1 x_1 + f^*_{cen,m}(x_1, \bar{x}_2)$. We originally tried to solve this problem in terms of stage 1 and stage 2 variables, but found that the solution lent itself more easily to stage 1 and **system** variables. Define the system inventory position as $x_s \equiv x_1 + \bar{x}_2$ and the system inventory level as $y_s \equiv y_1 + y_2$. Note that we may occasionally interchange the inventory information $(x_1, \bar{x}_2)$ and (x_1, x_s). We substitute these variables and rewrite $g_{cen,m}(\cdot)$ as

$$
\begin{aligned}
& g_{cen,m}(x_1, x_s, y_1, y_s, D) \\
&= \alpha(1-\alpha)c_1 y_1 + \alpha^2 c_1 D + K_o \delta(y_1 - x_s) + c_o(y_1 - x_s)^+ + \\
&\quad (h_2 - \alpha c_2)(x_s - y_1)^+ + \alpha c_2(y_s - y_1) + \alpha(h_1(y_1 - D)^+ + b_1(y_1 - D)^-) \\
&= \alpha(1-\alpha)c_1 y_1 + \alpha^2 c_1 D + \alpha(h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + \\
&\quad K_o \delta(y_1 - x_s) + c_o(y_1 - x_s)^+ + (h_2 - \alpha c_2)(x_s - y_1)^+ + \\
&\quad \alpha c_2(y_s - y_1) + \alpha c_2(x_s - y_1) - \alpha c_2(x_s - y_1) \\
&= \alpha(1-\alpha)c_1 y_1 + \alpha^2 c_1 D + \alpha(h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + && (3.1) \\
&\quad K_o \delta(y_1 - x_s) + (c_o - \alpha c_2)(y_1 - x_s)^+ + h_2(x_s - y_1)^+ + && (3.2) \\
&\quad \alpha c_2(y_s - x_s)
\end{aligned}
$$

with $y_1 \geq x_1$ and $y_s \geq y_1 + (x_s - y_1)^+$. Note that the second restriction is equivalent to $y_s \geq \max\{y_1, x_s\}$. Also, we can rewrite $g_{cen,m}(\cdot)$ as

$$g_{cen,m}(x_1, y_1, x_s, y_s, D) = L_1(y_1, D) + L_2(y_1, x_s) + \alpha c_2(y_s - x_s)$$

where $L_1(y_1, D)$ represents the terms on line (3.1) and $L_2(y_1, x_s)$ represents the terms on line (3.2). Note that $L_1(y_1, D) = \alpha g_{1,dec,m}(\cdot, y_1, D)$ from Section 2.1. We can now rewrite the fully constrained optimal cost function which we would like to solve, namely,

$$f^*_{cen,m}(x_1, x_s) = \min_{\substack{y_1 \geq x_1 \\ y_s \geq \max\{x_s, y_1\}}} E_D \left[g_{cen,m}(x_1, y_1, x_s, y_s, D) + \alpha f^*_{cen,m}(y_1 - D, y_s - D) \right].$$

Again, we originally tried to solve this problem, but could not separate the variables as in typical two-stage problems (e.g. see [21]). So, in order to solve this equation, we relax some of the constraints; later we show that these constraints are always met by the optimal solution to the relaxed problem, and thus solve the original, fully constrained problem. First, we drop the constraint that $y_1 \geq x_1$. Second, we drop the constraint that $y_s \geq y_1$ in the case when $y_1 > x_s$. For later reference, we label the relaxed assumptions:

(R1) $y_1 \geq x_1$ and

(R2) $y_s \geq y_1$ when $y_1 > x_s$.

After relaxing the constraints, our relaxed cost per period has the same costs as $g_{cen,m}(\cdot)$ but with only one constraint;

$$g_{cen,r}(y_1, x_s, y_s, D) \equiv L_1(y_1, D) + L_2(y_1, x_s) + \alpha c_2(y_s - x_s)$$

with $y_s \geq x_s$. Now, we show that $g_{cen,r}(\cdot) \geq 0$ and then apply the result from [7] to obtain our relaxed optimal cost function.

Lemma 9 $g_{cen,r}(y_1, x_s, y_s, D) \geq 0$

Proof: Every term of $g_{cen,r}(y_1, x_s, y_s, D)$ is non-negative, except possibly the first term if $y_1 < 0$ (which may occur since backorders are allowed at stage 1). When $y_1 < 0$,

$$g_{cen,r}(y_1, x_s, y_s, D) \quad \geq \quad \alpha(1-\alpha)c_1 y_1 - \alpha b_1 y_1 + \alpha b_1 E_D[D]$$

$$= \quad \alpha((1-\alpha)c_1 - b_1)y_1 + \alpha b_1 \mu \geq 0$$

The first inequality holds because we drop non-negative terms from $g_{cen,r}(y_1, x_s, y_s, D)$ and the second inequality holds by assumptions (A3) and (A8). $\square$

Because $g_{cen,r}(y_1, x_s, y_s, D) \geq 0$, Proposition 1.1 of Bertsekas [7] holds and the relaxed optimal cost function $f^*_{cen,r}$ satisfies

$$\begin{aligned}
& f^*_{cen,r}(x_s) \\
& = \min_{y_1, y_s \geq x_s} E_D\left[g_{cen,r}(y_1, x_s, y_s, D) + \alpha f^*_{cen,r}(y_s - D)\right] \qquad (3.3) \\
& = \min_{y_1, y_s \geq x_s} \left\{E_D[L_1(y_1, D)] + L_2(y_1, x_s) + \alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)].\right\}
\end{aligned}$$

Here it is important to notice that under the relaxed conditions, y_1 has no effect on either y_s or the costs to go, $\alpha E_D[f^*_{cen,r}(y_s - D)]$. Thus,

$$\begin{aligned}
& f^*_{cen,r}(x_s) \\
& = \min_{y_s \geq x_s} \left\{\min_{y_1} \left\{E_D[L_1(y_1, D)] + L_2(y_1, x_s)\right\} + \alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)]\right\} \\
& = \min_{y_s \geq x_s} \left\{m(x_s) + \alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)]\right\} \qquad (3.4)
\end{aligned}$$

where $m(x_s) \equiv \min_{y_1} \left\{E_D[L_1(y_1, D)] + L_2(y_1, x_s)\right\}$. Finding the optimal inventory policy for stage 1 has become a myopic problem which we solve in the next section. For later reference, we also define the related function $m'(x_s) \equiv m(x_s) - \alpha^2 c_1 \mu$, where $m'(x_s)$ represents all of the system costs that are not due to production. For convenience we study $m(x_s)$ below, but list our final results in terms of $m'(x_s)$.

3.2 Stage 1 Relaxed Optimal Policies

In this section, we determine the optimal inventory policy for stage 1 for the relaxed problem. We study the function $m(x_s)$ and show that the stage 1 policy depends only on the system inventory level x_s. Define $N_H(y_1)$ and $N_L(y_1)$ as

$$N_H(y_1) \equiv (\alpha(1-\alpha)c_1 - h_2)y_1 + \alpha E_D[\alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-]$$

and

$$N_L(y_1) \equiv (\alpha(1-\alpha)c_1 + c_o - \alpha c_2)y_1 + \alpha E_D[\alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-].$$

The function $N_H(y_1)$ corresponds to the stage 1 costs when $y_1 \leq x_s$ and the function $N_L(y_1)$ corresponds to the stage 1 costs when $y_1 > x_s$. We now have that

$$\begin{aligned}
m(x_s) &= \min_{y_1} \{E_D[L_1(y_1, D)] + L_2(y_1, x_s)\} \\
&= \min_{y_1} \begin{cases} E_D[L_1(y_1, D)] + h_2(x_s - y_1) & \text{if } y_1 \leq x_s \\ E_D[L_1(y_1, D)] + K_o + (c_o - \alpha c_2)(y_1 - x_s) & \text{if } y_1 > x_s \end{cases} \\
&= \min_{y_1} \begin{cases} \left\{ \begin{array}{l} h_2 x_s + (\alpha(1-\alpha)c_1 - h_2)y_1 + \\ \alpha E_D[\alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-] \end{array} \right\} & \text{if } y_1 \leq x_s \\ \left\{ \begin{array}{l} K_o - (c_o - \alpha c_2)x_s + (\alpha(1-\alpha)c_1 + c_o - \alpha c_2)y_1 \\ +\alpha E_D[\alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-] \end{array} \right\} & \text{if } y_1 > x_s \end{cases} \\
&= \min_{y_1} \begin{cases} h_2 x_s + N_H(y_1) & \text{if } y_1 \leq x_s \\ K_o - (c_o - \alpha c_2)x_s + N_L(y_1) & \text{if } y_1 > x_s \end{cases} \\
&= \min \begin{cases} h_2 x_s + \min_{y_1 \leq x_s}\{N_H(y_1)\} \\ K_o - (c_o - \alpha c_2)x_s + \min_{y_1 > x_s}\{N_L(y_1)\} \end{cases}
\end{aligned}$$

Before continuing our study of $m(x_s)$, we derive properties for $N_L(y_1)$ and $N_H(y_1)$ in the following lemma.

Lemma 10 *Define* $y_H = \arg\min_{y_1}\{N_H(y_1)\}$ *and* $y_L = \arg\min_{y_1}\{N_L(y_1)\}$.

(1) $N_L(y_1)$ *and* $N_H(y_1)$ *are convex in* y_1,

(2) $0 \leq y_L \leq y_H \leq \infty$, *and*

(3) $y_H \geq S^*_{1,dec}$.

Proof: The proof of (1) is straightforward. To prove (2), note that the middle inequality is satisfied because $-h_2 < 0 < c_o - \alpha c_2$. To prove the other inequalities, we define the differential of each function as $\Delta N_i(y_1) = N_i(y_1+1) - N_i(y_1)$ for $i = L, H$. To calculate y_L, we must solve $\Delta N_L(y_1) = 0$. If the solution to this equation is not integer, y_L will be either the ceiling or the floor of the solution to this equation. Consider

$$
\begin{aligned}
&\Delta N_L(y_1) = N_L(y_1+1) - N_L(y_1) \\
&\quad = \quad (\alpha(1-\alpha)c_1 + c_o - \alpha c_2)(y_1+1) + \\
&\qquad\quad \alpha E_D[\alpha c_1 D + h_1(y_1+1-D)^+ + b_1(y_1+1-D)^-] - \\
&\qquad\quad (\alpha(1-\alpha)c_1 + c_o - \alpha c_2)y_1 - \alpha E_D[\alpha c_1 D + h_1(y_1-D)^+ + b_1(y_1-D)^-] \\
&\quad = \quad (\alpha(1-\alpha)c_1 + c_o - \alpha c_2) + \\
&\qquad\quad \alpha E_D[h_1(y_1+1-D)^+ - h_1(y_1-D)^+ + b_1(y_1+1-D)^- - b_1(y_1-D)^-] \\
&\quad = \quad (\alpha(1-\alpha)c_1 + c_o - \alpha c_2) + \alpha h_1 F(y_1) - \alpha b_1(1 - F(y_1)) \\
&\quad = \quad \alpha(1-\alpha)c_1 + c_o - \alpha c_2 - \alpha b_1 + \alpha(h_1+b_1)F(y_1)
\end{aligned}
$$

Similarly,

$$\Delta N_H(y_1) = \alpha(1-\alpha)c_1 - h_2 - \alpha b_1 + \alpha(h_1+b_1)F(y_1).$$

Thus, at each respective minimum, $\Delta N_L(y_L) = \alpha(1-\alpha)c_1 + c_o - \alpha c_2 - \alpha b_1 + \alpha(h_1+b_1)F(y_L) \approx 0$ and $\Delta N_H(y_H) = \alpha(1-\alpha)c_1 - h_2 - \alpha b_1 + \alpha(h_1+b_1)F(y_H) \approx 0$. Or, $y_L \approx F^{-1}(\frac{\alpha b_1 - (\alpha(1-\alpha)c_1 + c_o - \alpha c_2)}{\alpha(h_1+b_1)})$ and $y_H \approx F^{-1}(\frac{\alpha b_1 + h_2 - \alpha(1-\alpha)c_1}{\alpha(h_1+b_1)})$. For y_L and y_H to exist, we require that the first fraction is non-negative and that the second fraction is less than or equal to one. So, we require that $\alpha b_1 - (\alpha(1-\alpha)c_1 + c_o - \alpha c_2) \geq 0$ and $\alpha b_1 + h_2 - \alpha(1-\alpha)c_1 \leq \alpha(h_1+b_1)$ which both hold by assumptions (A8) and (A9). Under these conditions, we have that $0 \leq y_L \leq y_H \leq \infty$.

Finally, to prove (3), consider that $y_H = \arg\min_{y_1}\{N_H(y_1)\}$ and $S^*_{1,dec} = \arg\min_{y_1}\{G_{1,dec,m}(y_1)\}$. There are two differences between $N_H(y_1)$ and $G_{1,dec,m}(y_1)$. The first difference is that most of the costs in $N_H(y_1)$ are multiplied by an additional factor of α. This is due to the decision timing in the centralized model, but does not affect the value of y_H. Dividing $N_H(y_1)$ through by α, we see that the second, important, difference is the coefficient in front of y_1, respectively $(1-\alpha)c_1 - \frac{h_2}{\alpha}$ and $(1-\alpha)c_1$. Since $(1-\alpha)c_1 - \frac{h_2}{\alpha} \leq (1-\alpha)c_1$, $y_H \geq S^*_{1,dec}$. $\square$

Returning to our study of $m(x_s)$ and defining $N(x_s) \equiv E_D[L_1(x_s, D)]$, we have that

$$m(x_s) = \min \begin{cases} h_2 x_s + \min_{y_1 \leq x_s}\{N_H(y_1)\} \\ K_o - (c_o - \alpha c_2)x_s + \min_{y_1 > x_s}\{N_L(y_1)\} \end{cases}$$

$$= \min \begin{cases} h_2 x_s + N_H(y_H) & \text{if } x_s \geq y_H \\ h_2 x_s + N_H(x_s) & \text{if } x_s < y_H \\ K_o - (c_o - \alpha c_2)x_s + N_L(x_s) & \text{if } x_s > y_L \\ K_o - (c_o - \alpha c_2)x_s + N_L(y_L) & \text{if } x_s \leq y_L \end{cases}$$

$$= \min \begin{cases} h_2 x_s + N_H(y_H) & \text{if } x_s \geq y_H \\ N(x_s) & \text{if } x_s < y_H \\ K_o + N(x_s) & \text{if } x_s > y_L \\ K_o - (c_o - \alpha c_2)x_s + N_L(y_L) & \text{if } x_s \leq y_L \end{cases}$$

$$= \min \begin{cases} h_2 x_s + N_H(y_H) & \text{if } x_s \geq y_H \\ N(x_s) & \text{if } x_s < y_H \\ K_o - (c_o - \alpha c_2)x_s + N_L(y_L) & \text{if } x_s \leq y_L \end{cases}$$

Define t_L as the smallest w such that $N(w) \leq K_o - (c_o - \alpha c_2)w + N_L(y_L)$. We get that

$$m(x_s) = \begin{cases} h_2 x_s + N_H(y_H) & \text{if } x_s \geq y_H \\ N(x_s) & \text{if } t_L \leq x_s < y_H \\ K_o - (c_o - \alpha c_2)x_s + N_L(y_L) & \text{if } x_s < t_L. \end{cases} \tag{3.5}$$

So, we have defined $m(x_s)$ explicitly and in the process we have determined the relaxed optimal inventory control policy at stage 1. If the system inventory is large, $x_s \geq y_H$, stage 1 orders up to y_H. If the system inventory is medium, $t_L \leq x_s < y_H$, stage 1 uses up the system inventory, x_s. Finally, if system inventory is small, $x_s < t_L$, stage 1 orders up to y_L.

Theorem 3 *Let $y^*_{cen,1}$ be the optimal inventory position at stage 1 for the relaxed* *problem. Then*

$$y^*_{cen,1} = \begin{cases} y_H & \text{if } x_s \geq y_H \\ x_s & \text{if } t_L \leq x_s < y_H \\ y_L & \text{if } x_s < t_L. \end{cases} \tag{3.6}$$

Proof: By definition of $m(x_s)$. □

Note that what has happened here, compared to the results of the previous chapter, is that stage 1 has become sensitive to the inventory available in the system (and hence to the inventory available at stage 2, as $x_s = x_1 + \bar{x}_2$). If there is plenty of inventory available in the system, stage 1 orders up to a high quantity. If there is a moderate amount of inventory available in the system, stage 1 basically "underorders" in order to avoid forcing overtime production at stage 2. However if the inventory available in the system is too low, stage 1 forces overtime production at stage 2 and orders up to specified quantity.

3.3 System Relaxed Optimal Policy

In this section, we determine the relaxed optimal policy for the system. Given $m(x_s)$, we now have the optimal relaxed cost function in terms of system variables only. From equation (3.4), we have

$$f^*_{cen,r}(x_s) = \min_{y_s \geq x_s} \left\{ m(x_s) + \alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)] \right\}.$$

We move the $m(x_s)$ and $-\alpha c_2 x_s$ terms back to the previous period as $\alpha m(y_s - D)$ and $-\alpha^2 c_2(y_s - D)$, respectively, and get

$f^*_{cen,s}(x_s)$

$$= \min_{y_s \geq x_s} \left\{ \alpha(1-\alpha)c_2 y_s + \alpha E_D[m(y_s - D)] + \alpha^2 c_2 \mu + \alpha E_D[f^*_{cen,s}(y_s - D)] \right\} \quad (3.7)$$

$$= \min_{y_s \geq x_s} \left\{ G_{cen,s}(y_s) + \alpha^2 c_2 \mu + \alpha E_D[f^*_{cen,s}(y_s - D)] \right\}.$$

where $G_{cen,s}(y_s) = \alpha(1-\alpha)c_2 y_s + \alpha E_D[m(y_s - D)]$. We need to justify two steps here. First, we can move the two terms back a period and $f^*_{cen,r}(\cdot)$ will have the same optimal policy as $f^*_{cen,s}(\cdot)$ using a similar argument as in Section 2.1. We have that

$$f^*_{cen,r}(x_s) = m(x_s) - \alpha c_2 x_s + f^*_{cen,s}(x_s).$$

Second, to prove the existence of $f^*_{cen,s}(\cdot)$, we must show that $g_{cen,s}(y_s) \equiv \alpha((1-\alpha)c_2 y_s + m(y_s - D)) \geq 0$. To prove $g_{cen,s}(y_s)$ is non-negative and to later prove that $G_{cen,s}(y_s)$ is quasiconvex, let us examine the function $g^+(w) \equiv (1-\alpha)c_2 w + m(w)$. Graphically, the function looks as in Figure 3.1.

Starting from the left, $g^+(\cdot)$ decreases at rate $-(c_o - c_2)$ until point $t_L - 1$. (The big dot on the left is $t_L - 1$, the big dot in the middle is y_L, and y_H is the big dot on the right). From t_L to $y_H - 1$, it follows $(1-\alpha)c_2 w + N(w)$, decreasing at first, then increasing. From y_H on, it increases at rate $h_2 + (1-\alpha)c_2$.

Lemma 11 *The function $g^+(\cdot)$ has exactly one minimum which occurs between t_L and $y_H - 1$ and is positive.*

Proof: To the left of t_L, the slope of $g^+(\cdot)$ is $-(c_o - c_2) < 0$ and to the right of $y_H - 1$, the slope of $g^+(\cdot)$ is $h_2 + (1-\alpha)c_2 > 0$. Also note that $g^+(t_L) \leq g^+(t_L - 1)$ by definition of t_L. Thus, any minima of the function occur between t_L and $y_H - 1$. Between these values, $g^+(\cdot)$ follows $(1-\alpha)c_2 w + N(w)$, a convex function, and thus

Figure 3.1: Graph of $g^{+}(w)$

there is exactly one minimum. The minimum value is positive by assumptions (A3) and (A8), following a similar argument as in Lemma 9. $\square$

Theorem 4 *For the relaxed problem, the optimal inventory control policy for the system inventory is a base-stock policy.*

Proof: Consider $g_{cen,s}(y_s)$:

$$\begin{aligned} g_{cen,s}(y_s) &= \alpha((1-\alpha)c_2 y_s + m(y_s - D)) \\ &= \alpha((1-\alpha)(c_2 y_s - c_2 D + c_2 D) + m(y_s - D)) \\ &= \alpha((1-\alpha)c_2 D + g^{+}(y_s - D)) \geq 0 \end{aligned}$$

where the inequality holds because the $g^+(\cdot) \geq 0$. Note from Figure 3.1 or Lemma 11 that $g^+(\cdot)$ is a quasiconvex function with a minimum point. Now consider $G_{cen,s}(y_s)$:

$$\begin{aligned} G_{cen,s}(y_s) &= \alpha E_D[(1-\alpha)c_2 y_s + m(y_s - D)] \\ &= \alpha((1-\alpha)c_2 E_D[D] + E_D[g^+(y_s - D)]). \end{aligned}$$

The first term is a constant, and the second term is a convolution of a quasiconvex function $(g^+(y_s - D))$ and a logconcave probability distribution by assumption (A2). Thus, according to [49] and [2], $G_{cen,s}(\cdot)$ is a quasiconvex function. Also, for $y_s < t_L$, the slope of $G_{cen,s}(y_s)$ is $\alpha(c_2 - c_o) < 0$, so as $y_s \to -\infty$, $G_{cen,s}(y_s) \to \infty$. As $y_s \to +\infty$, the slope of $G_{cen,s}(y_s)$ becomes $\alpha(h_2 + (1-\alpha)c_2) > 0$, and so $G_{cen,s}(y_s) \to \infty$. Hence, the desired result follows from Zheng [66]. $\square$

3.4 Fully Constrained Optimal Policies

In this section, we show that the optimal inventory control policy that solves the relaxed problem also solves the original, fully constrained problem. From Section 3.1, we have that

$$f^*_{cen}(x_1, \bar{x}_2) = -\alpha c_1 x_1 + f^*_{cen,m}(x_1, x_s).$$

From the previous section, we have that

$$f^*_{cen,r}(x_s) = m(x_s) - \alpha c_2 x_s + f^*_{cen,s}(x_s).$$

The missing piece of the puzzle is to show that

$$f^*_{cen,m}(x_1, x_s) = f^*_{cen,r}(x_s).$$

By doing so, we verify that the optimal policies listed in Sections 3.2 and 3.3 are truly optimal. We must show that the optimal policies for $f^*_{cen,r}(x_s)$ minimize $f^*_{cen,m}(x_1, x_s)$ and that both (R1) and (R2) are met.

Theorem 5

$$f^*_{cen,m}(x_1, x_s) = f^*_{cen,r}(x_s).$$

Proof: The optimal policies for $f^*_{cen,r}(x_s)$ minimize the costs for $f^*_{cen,m}(x_1, x_s)$ because both relaxed constraints are met and y_1 does not affect y_s or the costs-to-go. If $x_s < t_L$, $y_1^* = y_L \geq t_L \geq x_s \geq x_1$. If $t_L \leq x_s < t_H$, $y_1^* = x_s \geq x_1$. Finally, if $x_s \geq t_H$, $y_1^* = y_H \geq S^*_{1,dec} \geq x_1$ by assumption (A10) and (3) of Lemma 10. Thus, the first relaxation (R1) is satisfied. To show that (R2) is satisfied, define $y^*_{cen,s}$ to be the optimal system inventory position, S^*_{cen} to be the optimal system base-stock level, and $y^*_{cen,2}$ to be the optimal inventory position for stage 2. We must show that $y_s \geq y_1$ when $y_1 > x_s$. The only time when $y_1 > x_s$ is when $x_s < t_L$ (otherwise, $y_1 = x_s$ or $y_1 = y_H \leq x_s$). In this case, $y^*_{cen,1} = y_L \leq S^*_{cen} = y^*_{cen,s}$. The inequality holds because $0 \leq y^*_{cen,2} = y^*_{cen,s} - y^*_{cen,1} = S^*_{cen} - y_L$. □

For the original, fully constrained problem, we now know the optimal policies for stage 1, stage 2, and for the system.

$$y^*_{cen,1} = \begin{cases} y_H & \text{if } x_s \geq y_H \\ x_s & \text{if } t_L \leq x_s < y_H \\ y_L & \text{if } x_s < t_L, \end{cases}$$

$$\begin{aligned} y^*_{cen,s} &= \begin{cases} x_s & \text{if } x_s > S^*_{cen} \\ S^*_{cen} & \text{if } x_s \leq S^*_{cen}, \text{ and} \end{cases} \\ y^*_{cen,2} &= y^*_{cen,s} - y^*_{cen,1} \end{aligned}$$

Now, under the assumption that the initial system inventory is less than the system base-stock level, $x_s \leq S^*_{cen}$, we can further calculate $f^*_{cen}(x_1, x_s)$.

$$\begin{aligned} & f^*_{cen}(x_1, x_s) \\ &= -\alpha c_1 x_1 + f^*_{cen,m}(x_1, x_s) \\ &= -\alpha c_1 x_1 + m(x_s) - \alpha c_2 x_s + f^*_{cen,s}(x_s) \\ &= -\alpha c_1 x_1 + m(x_s) - \alpha c_2 x_s + \\ & \quad \min_{y_s \geq x_s} \left\{ \alpha(1-\alpha) c_2 y_s + \alpha E_D[m(y_s - D)] + \alpha^2 c_2 \mu + \alpha E_D[f^*_{cen,s}(y_s - D)] \right\} \\ &= -\alpha c_1 x_1 + m(x_s) - \alpha c_2 x_s + \\ & \quad \alpha(1-\alpha) c_2 S^*_{cen} + \alpha E_D[m(S^*_{cen} - D)] + \alpha^2 c_2 \mu + \alpha E_D[f^*_{cen,s}(S^*_{cen} - D)] \\ &= -\alpha c_1 x_1 + m(x_s) - \alpha c_2 x_s + \\ & \quad \alpha \left((1-\alpha) c_2 S^*_{cen} + E_D[m(S^*_{cen} - D)] + \alpha c_2 \mu \right) (1 + \alpha + \alpha^2 + \ldots) \\ &= -\alpha c_1 x_1 + m(x_s) - \alpha c_2 x_s + \frac{\alpha}{1-\alpha} \left((1-\alpha) c_2 S^*_{cen} + E_D[m(S^*_{cen} - D)] + \alpha c_2 \mu \right) \\ &= -\alpha c_1 x_1 + m'(x_s) + \alpha c_2 (S^*_{cen} - x_s) + \frac{\alpha}{1-\alpha} \left(E_D[m'(S^*_{cen} - D)] + \alpha(c_1 + c_2)\mu \right) \end{aligned}$$

We will use this relationship to compare costs in Chapter IV.

3.5 Optimal Policies with Premium Freight

In this section, we study the problem where both overtime production and premium freight are viable options. We follow the analysis from the previous four sections but add additional comments or proofs when necessary. In this problem, the overtime production decision is a real decision. If a shortage exists, the manager must choose how much of the shortage to fill with overtime production and how much to fill with premium freight. We will later show that it is optimal to pick one or the other expediting options, but never both. The decision timing occurs as described in Section 3.1, and the manager must decide $y_{1,t+1}$, $\tilde{y}_{2,t}$, and $y_{2,t+1}$ all at the same time.

When both options are viable, we require some additional assumptions. First, if both overtime production costs are less than the corresponding premium freight costs ($K_o < \alpha K_p$ and $c_o < \alpha(c_p+c_2)$), we have a situation where overtime production will always be utilized to fill shortages, and this is the problem we just considered. Similarly, if the premium freight costs are both less than the overtime costs, we have a situation where premium freight will always be utilized to fill shortages, and the analysis from the first four sections of this chapter applies. So, the problem is only interesting when one overtime cost is greater (e.g. $K_o \geq \alpha K_p$) and one overtime cost is less (e.g. $c_o \leq \alpha(c_p + c_2)$), or vice-versa. The discussion is the same in either case, so we will assume the case listed in the previous sentence. Second, we modify assumption (A8) from the beginning of the chapter.

(A8) $b_1 \geq (1-\alpha)c_1 + c_p$.

(A11) $K_o \geq \alpha K_p$

(A12) $c_o \leq \alpha(c_p + c_2)$

Using the same notation as before, we now develop the various cost equations.

$$g_{cen}(x_1, y_1, \bar{x}_2, y_2, \tilde{z}_2, D) \equiv$$

$$K_o\delta(\tilde{z}_2) + c_o\tilde{z}_2 + h_2(\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)^+$$

$$+\alpha K_p\delta((\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)^-) + \alpha c_p(\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)^- + \alpha c_1(y_1 - x_1) +$$

$$+\alpha c_2(y_2 - (\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)) + \alpha(h_1(y_1 - D)^+ + b_1(y_1 - D)^-)$$

with $y_1 \geq x_1$, $\tilde{z}_2 \geq 0$, and $y_2 \geq (\bar{x}_2 - (y_1 - \bar{x}_2) + \tilde{z}_2)^+$. Clearly, $g_{cen}(\cdot) \geq 0$ and hence by [7], the optimal cost function $f^*_{cen}(x_1, \bar{x}_2)$ satisfies

$$f^*_{cen}(x_1, \bar{x_2}) = \min_{\substack{y_1 \geq x_1 \\ \tilde{z}_2 \geq 0 \\ y_2 \geq (\bar{x}_2 - (y_1 - x_1))^+}} E[g_{cen}(x_1, y_1, \bar{x}_2, y_2, \tilde{z}_2, D) + \alpha f^*_{cen}(y_1 - D, y_2)].$$

The argument that minimizes this equation is the optimal inventory control policy which we seek. Moving the $-\alpha c_1 x_1$ back to the previous period as $-\alpha^2 c_1(y_1 - D)$ (as before) and moving some terms around, we get:

$$g_{cen,m}(x_1, y_1, \bar{x}_2, y_2, \tilde{z}_2, D) \equiv$$

$$\alpha(1-\alpha)c_1 y_1 + \alpha^2 c_1 D + K_o\delta(\tilde{z}_2) + c_o\tilde{z}_2 + h_2(\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)^+$$

$$+\alpha K_p\delta((\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)^-) + \alpha c_p(\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)^-$$

$$+\alpha c_2(y_2 - (\bar{x}_2 - (y_1 - x_1) + \tilde{z}_2)) + \alpha(h_1(y_1 - D)^+ + b_1(y_1 - D)^-)$$

under the same restrictions. Substituting system variables, we get:

$$g_{cen,m}(x_1, y_1, x_s, y_s, \tilde{z}_2, D) \equiv$$

$$\alpha(1-\alpha)c_1 y_1 + \alpha(\alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) \tag{3.8}$$

$$+K_o\delta(\tilde{z}_2) + (c_o - \alpha c_2)\tilde{z}_2 + h_2(x_s - y_1 + \tilde{z}_2)^+ \tag{3.9}$$

$$+\alpha K_p\delta((x_s - y_1 + \tilde{z}_2)^-) + \alpha c_p(x_s - y_1 + \tilde{z}_2)^- \tag{3.10}$$

$$+\alpha c_2(y_s - x_s)$$

with $y_1 \geq x_1$, $\tilde{z}_2 \geq 0$, and $y_s \geq y_1 + (x_s - y_1 + \tilde{z}_2)^+ = \max\{y_1, x_s + \tilde{z}_2\}$. We can rewrite $g_{cen,m}(\cdot)$ as

$$g_{cen,m}(x_1, y_1, x_s, y_s, \tilde{z}_2, D) = L_1(y_1, D) + L_2(y_2, x_s, \tilde{z}_2) + \alpha c_2(y_s - x_s)$$

where $L_1(y_1, D)$ represents the terms on line (3.8) and $L_2(y_2, \bar{x}_2, \tilde{z}_2)$ represents the terms on lines (3.9) and (3.10).

Again, we relax some constraints. First, we drop the constraint that $y_1 \geq x_1$. Second, we relax the constraint on the system inventory position so that only $y_s \geq x_s$. For later reference, we label the relaxed assumptions as:

(R1) $y_1 \geq x_1$, and

(R2) $y_s \geq \max\{y_1, x_s + \tilde{z}_2\} \rightarrow y_s \geq x_s$.

After relaxing the constraints, our cost per period becomes

$$g_{cen,r}(y_1, x_s, y_s, \tilde{z}_2, D) \equiv L_1(y_1, D) + L_2(y_1, x_s, \tilde{z}_2) + \alpha c_2(y_s - x_s)$$

with $\tilde{z}_2 \geq 0$ and $y_s \geq x_s$. The function $g_{cen,r}(\cdot)$ can be shown to be non-negative by analysis similar to Lemma 9 and we can thus use the same result from Bertsekas [7] for the optimal cost function $f^*_{cen,r}$.

$$\begin{aligned}
& f^*_{cen,r}(x_s) \\
& \quad = \min_{y_s \geq x_s, \tilde{z}_2 \geq 0, y_1} E_D\left[g_{cen,r}(y_1, x_s, y_s, \tilde{z}_2, D) + \alpha f^*_{cen,r}(y_s - D)\right] \qquad (3.11) \\
& \quad = \min_{y_s \geq x_s, \tilde{z}_2 \geq 0, y_1} \left\{ \begin{array}{l} E_D[L_1(y_1, D)] + L_2(y_1, x_s, \tilde{z}_2) \\ +\alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)] \end{array} \right\}
\end{aligned}$$

Again, it is important to notice that under the relaxed conditions, y_1 and $\tilde{z}_2$ have no effect on either y_s or the cost to go, $\alpha E_D[f^*_{cen,r}(y_s - D)]$. Thus,

$$\begin{aligned}
& f^*_{cen,r}(x_s) \\
&= \min_{y_s \geq x_s} \left\{ \begin{array}{l} \min_{\tilde{z}_2 \geq 0, y_1} \{E_D[L_1(y_1, D)] + L_2(y_1, x_s, \tilde{z}_2)\} \\ \\ +\alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)] \end{array} \right\} \\
&= \min_{y_s \geq x_s} \left\{ m(x_s) + \alpha c_2(y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)] \right\} \qquad (3.12)
\end{aligned}$$

where $m(x_s) = \min_{\tilde{z}_2 \geq 0, y_1} \{E_D[L_1(y_1, D)] + L_2(y_1, x_s, \tilde{z}_2)\}$. Finding the optimal inventory policy for stage 1 has become a myopic problem that now depends on y_1 and $\tilde{z}_2$. Now consider $m(x_s)$ under two cases, when stage 1 does not order more than the system inventory on hand ($y_1 \leq x_s$) and when stage 1 does order more than the system inventory on hand ($y_1 > x_s$). In the first case, we get that

$$\begin{aligned}
L_2(y_1, x_s, \tilde{z}_2) &= K_o \delta(\tilde{z}_2) + (c_o - \alpha c_2)\tilde{z}_2 + h_2(x_s - y_1 + \tilde{z}_2) \\
&= K_o \delta(\tilde{z}_2) + (h_2 + c_o - \alpha c_2)\tilde{z}_2 + h_2(x_s - y_1)
\end{aligned}$$

which is minimized when $\tilde{z}_2 = 0$ and thus $L_2(y_1, x_s, \tilde{z}_2) = h_2(x_s - y_1) - \alpha c_2 x_s$ when $y_1 \leq x_s$. In other words, if there isn't a shortage, don't use overtime production.

In the second case, there are four options.

$$\begin{aligned}
& L_2(y_1, x_s, \tilde{z}_2) = \\
& \begin{cases}
\alpha K_p + \alpha c_p(y_1 - x_s) & \text{if } \tilde{z}_2 = 0 \\
K_o + \alpha K_p + (c_o - \alpha(c_p + c_2))\tilde{z}_2 + \alpha c_p(y_1 - x_s) & \text{if } 0 < \tilde{z}_2 < y_1 - x_s \\
K_o + (c_o - \alpha c_2)(y_1 - x_s) & \text{if } \tilde{z}_2 = y_1 - x_s. \\
K_o + (h_2 + c_o - \alpha c_2)\tilde{z}_2 + h_2(x_s - y_1) & \text{if } \tilde{z}_2 > y_1 - x_s.
\end{cases}
\end{aligned}$$

It is easy to show that the third option is less expensive than both the second and fourth options using assumption (A12). Thus, either the first option or the third option are optimal. Either $\tilde{z}_2 = 0$ (use premium freight) with $L_2(y_1, x_s, \tilde{z}_2) = \alpha K_p + \alpha c_p(y_1 - x_s)$, or $\tilde{z}_2 = y_1 - x_s$ (use overtime production up to 0) with $L_2(y_1, x_s, \tilde{z}_2) = K_o + (c_o - \alpha c_2)(y_1 - x_s)$.

Before returning to $m(x_s)$, define $N_M(y_1)$ as

$$N_M(y_1) = \alpha((1-\alpha)c_1 + c_p)y_1 + \alpha E_D[\alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-].$$

Similar to the results of Lemma 10, define $y_M = \arg\min_{y_1}\{N_M(y_1)\}$. As before, $N_M(y_1)$ is convex and we have that $0 \le y_M \le y_L \le y_H < \infty$. [$0 \le y_M$ by assumption (A8), $y_M \le y_L$ by assumption (A12), $y_L \le y_H$ by algebra, and $y_H < \infty$ as before.]

We have

$$\begin{aligned}
m(x_s) &= \min_{\tilde{z}_2 \ge 0, y_1} \{E_D[L_1(y_1, D)] + L_2(y_1, x_s, \tilde{z}_2)\} \\
&= \min \begin{cases} \min_{y_1 \le x_s}\{h_2 x_s + N_H(y_1)\} & \text{if } y_1 \le x_s \\ \min_{y_1 > x_s}\{\alpha K_p - \alpha c_p x_s + N_M(y_1)\} & \text{if } y_1 > x_s \\ \min_{y_1 > x_s}\{K_o - (c_o - \alpha c_2)x_s + N_L(y_1)\} & \text{if } y_1 > x_s. \end{cases} \\
&= \min \begin{cases} h_2 x_s + \min_{y_1 \le x_s}\{N_H(y_1)\} \\ \alpha K_p - \alpha c_p x_s + \min_{y_1 > x_s}\{N_M(y_1)\} \\ K_o - (c_o - \alpha c_2)x_s + \min_{y_1 > x_s}\{N_L(y_1)\} \end{cases}
\end{aligned}$$

$$= \min \begin{cases} h_2 x_s + N_H(y_H) & \text{if } x_s \geq y_H \\ N(x_s) & \text{if } x_s < y_H \\ \alpha K_p + N(x_s) & \text{if } x_s > y_M \\ \alpha K_p - \alpha c_p x_s + N_M(y_M) & \text{if } x_s \leq y_M \\ K_o + N(x_s) & \text{if } x_s > y_L \\ K_o - (c_o - \alpha c_2) x_s + N_L(y_L) & \text{if } x_s \leq y_L \end{cases}$$

$$= \begin{cases} h_2 x_s + N_H(y_H) & \text{if } x_s \geq y_H \\ N(x_s) & \text{if } t_M \leq x_s < y_H \\ \alpha K_p - \alpha c_p x_s + N_M(y_M) & \text{if } t_L \leq x_s < t_M \\ K_o - (c_o - \alpha c_2) x_s + N_L(y_L) & \text{if } x_s < t_L \end{cases}$$

where t_M is defined as the smallest w such that $N(w) \leq \alpha K_p - \alpha c_p w + N_M(y_M)$ and t_L is redefined as

$$t_L \equiv \left\lceil \frac{K_o + N_L(y_L) - \alpha K_p - N_M(y_M)}{c_o - \alpha(c_p + c_2)} \right\rceil . \tag{3.13}$$

We have now defined $m(x_s)$ explicitly and again determined the relaxed optimal inventory control policy at stage 1.

$$y^*_{cen,1} = \begin{cases} y_H & \text{if } x_s \geq y_H \\ x_s & \text{if } t_M \leq x_s < y_H \\ y_M & \text{if } t_L \leq x_s < t_M \\ y_L & \text{if } x_s < t_L . \end{cases} \tag{3.14}$$

From equation (3.12), we have that

$$f^*_{cen,r}(x_s) = \min_{y_s \geq x_s} \left\{ m(x_s) + \alpha c_2 (y_s - x_s) + \alpha E_D[f^*_{cen,r}(y_s - D)] \right\}$$

As earlier in the chapter, we move the $m(x_s)$ and $-\alpha c_2 x_s$ terms back to get

$$f^*_{cen,s}(x_s) \quad = \quad \min_{y_s \geq x_s} \left\{ G_{cen,s}(y_s) + \alpha^2 c_2 \mu + \alpha E_D[f^*_{cen,s}(y_s - D)] \right\}$$

where $G_{cen,s}(y_s) = \alpha((1-\alpha)c_2 y_s + E_D[m(y_s - D)])$. We justify this step as before. The only difference is that now the $g^+(\cdot)$ function has an extra kink on the left. From the left, $g^+(\cdot)$ starts out decreasing at rate $-(c_o - c_2)$ up to point t_L; next, by assumption (A12), the function decreases at the steeper rate $-(\alpha(c_p + c_2) - c_2)$ up to the point t_M; after this point, the function behaves as before. Analogous results to Lemma 11 hold, $G_{cen,s}(\cdot)$ is a quasiconvex function, and the relaxed optimal system inventory control policy is a base-stock policy.

We have just shown that the optimal policies for the relaxed problem minimize the fully constrained problem, assuming the original constraints are met. In other words, as before, $f_{cen}(x_1, \bar{x_2})$ has the same solution as $f_{cen,m}(x_1, x_s)$ and $f_{cen,r}(x_s)$ has the same solution as $f_{cen,s}(x_s)$, and we need to show the relaxed constraints are met to link $f_{cen,m}(x_1, x_s)$ and $f_{cen,r}(x_s)$. The first relaxed constraint (R1) is satisfied by similar analysis as before. To show that the second relaxed constraint (R2) is satisfied, note that in the relaxed optimal solution, $\tilde{z}_2 = 0$ or $\tilde{z}_2 = y_1 - x_s$. Thus, the original constraint $y_s \geq \max\{y_1, x_s + \tilde{z}_2\}$ is equivalent to $y_s \geq \max\{y_1, x_s\}$ under optimality and similar analysis as before yields that (R2) is satisfied. Thus, we have solved our original problem when both methods of expediting are available.

3.6 Conclusion and Insights

In this chapter we have studied the two-stage supply chain under centralized control. We have shown that the optimal inventory control policies for both stages depend only on the system inventory, x_s, and that the optimal policy for the system inventory is a base-stock policy. In the first four sections, we assumed that overtime production was the only method of expediting. In Section 3.1, we defined our relaxed cost functions. In Section 3.2, we proved the relaxed optimal policy for stage 1 and in Section 3.3, we proved the relaxed optimal policy for the system. In Section 3.4, we showed that the solution for the relaxed problem also solves the original, fully constrained problem and in the last section we repeated the analysis when the premium freight option is included.

The obvious question at this point is how much does centralized control save over decentralized control? The centralized optimal policies are more complicated than the two base-stock policies of the decentralized model, and they require that the two firms share inventory information. In Chapter V, we address this issue with numerical analysis. We show that the centralized optimal policies do affect significant savings over the decentralized optimal policies, particularly if demand variation is high or if the fixed costs of expediting are expensive.

We feel that some of our analyses and results are distinctive when compared to traditional inventory literature. Traditional two-echelon proofs proceed by separating variables and then solving two independent problems (e.g., see [21]). We tried this approach at first, but we were unable to decouple the equations. However, we found that by substituting system variables and relaxing some constraints, we could first solve a myopic problem then solve a straightforward dynamic program. Our optimal policies also vary from traditional optimal inventory policies. Our stage 1 policy of ordering up to two (or three) separate inventory levels and occasionally underordering is quite different from traditional inventory policies. Hence, we feel that our base-stock result for the system is also interesting.

The main managerial insight gained from this chapter is that to cut costs in this kind of supply chain, stage 1 must be sensitive to the amount of inventory available at stage 2. Stage 1 must be willing to occasionally underorder in order to save significant overtime production costs (or premium freight shipping costs) at stage 2. By the same token, stage 2 must be willing to produce extra units when stage 1 underorders, trusting that stage 1 will want those additional units the next period. Here it is interesting to compare our centralized results with those of Federgruen and Zipkin [21]. In their model, stage 1 completely ignores stage 2 and follows a base-stock model dependent on only stage 1 cost parameters; stage 2 also follows a base-stock policy, but with a higher base-stock level to reduce the chance of not filling supply requests from stage 1. In our model, stage 1 is sensitive to the costs and inventory available at stage 2, and orders accordingly; stage 2 orders more when stage 1 underorders, bringing the system inventory up to a base-stock level.

CHAPTER IV

COORDINATED MODEL

In this chapter, we study a coordinated system. We consider how to coordinate the decisions made at both stages so that each stage reduces its own individual costs and together the two stages achieve system optimal (or near-optimal) results. We investigate two related, but slightly different contracts which we refer to as Contract A and Contract B. In Section 4.1, we describe the two contracts and the associated costs, and we describe the decision timing. In Section 4.2, we show that under Contract A, both stages follow the centralized optimal policies. In Section 4.3, we show that under Contract B, stage 1 follows the centralized optimal policy and the system policy will be base-stock, although the base-stock level may be too high. In Section 4.4, we discuss appropriate values for the linear transfer payment W. We consider the average cost case in Section 4.5, and show that Contract B performs optimally under this criterion. In Section 4.6, we reconsider the problem when both options of expediting are available and we conclude the chapter in the last section.

4.1 Contract Descriptions

In both contracts, stage 2 charges a two-tiered wholesale cost to stage 1, but also offers a negotiable linear transfer payment to stage 1 to sweeten the deal. When overtime production is the only method of expediting available, the wholesale cost and transfer payment combined are $w(z_1 - \bar{x}_2)$, where z_1 is the amount ordered by stage 1 and $\bar{x}_2$ is the amount stage 2 has on hand before the order:

$$w(z_1 - \bar{x}_2) \equiv -W + \begin{cases} h_2(\bar{x}_2 - z_1) & \text{if } z_1 \leq \bar{x}_2 \\ K_o + (c_o - \alpha c_2)(z_1 - \bar{x}_2) & \text{if } z_1 > \bar{x}_2 \end{cases}$$

where W is the side payment from stage 2 to stage 1. We will discuss appropriate values for W later in the chapter. Note that in terms of previously defined variables, $z_1 - \bar{x}_2 = (y_1 - x_1) - \bar{x}_2 = y_1 - (x_1 + \bar{x}_2) = y_1 - x_s$. So, we can also write the combined cost as

$$w(y_1 - x_s) = -W + \begin{cases} h_2(x_s - y_1) & \text{if } y_1 \leq x_s \\ K_o + (c_o - \alpha c_2)(y_1 - x_s) & \text{if } y_1 > x_s \end{cases}$$

The combined cost depends on whether the stage 1 order quantity is greater than how much is available at stage 2, $z_1 > \bar{x}_2$, or, in other words, whether the stage 1 order-up-to level is greater than how much is available in the system, $y_1 > x_s$. Under either contract, stage 1 will receive the side payment every period, but will face additional production costs that depend on the stage 2 inventory. These costs will induce stage 1 to make its order quantity close to the stage 2 inventory. However, stage 1 will now have some control over what that stage 2 inventory will be.

For both contracts, we assume that stage 2 proposes the contract to stage 1 and the two firms negotiate over a linear transfer payment W. Assuming that the two stages are currently under decentralized control, there are two reasons why stage 2 should desire a contract. First, stage 2 pays all the costs of expediting, which may be quite high, whereas stage 1 ignores these costs and is not affected by them. Second, in our original problem, stage 2 is a parts supplier for a stage 1, a major automobile assembler. Stage 1 is the more powerful firm, and an appropriate contract will help even out the power structure between the two firms, helping stage 2. Stage 1 should desire a contract if it reduces costs compared to the stage 1 decentralized costs. Also, in both contracts, stage 1 will have some control over the inventory decisions at stage 2. We assume that both stages will follow the contract in good faith; however, in reality, either stage may not follow the contract or may misreport their inventory levels, and the possibility of gaming exists.

In Contract A, stage 1 determines its own inventory policy and stage 2 guarantees that it will bring the system inventory up to the optimal level $y^*_{cen,s}$ as determined in Chapter III. The advantage of this contract, as we will show, is that the system optimal policies will be followed by both stages. The disadvantage of this contract is that stage 2 must know the optimal centralized system policy prior to proposing the contract. In Contract B, stage 1 is allowed to determine its own inventory policy **and** the inventory policy for the system (and hence the stage 2 policy). The advantage of Contract B is that stage 2 only needs to know its own costs to propose the contract, and all of the inventory control parameters are determined explicitly using the contract. The disadvantage of Contract B, as we will show, is that stage 1 may tend to overestimate the optimal system base-stock level, yielding non-optimal results. However, we show in Section 4.5 that Contract B is optimal for the average cost case and in Chapter V, we show that for the discounted cost case Contract B yields near-optimal results.

We have attempted to design contracts that follow the three “good” contract properties discussed in Lee and Whang [38]: cost conservation, incentive compatibility, and informational decentralizability. Both contracts follow the first property: cost conservation holds when all relevant system costs are paid by the two stages, without requiring a “headquarters” to disburse payments. Incentive compatibility holds when each stage is induced to follow the centralized optimal policy. This property holds for stage 1 for both contracts, but not for stage 2: under Contact A, stage 2 follows the optimal centralized policy as a condition of the contract (but not because it is *induced* to do so), and under Contract B, stage 2 follows the policy determined by stage 1, which may not be optimal. Informational decentralizability holds when each stage makes decisions considering only their own information, i.e, inventory levels. Neither of our contracts have this property for good reason. The centralized optimal policies for both stages depend explicitly on the amount of inventory available in the system, $x_s = x_1 + \bar{x}_2$, and thus sharing information is necessary in order to achieve system optimal results.

When we analyze the contracts in this chapter (and compare costs in Chapter V), we must be very careful about the timing of the decisions. Recall from Chapter II that when stage 1 and stage 2 make their decisions in a decentralized fashion, their respective inventory information is $x_{1,t}$ and $x_{2,t}$. In Chapter III, centralized decisions are made with inventory information $x_{1,t+1}$ and $y_{2,t} = \bar{x}_{2,t}$. Thus the total costs $f_{dec,1}(x_{1,0})$ and $f_{dec,2}(x_{2,0})$ contain additional costs compared to $f_{cen}(x_{1,1}, \bar{x}_{2,0})$, which we must bear in mind when we compare them. Referring to Table 1.1, the additional costs at stage 1 consist of all costs during time period 0 which are $c_1(y_{1,0} - x_{1,0}) + h_1(y_{1,0} - D_0)^+ + b_1(y_{1,0} - D_0)^-$. In other words, stage 1 pays for an extra time period under the decentralized total cost calculation; to account for these costs, we multiply the stage 1 total costs by a factor of α when we compare them to the total costs under contract, which has the same decision timing as the centralized model. The additional costs at stage 2 consist of the production costs during time period 0 which are $c_2(y_{2,0} - x_{2,0}^+)$; to account for these costs, we simply subtract them from the stage 2 decentralized costs when we compare contract costs.

4.2 Contract A

Under Contract A, stage 1 pays its original production, holding, and backordering costs plus the combined cost $w(y_1 - x_s)$ to stage 2. Stage 2 pays its original production, holding, and overtime costs and receives $w(y_1 - x_s)$ from stage 1; stage 2 also guarantees to bring the system inventory up to the optimal centralized system inventory level. For a range of appropriate W, we show that under Contract A, both stage 1 and stage 2 will follow the optimal policies for the centralized model and will each spend less than under decentralized control.

Having adjusted the original stage 1 costs by α to equalize the timing, stage 1 pays the following one period costs:

$$g_{1,CA}(x_1, x_s, y_1, D) = \alpha(c_1(y_1 - x_1) + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w(y_1 - x_s).$$

with $y_1 \geq x_1$. Again by [7], the optimal cost function $f^*_{1,CA}(x_1, x_s)$ satisfies

$$f^*_{1,CA}(x_1, x_s) = \min_{y_1 \geq x_1} E_D[g_{1,CA}(x_1, x_s, y_1, D) + \alpha f^*_{1,CA}(y_1 - D, y^*_{cen,s} - D)].$$

since stage 2 ensures the centralized system inventory. The solution to this equation is the optimal inventory control policy which we seek. As before, we move the $-c_1x_1$ term back and get

$$\begin{aligned} & g_{1,CA,m}(x_1, x_s, y_1, D) = \\ & \qquad \alpha((1-\alpha)c_1y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w(y_1 - x_s) \end{aligned}$$

with $y_1 \geq x_1$. Note that

$$f^*_{1,CA}(x_1, x_s) = -\alpha c_1 x_1 + f^*_{1,CA,m}(x_1, x_s).$$

We now relax the constraint that $y_1 \geq x_1$ and write our relaxed one period costs and optimal cost functions:

$$\begin{aligned} & g_{1,CA,r}(x_s, y_1, D) = \\ & \qquad \alpha((1-\alpha)c_1y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w(y_1 - x_s) \end{aligned}$$

and again by [7] we have

$$\begin{aligned} f^*_{1,CA,r}(x_s) &= \min_{y_1} E_D\Big[\, g_{1,CA,r}(x_s, y_1, D) + \alpha f^*_{1,C1,r}(y^*_{cen,s} - D)\Big] \\ &= \min_{y_1} E_D\Big[\, g_{1,CA,r}(x_s, y_1, D)\Big] \; + \alpha E_D\Big[\, f^*_{1,C1,r}(y^*_{cen,s} - D)\Big] \end{aligned}$$

since y_1 has no effect on the costs to go. We now show that stage 1 follows the optimal policy of the centralized model.

Lemma 12 *Under Contract A, stage 1 follows the optimal policies of the centralized model.*

Proof: The stage 1 relaxed optimal policy under Contract A is the solution to the myopic problem:

$$\begin{aligned}
&\min_{y_1} E_D[g_{1,CA,r}(x_s, y_1, D)] \\
&\quad = \min_{y_1} E_D[\alpha(1-\alpha)c_1 y_1 + \alpha^2 c_1 D + \alpha h_1(y_1 - D)^+ + \alpha b_1(y_1 - D)^- + w(y_1 - x_s)] \\
&\quad = \min_{y_1}\{\alpha(1-\alpha)c_1 y_1 + E_D[\alpha^2 c_1 D + \alpha h_1(y_1 - D)^+ + \alpha b_1(y_1 - D)^-] - \\
&\qquad\qquad W + h_2(x_s - y_1)^+ + K_o\delta(y_1 - x_s) + (c_o - \alpha c_2)(y_1 - x_s)^+\} \\
&\quad = m(x_s) - W.
\end{aligned}$$

Stage 1 minimizes the same problem, $m(x_s)$, as in the centralized model, and hence stage 1 will follow the centralized optimal policy for the relaxed problem. Under the assumption (A10) that $x_1 \leq S^*_{1,dec}$, we can show that the solution to the relaxed problem will always have $y_1 \geq x_1$ by the same argument as in Theorem 5. Thus we have solved the original, fully constrained problem as well. Stage 1 follows the optimal policies of the centralized model. $\square$

From the previous lemma and under the assumption that $x_s \leq S^*_{cen}$ we have that

$$\begin{aligned}
f^*_{1,CA}(x_1, x_s) &= -\alpha c_1 x_1 + f^*_{1,CA,m}(x_1, x_s) \\
&= -\alpha c_1 x_1 + f^*_{1,CA,r}(x_s)
\end{aligned}$$

$$
\begin{aligned}
&= -\alpha c_1 x_1 + \min_{y_1} E_D[g_{1,CA,r}(x_s, y_1, D)] + \alpha E_D[f^*_{1,CA,r}(S^*_{cen} - D)] \\
&= -\alpha c_1 x_1 + m(x_s) - W + \alpha E_D[f^*_{1,CA,r}(S^*_{cen} - D)] \\
&= -\alpha c_1 x_1 + m(x_s) - W + \\
&\quad \alpha E_D[m(S^*_{cen} - D) - W + \alpha E_D[f^*_{1,CA,r}(S^*_{cen} - D)]] \\
&= -\alpha c_1 x_1 + m(x_s) + (\alpha E_D[m(S^*_{cen} - D)] - W) \\
&\quad +\alpha(\alpha E_D[m(S^*_{cen} - D)] - W) + \alpha^2(\alpha E_D[m(S^*_{cen} - D)] - W) + \ldots \\
&= -\alpha c_1 x_1 + m(x_s) + \frac{1}{1-\alpha}(-W + \alpha E_D[m(S^*_{cen} - D)]) \\
&= -\alpha c_1 x_1 + m'(x_s) + \frac{1}{1-\alpha}(-W + \alpha(E_D[m'(S^*_{cen} - D)] + \alpha c_1 \mu)).
\end{aligned}
$$

Now consider stage 2 under Contract A. We know stage 2 will follow the centralized optimal policy (it is in the contract), but what will the cost be? Under decentralized control, stage 2 pays the following costs per period:

$$
g_{2,dec}(x_2, y_2, D) = c_2(y_2 - x_2^+) + K_o\delta((y_2 - D)^-) + c_o(y_2 - D)^- + h_2(y_2 - D)^+
$$

with $y_2 \geq x_2^+$. As usual we move the $-c_2 x_2^+$ term back and get

$$
g_{2,dec,m}(x_2, y_2, D) \;\; = \;\; c_2 y_2 + K_o\delta((y_2 - D)^-) + c_o(y_2 - D)^- + (h_2 - \alpha c_2)(y_2 - D)^+
$$

under the same restrictions. Now we replace y_2 with $\bar{x}_2$ and D with $y_1 - x_1$, then $x_1 + \bar{x}_2$ with x_s and get:

$$
g_{2,dec,m}(\bar{x}_2, x_s, y_1) \;\; = \;\; c_2\bar{x}_2 + K_o\delta(y_1 - x_s) + c_o(y_1 - x_s)^+ + (h_2 - \alpha c_2)(x_s - y_1)^+.
$$

The decision under Contract A takes places after $\bar{x}_2$ has been chosen. We now move the $c_2\bar{x}_2$ back a period as $\alpha c_2 y_2$ and define our one period cost at stage 2 under

Contract A as:

$$
\begin{aligned}
& g_{2,C2}(x_s, y_1, y_2) \\
&= K_o\delta(y_1 - x_s) + c_o(y_1 - x_s)^+ + (h_2 - \alpha c_2)(x_s - y_1)^+ + \alpha c_2 y_2 - w(y_1 - x_s) \\
&= (h_2 - \alpha c_2)(x_s - y_1)^+ + c_o(y_1 - x_s)^+ + K_o\delta(y_1 - x_s) + \alpha c_2 y_2 \\
&\quad + W - h_2(x_s - y_1)^+ - (c_o - \alpha c_2)(y_1 - x_s)^+ - K_o\delta(y_1 - x_s) \\
&= \alpha c_2 y_2 + \alpha c_2(y_1 - x_s) + W \\
&= \alpha c_2(y_1 + y_2 - x_s) + W \\
&= \alpha c_2(S^*_{cen} - x_s) + W
\end{aligned}
$$

since stage 2 brings the system inventory up to S^*_{cen} every period. We can calculate $f^*_{2,CA}(x_s)$.

$$
\begin{aligned}
f^*_{2,CA}(x_s) &= \alpha c_2(S^*_{cen} - x_s) + W + E_D[\sum_{k=1}^{\infty} \alpha^k(\alpha c_2(S^*_{cen} - x_{s,k}) + W)] \\
&= \alpha c_2(S^*_{cen} - x_s) + W + E_D[\sum_{k=1}^{\infty} \alpha^k(\alpha c_2(S^*_{cen} - (S^*_{cen} - D_{k-1})) + W)] \\
&= \alpha c_2(S^*_{cen} - x_s) + W + E_D[\sum_{k=1}^{\infty} \alpha^k(\alpha c_2 D_{k-1}) + W)] \\
&= \alpha c_2(S^*_{cen} - x_s) + W + \sum_{k=1}^{\infty} \alpha^k(E_D[\alpha c_2 D_{k-1}] + W) \\
&= \alpha c_2(S^*_{cen} - x_s) + W + \sum_{k=1}^{\infty} \alpha^k(\alpha c_2 \mu + W) \\
&= \alpha c_2(S^*_{cen} - x_s) + \frac{1}{1-\alpha}(W + \alpha^2 c_2 \mu).
\end{aligned}
$$

Note that we can interchange the expectation and the infinite sum by Fubini's Theorem since the terms are all positive. A quick algebraic check shows that $f^*_{1,CA}(x_1, x_s) + f^*_{2,CA}(x_s) = f^*_{cen}(x_1, x_s)$, so Contract A does achieve system optimality. Lastly, neither stage will bother considering Contract A unless it decreases costs. There is a range of W for which both stages will improve over the decentralized model. In Section 4.4, we discuss the values of W such that $f^*_{1,CA}(x_1, x_s) < \alpha f^*_{1,dec}(x_1)$ and $f^*_{2,CA}(x_s) < f^*_{2,dec}(x_2) - c_2(S^*_{2,dec} - x_2^+)$. Note that the values have been adjusted so equivalent costs are considered, given the different decision timing.

4.3 Contract B

Under Contract B, stage 1 pays its original production, holding, and backordering costs plus the combined cost $w(y_1 - x_s)$ to stage 2. Stage 1 determines its own inventory policy, the system inventory policy, and therefore stage 2's inventory policy. Stage 2 pays its original costs less $w(y_1 - x_s)$ and follows the inventory control policy determined by stage 1. We show that under Contract B, stage 1 will follow the optimal policies for the centralized model, stage 1 will determine that a base-stock policy is optimal for the system inventory, but that the base-stock level determined by stage 1, S^*_{CB}, may be greater than the optimal base-stock level S^*_{cen}. Thus, the system performance under Contract B may not be optimal.

Stage 1 pays the same one period costs as under Contract A:

$$g_{1,CB}(x_1, x_s, y_1, D) = \alpha(c_1(y_1 - x_1) + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w(y_1 - x_s).$$

with $y_1 \geq x_1$. However, the optimal cost function $f^*_{1,CB}(x_1, x_s)$ which satisfies

$$f^*_{1,CB}(x_1, x_s) = \min_{\substack{y_1 \geq x_1 \\ y_s \geq \max\{x_s, y_1\}}} E_D[g_{1,CB}(x_1, x_s, y_1, D) + \alpha f^*_{1,CB}(y_1 - D, y_s - D)]$$

is different since stage 1 must also choose the system inventory level y_s. As before, we move the $-c_1x_1$ term back and get

$$g_{1,CB,m}(x_1, x_s, y_1, D) =$$

$$\alpha((1-\alpha)c_1y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w(y_1 - x_s)$$

with $y_1 \geq x_1$. Note that

$$f^*_{1,CB}(x_1, x_s) = -\alpha c_1 x_1 + f^*_{1,CB,m}(x_1, x_s).$$

We now relax the constraints that $y_1 \geq x_1$ and that $y_s \geq y_1$ and write our relaxed one period costs and optimal cost functions:

$$g_{1,CB,r}(x_s, y_1, D) =$$

$$\alpha((1-\alpha)c_1y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w(y_1 - x_s)$$

and again by [7] we have

$$\begin{aligned} f^*_{1,CB,r}(x_s) &= \min_{y_1, y_s \geq x_s} E_D\Big[\ g_{1,CB,r}(x_s, y_1, D) + \alpha f^*_{1,CB,r}(y_s - D)\Big] \\ &= \min_{y_s \geq x_s} \Big\{\ \min_{y_1} E_D[g_{1,CB,r}(x_s, y_1, D)] + \alpha E_D[f^*_{1,CB,r}(y_s - D)]\Big\} \end{aligned}$$

since y_1 has no effect on the costs to go. Following the argument from Lemma 12, we get that under the relaxed conditions, $\min_{y_1} E_D[g_{1,CB,r}(x_s, y_1, D)] = m(x_s) - W$, stage 1 follows the centralized optimal policy, and

$$f^*_{1,CB,r}(x_s) = \min_{y_s \geq x_s} \left\{m(x_s) - W + \alpha E_D[f^*_{1,CB,r}(y_s - D)]\right\}.$$

Now we move the $m(x_s)$ term back as $\alpha m(y_s - D)$, define $g_{1,CB,r2}(y_s, D) \equiv \alpha m(y_s - D)$, and get

$$\begin{aligned} f^*_{1,CB,r2}(x_s) &= \min_{y_s \geq x_s} \left\{E_D[g_{1,CB,r2}(y_s, D)] - W + \alpha E_D[f^*_{1,CB,r2}(y_s - D)]\right\} \\ &= \min_{y_s \geq x_s} \left\{\alpha E_D[m(y_s - D)] - W + \alpha E_D[f^*_{1,CB,r2}(y_s - D)]\right\} \end{aligned}$$

Note that $f^*_{1,CB,r}(x_s) = m(x_s) + f^*_{1,CB,r2}(x_s)$ and that $f^*_{1,CB,r2}(x_s)$ is similar to a function we have previously studied. Besides the constant W, the term we are trying to minimize every period is $\alpha E_D[m(y_s - D)]$; in equation (3.7) from Chapter III (besides the constant $\alpha^2 c_2 \mu$), the terms we are trying to minimize are $\alpha E_D[m(y_s - D)] + \alpha(1-\alpha)c_2 y_s$. The minimization problem differs by the term $\alpha(1-\alpha)c_2 y_s$, the coefficient of which will be relatively small for high values of α.

Lemma 13 *Under Contract B, stage 1 determines a base-stock policy for the system with base stock level $S^*_{CB} \geq S^*_{cen}$, and the optimal policies for the relaxed problem also solve the original, fully constrained problem.*

Proof: The base-stock result follows from the same arguments as Lemma 11 and Theorem 4 from Chapter III. The difference is that the $\alpha(1-\alpha)c_2 y_s$ term is missing. We redefine $g^+(w) \equiv m(w)$, which is still quasiconvex, although the slopes have changed slightly (the slope on the left is now $-(c_o - \alpha c_2) < 0$, in the middle the function follows $N(w)$, and the slope on the right is $h_2 > 0$). Because $0 < \alpha(1-\alpha)c_2 y_s$, the "slope" under Contract B is less than in the centralized case, and $S^*_{CB} \geq S^*_{cen}$. To show that $f_{1,CB,m}(x_1, x_s) = f_{1,CB,r}(x_s)$, we follow the exact same proof as Theorem 5 and use the fact that $S^*_{CB} \geq S^*_{cen}$. □

How different are S^*_{CB} and S^*_{cen} and how different are the total costs under Contract B and under centralized control? First, since S^*_{CB} and S^*_{cen} are discrete, they may turn out to be the same. Second, as $\alpha \uparrow 1$, the values should converge together. We show that this is the case for $\alpha = 1$ in Section 4.5 and in Chapter V, we show that the values of S^*_{CB} and S^*_{cen} and the total costs are very close by numerical analysis.

Under the assumption that $x_s \leq S^*_{CB}$ we have that

$$f^*_{1,CB}(x_1, x_s) = -\alpha c_1 x_1 + m'(x_s) + \frac{1}{1-\alpha}(-W + \alpha(E_D[m'(S^*_{CB} - D)] + \alpha c_1 \mu))$$

and

$$f^*_{2,CB}(x_s) = \alpha c_2 (S^*_{CB} - x_s) + \frac{1}{1-\alpha}(W + \alpha^2 c_2 \mu).$$

The total system cost under Contract B is the sum of these two costs. By some algebra, we get that

$$\begin{aligned}
& f^*_{1,CB}(x_1, x_s) + f^*_{2,CB}(x_s) \\
= \;& -\alpha c_1 x_1 + m'(x_s) + \alpha c_2 (S^*_{CB} - x_s) + \frac{\alpha}{1-\alpha}\left(E_D[m'(S^*_{CB} - D)] + \alpha(c_1 + c_2)\mu\right) \\
\geq \;& -\alpha c_1 x_1 + m'(x_s) + \alpha c_2 (S^*_{cen} - x_s) + \frac{\alpha}{1-\alpha}\left(E_D[m'(S^*_{cen} - D)] + \alpha(c_1 + c_2)\mu\right) \\
= \;& f_{cen}(x_1, x_s).
\end{aligned}$$

It is possible to guarantee that Contract B will yield the centralized optimal results, by adding the term $(1-\alpha)x_s$ to $w(y_1 - x_s)$. However, now the combined cost $w(y_1 - x_s)$ depends not only on the difference $y_1 - x_s$, but also on the actual value of x_s. We feel that this kind of contract is less reasonable than one that simply depends on the difference. As mentioned earlier, neither stage 1 nor stage 2 will even consider Contract B unless it saves them money. In the next section, we explore the appropriate values of the transfer payment W that benefit both stages.

4.4 Transfer Payment W

For either contract, the two stages need to negotiate an appropriate transfer payment W. Stage 1 will determine a minimum value of the transfer payment and strive for the highest possible payment during negotiations. Alternatively, stage 2 will determine a maximum value of the transfer payment and will strive to make it as low as possible. We will first discuss appropriate values for W_A under Contract A. The analysis for Contract B will be very similar.

For stage 2 to consider proposing the contract, the total cost under contract must be less expensive for than under decentralized control. Taking into account the different decision timing, stage 2 will require that $f^*_{2,CA}(x_s) < f^*_{2,dec}(x_2) - c_2(S^*_{2,dec} - x_2^+)$, or $f^*_{2,CA}(x_s) - (f^*_{2,dec}(x_2) - c_2(S^*_{2,dec} - x_2^+)) < 0$.

$$\begin{aligned}
& f^*_{2,CA}(x_s) - (f^*_{2,dec}(x_2) - c_2(S^*_{2,dec} - x_2^+)) \\
= \quad & \alpha c_2(S^*_{cen} - x_s) + \frac{1}{1-\alpha}(W_A + \alpha^2 c_2 \mu) - \\
& (-c_2 x_2^+ + \frac{G_{2,dec,m}(S^*_{2,dec})}{1-\alpha} - c_2(S^*_{2,dec} - x_2^+)) \\
= \quad & c_2(\alpha(S^*_{cen} - x_s) + S^*_{2,dec}) + \frac{1}{1-\alpha}(W_A + \alpha^2 c_2 \mu - G_{2,dec,m}(S^*_{2,dec})) \\
< \quad & 0
\end{aligned}$$

Solving for W_A, we get that

$$W_A < G_{2,dec,m}(S^*_{2,dec}) - \alpha^2 c_2 \mu - (1-\alpha) c_2(\alpha(S^*_{cen} - x_s) + S^*_{2,dec}). \tag{4.1}$$

Note that W_A does depend on x_s. However, for a good approximation, if we let $\alpha \uparrow 1$, we get that stage 2 desires $W_A < G_{2,dec,m}(S^*_{2,dec}) - c_2\mu$. In other words, the transfer payment must be less that stage 2's per period expected decentralized costs that are not due to production.

For stage 1 to consider accepting the contract, again taking into account the different decision timing, stage 1 will require that $f^*_{1,CA}(x_1, x_s) < \alpha f^*_{1,dec}(x_1)$, or

$f^*_{1,CA}(x_1,x_s) - \alpha f^*_{1,dec}(x_1) < 0$.

$$
\begin{aligned}
& f^*_{1,CA}(x_1,x_s) - \alpha f^*_{1,dec}(x_1) \\
= \;& -\alpha c_1 x_1 + m'(x_s) + \frac{1}{1-\alpha}(-W_A + \alpha(E_D[m'(S^*_{cen} - D)] + \alpha c_1 \mu) \\
& -\alpha\left(-c_1 x_1 + \frac{G_{1,dec,m}(S^*_{1,dec})}{1-\alpha}\right) \\
= \;& m'(x_s) + \frac{1}{1-\alpha}(-W_A + \alpha(E_D[m'(S^*_{cen} - D)] - (G_{1,dec,m}(S^*_{1,dec}) - \alpha c_1 \mu)) \\
< \;& 0.
\end{aligned}
$$

Solving for W_A, we get that

$$W_A > \alpha(E_D[m'(S^*_{cen} - D)] - (G_{1,dec,m}(S^*_{1,dec}) - \alpha c_1 \mu)) + (1-\alpha)m'(x_s). \tag{4.2}$$

Note that W_A does depend on x_s. Again, for a good approximation, if we let $\alpha \uparrow 1$, we get that stage 1 desires $W_A > E_D[m'(S^*_{cen} - D)] - (G_{1,dec,m}(S^*_{1,dec}) - c_1\mu)$. In other words, the transfer payment must be more than the difference between the per period expected centralized non-production costs and the per period stage 1 decentralized non-production costs.

So, under the assumption of risk neutrality, both stage 1 and stage 2 will accept Contract A for values of W_A defined by equations (4.1) and (4.2). However, as mentioned previously, the inventory control policies under Contract A are more complicated than the two base-stock policies of the decentralized model. Further, the contract requires that both stages trust each other and share information. Realistically, for both stages to actually endorse this contract, a fairly significant savings would have to exist for both parties. In the next chapter, we show that these savings can be significant with numerical analysis.

Now we consider Contract B. The analysis of W_B is nearly identical, and here we list the results. For stage 2 to propose Contract B, we must have $f^*_{2,CB}(x_s) < f^*_{2,dec}(x_2) - c_2(S^*_{2,dec} - x_2^+)$, or that

$$W_B < G_{2,dec,m}(S^*_{2,dec})) - \alpha^2 c_2 \mu - (1-\alpha)c_2(\alpha(S^*_{CB} - x_s) + S^*_{2,dec}).$$

This condition is nearly the same as before. For stage 1 to accept contract B, we must have that $f^*_{1,CA}(x_1, x_s) < \alpha f^*_{1,dec}(x_1)$, or that

$$W_B > \alpha(E_D[m'(S^*_{CB} - D)] - (G_{1,dec,m}(S^*_{1,dec}) - \alpha c_1 \mu)) + (1-\alpha)m'(x_s).$$

4.5 Average Cost Case

In this section, we consider the average cost case of some of our previous models, under the assumption that overtime production is the only method of expediting. In the previous sections of this chapter, we showed that differences may exist between Contract A and Contract B for $0 < \alpha < 1$; when $\alpha = 1$, we will show that the structure of the optimal policies remains the same for the decentralized, centralized, and contract coordinated models and that the two contracts induce exactly the same behavior from both stages. We are now interested in minimizing the average cost function $f_\pi(x_0)$:

$$f_\pi(x_0) \equiv \lim_{N\to\infty} \frac{1}{N} E\left[\sum_{k=0}^{N-1} g(\text{period ``}k\text{'' variables})\right].$$

We discuss the existence of the limit later in the section. We wish to find the optimal policy π out of all possible admissible policies Π and hence the optimal average cost per period over the infinite horizon, $f^\star(x_0) \equiv \min_{\pi\in\Pi} f_\pi(x_0)$, where the * indicates average cost optimality.

We now revisit the results from the first two sections of Chapter II, the first four sections of Chapter III, and the previous sections of the current chapter. The results from Sections 2.1 and 2.2 continue to hold for the average cost case by Section 4 of Zheng's paper [66]. The optimal policy at stage 1 is to produce up to a base-stock level of $S^\star_{1,dec}$ and the optimal policy at stage 2 is to produce up to a base-stock level of $S^\star_{2,dec}$. In fact, for any initial inventories x_1 and x_2,

$$\begin{aligned} f^\star_{1,dec}(x_1) &= G_{1,dec,m}(S^\star_{1,dec}), \text{ and} \\ f^\star_{2,dec}(x_2) &= G_{2,dec,m}(S^\star_{2,dec}). \end{aligned}$$

The analysis of the centralized and coordinated models is not as straightforward. For this analysis, we restrict our attention to finite demand distributions and finite inventory state spaces. We feel that these are not restrictive assumptions as in reality, an upper limit on demand must exist and there are physical limits on inventory at both stages. For the remainder of this section, we assume that the demand $D \in [0, \widehat{D}]$. (The assumption that the minimum value of $D = 0$ is not necessary, but clarifies the exposition.) Since the demand distribution is logconcave, the random variable D has a positive probability of taking on all values in $[0, \widehat{D}]$. We also assume that the inventory positions for the centralized model, x_1 and $\bar{x}_2$, have finite state spaces with $-\widehat{D} \le x_1 \le \widehat{x_1}$ and $0 \le \bar{x}_2 \le \widehat{\bar{x}_2}$, where $\widehat{x_1}$ and $\widehat{\bar{x}_2}$ are large numbers. The lower bound on x_1 is $-\widehat{D}$ for ease of exposition; this lower bound could be any negative number and the analysis would still hold with a slight modification of the policy π. The lower bound on $\bar{x}_2$ is 0 since backorders are not allowed at stage 2.

We now reconsider the centralized model with $\alpha = 1$. Since we have restricted our attention to finite state spaces, the average cost exists by Proposition 1.1 of Bertsekas ([7], page 187). We next use Proposition 2.6 from Bertsekas ([7], page 198) to show that the optimal average cost per period $f^{\star}_{cen}$ satisfies Bellman's equation and then show that similarly structured inventory policies as those from Chapter III minimize the average cost case. Proposition 2.6 of [7] states the following: Let $i_1, j_1 \in [-\widehat{D}, \widehat{x_1}]$ and $i_2, j_2 \in [0, \widehat{\bar{x}_2}]$. If for every two inventory positions (i_1, i_2) and (j_1, j_2), there exists a stationary policy π such that, for some k, $P((x_{1,k}, \bar{x}_{2,k}) = (j_1, j_2)|(x_{1,0}, \bar{x}_{2,0}) = (i_1, i_2), \pi) > 0$, then $f^{\star}_{cen}$ has the same value for all initial states, $f^{\star}_{cen} = \lim_{\alpha \to 1}(1-\alpha) f^{*}_{cen}(i_1, i_2)$, and $f^{\star}_{cen}$ satisfies Bellman's equation.

Consider the following stationary policy π:

$$\pi(x_1, \bar{x}_2) = \begin{cases} (x_1, \bar{x}_2) & \text{if } x_1 \geq 0 \\ (\widehat{x_1}, \bar{x}_2 - 1) & \text{if } x_1 < 0 \text{ and } \bar{x}_2 > 0 \\ (\widehat{x_1}, \widehat{\bar{x}_2}) & \text{if } x_1 < 0 \text{ and } \bar{x}_2 = 0 \end{cases}$$

The policy works as follows: If the inventory at stage 1 is non-negative, the policy π keeps both values the same, but then the stage 1 inventory decreases by a random amount D, and the inventory at the start of the next period is $(x_1 - D, \bar{x}_2)$. Eventually, the stage 1 inventory goes negative, at which point stage 1 places a large order up to $\widehat{x_1}$, stage 2 experiences demand $\widehat{x_1} - x_1$ from stage 1, then stage 2 reorders up to its previous inventory level minus 1 unit. The stage 1 inventory quantity rapidly cycles from its upper bound to its lower bound, while the stage 2 inventory quantity slowly cycles by 1 unit each time stage 1 completes a cycle.

The probability that $D = 1$ for the first $(\widehat{x_1}+1)\widehat{\bar{x}_2}$ periods is small, but positive. Thus, every possible value of $\bar{x}_2$ has a positive probability of being paired up with every non-negative value of x_1, and with $x_1 = -1$. There is also a positive probability that for the next $(\widehat{x_1}+1)\widehat{\bar{x}_2}$ periods, $D = 1$ every period, except when $x_1 = 0$, when $D = 2$. Thus, every possible value of $\bar{x}_2$ has a positive probability of being paired up with $x_1 = -2$. These small but positive probabilities continue through the case where when $x_1 = 0$, $D = \widehat{D}$. At this point, after $k = (\widehat{x_1}+1)\widehat{\bar{x}_2}\widehat{D}$ periods, there is a positive probability of reaching every possible inventory position. Thus, Proposition 2.6 of [7] holds.

From Proposition 2.6 of [7], we know that $f^\star_{cen}$ satisfies Bellman's equation. We now need to show that the structure of the optimal policies for the average cost case is the same as the structure for the discounted cost case, or in other words, we need to show that the results from Chapter III do not depend on the fact that $0 < \alpha < 1$. Following Section 3.1, we define $g_{cen}(\cdot)$ and $f^\star_{cen}(x_1, \bar{x}_2)$ the same way, just with $\alpha = 1$. We can move the $-c_1 x_1$ term back as $-c_1(y_1 - D)$ by the same reasoning, and we substitute the same system variables as before. We make the same relaxations and get that

$$
\begin{aligned}
g_{cen,r}(x_s, y_1, y_s, D) \quad = \quad & c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^- \\
& + K_o \delta(y_1 - x_s) + (c_o - c_2)(y_1 - x_s)^+ + h_2(x_s - y_1)^+ \\
& + c_2(y_s - x_s)
\end{aligned}
$$

with $y_s \geq x_s$. We define $f^\star_{cen,r}(x_s)$ the same way, and note that since $x_s = x_1 + \bar{x}_2$, Proposition 2.6 of [7] still holds, since we have shown that that we we can reach every inventory pair, we can surely reach the sum of every pair. We move the minimization over y_1 inside the minimization over y_s for the same reason, defining the functions $m(x_s)$ and $m'(x_s)$ for the average cost case.

In Section 3.2, the definitions of $N_L(y_1)$, $N_H(y_1)$, and $N(x_s)$ do not change (they actually simplify since $\alpha = 1$) and our study of $m(x_s)$ remains the same. Theorem 3 still holds and the structure of the optimal policy for stage 1 is the same. In Section 3.3, we define $f^{\star}_{cen,s}(x_s)$ as before and Lemma 11 still holds. Theorem 4 continues to hold because $G_{cen,s}(y_s)$ is only different by the factor of α, and the base-stock result again holds from Zheng [66], only this time from Section 4 rather than Section 3. Finally, Theorem 5 in Section 3.4 holds by the same logic, and thus the structure of the optimal policies for the average cost case is the same as for the discounted cost case. Using Proposition 2.6 of [7], we get that:

$$
\begin{aligned}
f^{\star}_{cen} &= \lim_{\alpha \to 1} (1-\alpha) f^{*}_{cen}(x_1, \bar{x}_2) \\
&= E_D[m'(S^{\star}_{cen} - D)] + (c_1 + c_2)\mu
\end{aligned}
$$

At this point, it is interesting to compare the average cost for the system under centralized control and under decentralized control. Under centralized control, the average cost for the system is

$$
\begin{aligned}
f^{\star}_{cen} &= (c_1 + c_2)\mu + E_D[m'(S^{\star}_{cen} - D)] \\
&= (c_1 + c_2)\mu + \\
&\quad E_D \min_{y_1} \left\{ \begin{array}{l} h_1(y_1 - D)^+ + b_1(y_1 - D)^- + K_o \delta(y_1 - (S^{\star}_{cen} - D)) + \\ (c_o - c_2)(y_1 - (S^{\star}_{cen} - D))^+ + h_2((S^{\star}_{cen} - D) - y_1)^+ \end{array} \right\}.
\end{aligned}
$$

Under decentralized control, the average cost for the system is

$$\begin{aligned}
f^\star_{1,dec} + f^\star_{2,dec} &= G_{1,dec}(S^\star_{1,dec}) + G_{2,dec}(S^\star_{2,dec}) \\
&= c_1\mu + \min_{y_1} E_D[h_1(y_1-D)^+ + b_1(y_1-D)^-] + \\
&\quad c_2\mu + \min_{y_2} E_D[K_o\delta((y_2-D)^-) + (c_o-c_2)(y_2-D)^- + h_2(y_2-D)^+] \\
&= (c_1+c_2)\mu + \\
&\quad \min_{y_1} E_D[h_1(y_1-D)^+ + b_1(y_1-D)^-] + \\
&\quad \min_{y_2} E_D[K_o\delta((y_2-D)^-) + (c_o-c_2)(y_2-D)^- + h_2(y_2-D)^+].
\end{aligned}$$

As one would expect, both systems pay the average production costs at both stages, plus each system minimizes inventory and overtime production costs. Under decentralized control, stage 1 independently minimizes its own holding and backordering costs and stage 2 independently minimizes its own holding and overtime production costs. Under centralized control, stage 1 minimizes all of these costs together, and hence potential savings exist for the centralized system.

We now reconsider the earlier sections of this chapter for the average cost model. We first consider Contract A from Section 4.2. We define $g_{1,CA}(x_1, x_s, y_1, D)$ and $f^\star_{1,CA}(x_1, x_s)$ the same as we did earlier in the chapter. Note that the inventory state for $f^\star_{1,CA}$ is (x_1, x_s), rather than $(x_1, \bar{x}_2)$ as earlier in this section. However, since $x_s = x_1 + \bar{x}_2$, the possible inventory positions have a one-to-one correlation and Proposition 2.6 of [7] still holds. Following the rest of Section 4.2, we define the moved and relaxed average cost functions as before. Lemma 12 still holds because we first consider the myopic problem for stage 1 and then solve for the fully constrained problem exactly as in Chapter III. Thus, under Contract A, stage 1 follows the same kind of optimal policy as in the discounted cost case. We define $f^{\star}_{2,CA}(x_s)$ as before, and note that Proposition 2.6 of [7] holds because $x_s = x_1 + \bar{x}_2$. Stage 2 follows the centralized policy (because it's in the contract) and ensures that the system optimal base-stock level will be met. By Proposition 2.6 of [7],

$$\begin{aligned} f^{\star}_{1,CA} &= \lim_{\alpha \to 1}(1-\alpha) f^{*}_{1,CA}(x_1, x_s) \\ &= -W + E_D[m'(S^{\star}_{cen} - D)] + c_1\mu \end{aligned}$$

Similarly,

$$\begin{aligned} f^{\star}_{2,CA} &= \lim_{\alpha \to 1}(1-\alpha) f^{*}_{2,CA}(x_s) \\ &= W + c_2\mu \end{aligned}$$

In other words, under Contract A, stage 2 pays only its own production costs and the transfer payment. Stage 1 pays for every other cost in the system, and thus will require a significant transfer payment to enter into the contract.

Now consider Contract B. Following Section 4.3, we define $f^{\star}_{1,CB}(x_1, x_s)$, $f^{*}_{1,CB,m}(x_1, x_s)$, and $f^{\star}_{1,CB,r}(x_s)$ as before. For $f^{\star}_{1,CB,r}(x_s)$, we move the $m(x_s)$ term back as before; however, since $\alpha = 1$, the coefficient of $\alpha(1-\alpha)c_2 y_s$ is 0 and hence the two minimization problems are the same. Thus, Contract B yields the same results as Contract A and the centralized model for the average cost case. Under either contract, the transfer payment W has the following bounds:

$$E_D[m'(S^{\star}_{cen} - D)] - (G_{1,dec,m}(S^{\star}_{1,dec}) - c_1\mu) < W < G_{2,dec,m}(S^{\star}_{2,dec}) - c_2\mu.$$

The transfer payment is bounded below by the difference between the expected centralized system non-production costs and the expected decentralized stage 1 non-production costs, and is bounded above by the expected decentralized stage 2 non-production costs. In this section, we have shown that under the average cost criterion, our previous results hold (assuming only one method of expediting) and that both contracts achieve system optimality.

4.6 Contracts with Premium Freight

In this section we include premium freight as an expediting option and reconsider our contracts. We show that a combined cost function similar to the one described in Section 4.1, with an additional payment choice for stage 1, will achieve system optimality. We do not specify a particular contract (A or B) because the analysis is the same for both.

The combined cost consists of the transfer payment W and the wholesale cost that stage 2 will charge stage 1. For this case, the wholesale cost will be $h_2(x_s - y_1)$ if $y_1 \leq x_s$, as before. However, if stage 1 decides to order more than the system has available, stage 1 will actually choose from a menu of two different costs, either $\alpha K_p + \alpha c_p(y_1 - x_s)$ or $K_o + (c_o - \alpha c_2)(y_1 - x_s)$. At first glance, the choice may seem obvious: pick the minimum of the two, depending on the values of y_1 and x_s. However, this choice may not be optimal. When stage 1 chooses to force expediting at stage 2, the optimal value of y_1 will be either y_M or y_L (refer to Section 3.5). These two values depend on the slopes of the functions to be minimized, which depend on the coefficients of y_1 in the choices above.

We now show why picking the minimum of the two costs, without considering other costs, may not be optimal. Define

$$w_m(y_1 - x_s) = -W + \begin{cases} h_2(x_s - y_1) & \text{if } y_1 \leq x_s \\ \min \left\{ \begin{array}{l} \alpha K_p + \alpha c_p(y_1 - x_s) \\ K_o + (c_o - \alpha c_2)(y_1 - x_s) \end{array} \right\} & \text{if } y_1 > x_s \end{cases}$$

The analysis of the contract turns out to be the same as in Section 4.2 up until Lemma 12. Now note that we can rewrite

$$w_m(y_1 - x_s) = -W + \begin{cases} h_2(x_s - y_1) & \text{if } y_1 \leq x_s \\ \alpha K_p + \alpha c_p(y_1 - x_s) & \text{if } 0 > y_1 - x_s \geq t_{min} \\ K_o + (c_o - \alpha c_2)(y_1 - x_s) & \text{if } y_1 - x_s < t_{min} \end{cases}$$

where $t_{min} \equiv \frac{K_0 - \alpha K_p}{c_o - \alpha(c_p + c_2)}$. Following the steps of Lemma 12,

$$\begin{aligned}
&\min_{y_1} E_D[g_{1,C,r}(x_s, y_1, D)] \\
&= \min_{y_1} \quad E_D[\alpha((1-\alpha)c_1 y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w_m(y_1 - x_s)] \\
&= \min_{y_1} \quad \{\alpha(1-\alpha)c_1 y_1 + E_D[\alpha^2 c_1 D + \alpha h_1(y_1 - D)^+ + \alpha b_1(y_1 - D)^-] - \\
&\qquad W + h_2(x_s - y_1)^+ + (\alpha K_p + c_p(y_1 - x_s))1(0 > y_1 - x_s \geq t_{min}) + \\
&\qquad (K_o + (c_o - \alpha c_2)(y_1 - x_s))1(y_1 - x_s < t_{min})\} \\
&= \min_{y_1} \quad \{\alpha(1-\alpha)c_1 y_1 + E_D[\alpha^2 c_1 D + \alpha h_1(y_1 - D)^+ + \alpha b_1(y_1 - D)^-] \\
&\qquad h_2(x_s - y_1)^+ + (\alpha K_p + c_p(y_1 - x_s))1(0 > y_1 - x_s \geq t_{min}) + \\
&\qquad (K_o + (c_o - \alpha c_2)(y_1 - x_s))1(y_1 - x_s < t_{min})\} - W \\
&\neq \qquad m(x_s) - W.
\end{aligned}$$

Referring to Section 3.5, we see that by minimizing too early, we have limited the options that stage 1 has. Note that the real threshold between premium freight and overtime production should be t_L, as defined in equation (3.13), not t_{min} as defined above. The combined cost $w_m(y_1 - x_s)$ does not induce stage 1 to follow the centralized optimal policy.

Now redefine the wholesale cost as below, where the "choice" indicates that stage 1 may choose either of the two costs if $y_1 > x_s$.

$$w_c(y_1 - x_s) \quad = \quad -W + \begin{cases} h_2(x_s - y_1) & \text{if } y_1 \leq x_s \\ \text{choice} \left\{ \begin{array}{l} \alpha K_p + \alpha c_p(y_1 - x_s) \\ K_o + (c_o - \alpha c_2)(y_1 - x_s) \end{array} \right\} & \text{if } y_1 > x_s \end{cases}$$

Note that we could write $w_c(y_1 - x_s)$ without the choice, but it would depend on both the difference $y_1 - x_s$ and the actual value of x_s. Again following Lemma 12,

$$\begin{aligned}
&\min_{y_1} E_D[g_{1,C,r}(x_s, y_1, D)] \\
&= \min\nolimits_{y_1} \quad E_D[\alpha((1-\alpha)c_1 y_1 + \alpha c_1 D + h_1(y_1 - D)^+ + b_1(y_1 - D)^-) + w_c(y_1 - x_s)] \\
&= \min\nolimits_{y_1} \quad \{\alpha(1-\alpha)c_1 y_1 + E_D[\alpha^2 c_1 D + \alpha h_1(y_1 - D)^+ + \alpha b_1(y_1 - D)^-] - \\
&\qquad W + h_2(x_s - y_1)^+ + \{(\alpha K_p \delta(y_1 - x_s) + \alpha c_p(y_1 - x_s)^+), \\
&\qquad (K_o \delta(y_1 - x_s) + (c_o - \alpha c_2)(y_1 - x_s)^+)\}\} \\
&= \min\nolimits_{y_1} \quad \{\alpha(1-\alpha)c_1 y_1 + E_D[\alpha^2 c_1 D + \alpha h_1(y_1 - D)^+ + \alpha b_1(y_1 - D)^-] \\
&\qquad + h_2(x_s - y_1)^+ + \{(\alpha K_p \delta(y_1 - x_s) + \alpha c_p(y_1 - x_s)^+), \\
&\qquad (K_o \delta(y_1 - x_s) + (c_o - \alpha c_2)(y_1 - x_s)^+)\}\} - W \\
&= \min \quad \{(h_2 x_s + \min_{y_1 \leq x_s}\{N_H(y_1)\}), \\
&\qquad (\alpha K_p - \alpha c_p x_s + \min_{y_1 > x_s}\{N_M(y_1)\}), \\
&\qquad (K_o - (c_o - \alpha c_2)x_s + \min_{y_1 > x_s}\{N_L(y_1)\})\} - W
\end{aligned}$$

$$= \quad m(x_s) - W.$$

Thus, the combined cost $w_c(y_1 - x_s)$ does induce stage 1 to follow the centralized optimal policy when both premium freight and overtime production exist as expediting options. By giving stage 1 the choice of which payment to make, not only do we achieve optimality, but we also help convince stage 1 to actually consider the contract by offering more flexibility in payments. The rest of the analysis for this case follows exactly as in Sections 4.2, 4.3, and 4.4.

4.7 Conclusion and Insights

In this chapter, we have considered two contracts that will induce system optimality (or near-optimality). In both contracts, stage 1 pays a two-tiered wholesale cost to stage 2, and stage 2 makes a linear transfer payment to stage 1. Under Contract A, the two stages achieve system optimality, but Contract A requires that stage 2 has prior knowledge of the optimal system base-stock level. Under Contract B, stage 1 achieves centralized optimality, the two stages together follow the right kind of policy for the system, but the actual system inventory may be too high. For each stage to be interested in a contract, the transfer payment W will have to be within certain boundaries; above one threshold for stage 1 and below another threshold for stage 2. Using the average cost criterion, we showed that the structure of the optimal policies are the same as for the discounted cost case and then showed that Contract B will achieve system optimality. Finally, when premium freight is included, an interesting three-tiered wholesale cost with a choice will induce stage 1 to behave as in the centralized model.

The main insight from this chapter is that it is possible for the two stages to achieve system optimal (or near-optimal) performance by working together under one of the two contracts. Because Contract A requires prior knowledge of optimal system parameters, it would be more likely to work as a coordinating scheme between managers at two stages that are both part of the same firm. On the other hand, for the manager at stage 2 to propose Contract B, that manager must only know the various costs associated with stage 2 operations. Hence, Contract B is more likely to work between two independent firms than Contract A. In either contract, the wholesale cost is pre-determined, but the transfer payment W is negotiable. There exists a range of W, and we feel that somewhere within this range the two stages can agree on a value that benefits both stages enough so that it is worthwhile to follow the contractual policies, rather than the decentralized base-stock policies.

CHAPTER V

NUMERICAL ANALYSIS

In this chapter, we perform a numerical analysis. We first discuss our numerical experiment and then use the results of the experiment in the following sections. In Section 5.1, we compare costs under centralized and decentralized control. We show that the savings of the centralized model are significant, particularly if the demand variance is high, if the holding costs are high, or if the setup cost for overtime production K_o is large. In Section 5.2, we compare costs under Contract B to the centralized costs. We show that in general, Contract B results in very near-optimal system performance. In Section 5.3, we study two example problems to see how the centralized model saves costs over the decentralized model and to gain insight about the transfer payment W under both contracts. We conclude the chapter in Section 5.4.

We performed numerical experiments written in C++ code, under the assumption that overtime production is the only method of expediting in order to reduce the total number of variables. During each experiment, we assign various values for each parameter except the cost of production at stage 1, which we arbitrarily set to

$c_1 = 10$. For the rest of the parameters, we let:

$h_1 = 1, 2, 4, 8,$

$b_1 = 10, 20, 40,$

$c_2 = 3, 5, 7,$

$h_2 = 0.5, 1, 2, 4,$

$c_o = 4, 6, 10,$

$K_o = 0, 50, 200,$ and

$\alpha = 0.90, 0.95, 0.99.$

The holding costs are relatively high in order to represent a lean inventory environment and we chose the α values as 0.99 or less because for values of $\alpha > 0.99$, the numerical results were very similar but took longer to obtain. These variations lead to a total of $3^5 4^2 = 3888$ possible combinations. However, recall that we require that the per unit cost of overtime production is greater than the per unit cost of regular production at stage 2, the backordering cost at stage 1 is not too small, and the holding cost at stage 2 is not more than the holding cost at stage 1. The combinations that violate assumption (A4), (A8), or (A9) are not included in the experimental results.

We compare the results for four different demand distributions: Poisson(mean), Uniform(lower bound, upper bound), Normal(mean, standard deviation), and Exponential(mean). Since we consider discrete demand, we use discrete approximations for the last three distributions. Also, we truncate each distribution below at zero and above at forty-nine to fit into our probability array, and adjust the probabilities appropriately to ensure the total probability is one. We use a relatively small probability array in order to decrease computation times, but we feel that the array is still large enough to reflect realistic outcomes. In reality, each unit in the array may represent a batch of 10, 100, or even 1000 units.

5.1 Comparison of Centralized and Decentralized Policies

To compare the centralized and decentralized models, we made several calculations for each combination. Under decentralized control, we calculated the optimal base-stock levels and the expected total discounted costs. For the centralized case, we first calculated the optimal inventory control parameters for stage 1: t_L, y_L, and y_H. Using these parameters, we calculated the system base-stock level S^*_{cen} and the total expected discounted cost for the system. Finally, for each combination we calculated two statistics comparing the centralized case and the decentralized case: the percentage savings in total costs and the percentage reduction in system inventory. The percentage savings in total costs is the difference between the sum of the decentralized costs and the centralized cost, divided by the sum of the decentralized costs. The inventory reduction percentage is determined by $\frac{S^*_{1,dec}+S^*_{2,dec}-S^*_{cen}}{S^*_{1,dec}+S^*_{2,dec}}$. For several different distributions, we averaged these savings over all feasible combinations and we determined the maximum total savings out of all the trials; the results are in Table 5.1 below.

Note that we ran the experiment for constant demand (Uniform(25,25)) as one way to check the accuracy of our computer code. Our first observation from the

Demand Distribution	Average Savings	Maximum Savings	Inv. Reduction
Uniform(25,25)	0.00%	0.00%	0.00%
Normal(25,1)	0.47%	2.30%	1.74%
Normal(25,5)	1.86%	9.24%	6.31%
Normal(25,10)	2.72%	13.58%	8.56%
Poisson(25)	1.94%	9.60%	6.52%
Uniform(0,49)	2.81%	16.43%	9.92%
Exponential(15)	5.43%	23.25%	13.17%

Table 5.1: Average Savings of the Centralized Optimal Policy

data is that in general, the centralized savings are somewhat attractive, around 2% or 3% of the total cost. We feel that this is a significant enough value to make the centralized policy worth considering. A second observation is that as demand variance increases, so do the total savings. This trend is particularly evident by looking at the Normal distribution results. Clearly, as demand variance increases, it becomes more difficult to manage inventory and the centralized policy adds more flexibility by allowing stage 1 to underorder. A third observation is that in general, system inventory levels are reduced by more than 5% under the centralized policy. It turns out that leaner inventory is a pleasant by-product of the centralized policy.

After comparing results for the different parameters, we found that the parameters that had the greatest effect on total cost savings were the overtime production cost, K_o, and the holding costs, h_1 and h_2. For the Poisson(25) distribution, we

Poisson(25) with:	Average Savings	Maximum Savings	Inv. Reduction
$K_o = 0$	0.59%	3.54%	-0.39%
$K_o = 50$	2.00%	6.60%	6.78%
$K_o = 200$	3.22%	9.60%	11.42%
$h_1 = 1, 2; h_2 = 0.5, 1$	1.39%	3.74%	7.78%
$h_1 = 4, 8; h_2 = 2, 4$	3.17%	9.60%	5.35%

Table 5.2: Savings Depend on Overtime Setup Costs and Holding Costs

broke down the various savings by the three different values of overtime production, and we broke down the costs into the low values of holding and the high values of holding for both stages. The results are in Table 5.2. Note that for $K_o = 0$, the total cost savings are only about half a percent, and in fact, the average system inventory increases under centralized control. These two results indicate that if stage 2 does not pay a setup cost for overtime production, it is probably not worthwhile considering a contract that would induce the centralized results. On the other hand, when K_o increases to 50 or 200, the opposite is true. When the overtime production setup cost is large, then the centralized policies save a significant amount. From the last two rows of Table 5.2, notice that when the holding costs are high at both stages, the average savings more than doubles compared to when the holding costs are low. Higher holding costs force stage 1 to increase the risk of backorders and stage 2 to increase the risk of overtime production, but these risks can be better managed under centralized control.

5.2 Comparison of Contract B and Centralized Policies

In this section, we show that Contract B from Chapter IV nearly achieves system optimality. As in the previous section, we calculated all the inventory control parameters and total costs for the centralized and decentralized models. For Contract B we calculated the system base-stock level S^*_{CB} and the total costs experienced by both stages. We calculated three different statistics comparing the total costs of the centralized, decentralized, and Contract B coordinated models. The first statistic is the same savings in total costs of the centralized model compared to the decentralized model as before. The second statistic is the savings in total costs of Contract B compared to the decentralized model. The third statistic is the savings in total cost of the centralized model compared to Contract B. We performed the experiment for the same demand distributions as before and the results are in Table 5.3

Demand Distribution	Centralized vs. Decentralized	Contract B vs. Decentralized	Centralized vs. Contract B
Normal(25,1)	0.47%	0.46%	0.0078%
Normal(25,5)	1.86%	1.82%	0.034%
Normal(25,10)	2.72%	2.66%	0.060%
Poisson(25)	1.94%	1.90%	0.035%
Uniform(0,49)	2.81%	2.73%	0.084%
Exponential(15)	5.43%	5.27%	0.16%

Table 5.3: Comparison of Contract B

Although Contract B may not be optimal, it is extremely close! The savings of Contract B over the decentralized model are just as significant as the savings of the centralized model. The two models differ by less than two tenths of one percent. Clearly the value of α will affect how close they are. As mentioned in the previous chapter, as $\alpha \uparrow 1$, the minimization problems for Contract B and the centralized model approach one another. To investigate the effect of α, we reconsidered the Poisson(25) distribution and broke it down into results for the three different values of the discount factor. The total costs do not actually vary by that much, but we compared the values of S^*_{cen} and S^*_{CB} for each α.

When $\alpha = 0.90$, the average value of $S^*_{cen} = 56.07$ and the average value of $S^*_{CB} = 57.44$. Under Contract B, the system base-stock level is 2.4% larger, on average. Also, $S^*_{CB} = S^*_{cen}$ in only 12% of the experimental trials. When α increases to 0.95, the average value of $S^*_{cen} = 57.16$ and the average value of $S^*_{CB} = 57.88$. The system base-stock level under Contract B is 1.3% larger on average and the two base-stock levels are equal in 37% of the trials. Finally, for $\alpha = 0.99$, the average value of $S^*_{cen} = 58.12$ and the average value of $S^*_{CB} = 58.31$. The system base-stock level under Contract B is only 0.033% larger on average and $S^*_{CB} = S^*_{cen}$ in 80% of the trials. Overall, Contract B sets a system base-stock level that is 1.3% larger, on average, than the centralized model, but the difference in total costs is much less, around 0.1%.

5.3 Two Examples

In this section, we look at two numerical examples to gain insight into centralized savings and appropriate values for W. We consider a "typical" example with Poisson demand that reflects results that are common to most of our experimental outcomes. We also consider an "atypical" example with Exponential demand that has uncommon characteristics, compared to the majority of our experimental outcomes.

As a typical example, consider a problem that has Poisson demand with mean 25. The per unit costs at stage 1 are 10 for production, 2 for holding, and 20 for backorders. The per unit costs at stage 2 are 5 for production, 1 for holding, and 6 for overtime production; the fixed cost for overtime production is 200. The discount factor is 0.95. Under decentralized control, the optimal policy at stage 1 is to order up to a base-stock level of 31 and the optimal policy at stage 2 is to order up to a base-stock level of 36, for a total system inventory of 67. Under centralized control, the optimal policy at stage 1 is to order up to 30 if the system inventory is 11 or less, use up the available system inventory if the system inventory is between 12 and 32, and order up to 33 if the system inventory is 33 or more. The optimal system base-stock level is 59 and the optimal policy at stage 2 is to order the difference between the system inventory and the inventory at stage 1.

So, under the centralized policy, the system carries $67-59=8$ less units of inventory. This is an inventory reduction of 11.9%, which is one factor that contributes to cost savings. Another factor that contributes is how often stage 2 is forced to run overtime production. In the decentralized case, stage 2 must run overtime when demand is greater than 36; the probability that $D > 36$ is 0.015. In the centralized case, stage 2 must run overtime when the system inventory after demand is less than the low threshold, or when $59 - D < 12$. The probability that stage 2 must run overtime equals the probability that $D > 47$, which is 0.000021. So, for this particular example, stage 2 is over seven hundred times more likely to run overtime production under decentralized control! These two factors lead to a total cost savings of 2.43%. These results are typical for the majority of our experimental outcomes.

The two stages can achieve these savings by working together under either Contract A or Contract B. Under Contract A, stage 1 and stage 2 will negotiate the value of W_A in order to determine how to split up the 2.43% savings on total system costs. Using equations (4.1) and (4.2) and making a reasonable assumption about the initial value of the system inventory , we calculate that the range for the transfer payment is $6.37 < W_A < 15.89$. Stage 2 will negotiate for a low value of W_A, receiving savings up to 4.13% as W_A approaches the low threshold. Stage 1 will negotiate for a high value of W_A, receiving savings up to 3.73% as W_A approaches the high threshold. A fair negotiation would likely end up with a W_A where both stages each save around the system savings of 2.43%. Under Contract B, the system base-stock level is 60, one unit higher than under centralized control, and the savings on total system costs is 2.42%. Again, the two stages will negotiate a value of W_B, this time with $6.19 < W_B < 15.66$.

Now we consider an atypical example that has Exponential demand with mean

15. The per unit costs at stage 1 are 10 for production, 4 for holding, and 20 for backorders. The per unit costs at stage 2 are 7 for production, 4 for holding, and 10 for overtime production. There is no fixed cost for overtime production and the discount factor is 0.99. Under decentralized control, the optimal policy at stage 1 is to order up to a base-stock level of 24 and the optimal policy at stage 2 is to order up to a base-stock level of 7. Note that the decentralized system carries a total of 31 units. Under centralized control, the optimal policy at stage 1 is to order up to 16 if the system inventory is 16 or less and to use up the available system inventory if the system inventory is 17 or more. Stage 1 will never order up to it's high inventory level of 48, because that value is greater than the optimal base-stock level for the system inventory, which is 35.

So, under the centralized policy, the system actually carries 35 - 31 = 4 more units of inventory than under the decentralized policy, or an inventory **increase** of 12.9%. However, there is still a significant savings in total cost for the centralized policy, 4.06%, despite the increase in system inventory. The reason for the savings is that under decentralized control, stage 2 uses overtime production almost 57% of the time, compared to 24% of the time under centralized control. Even though there is no fixed cost for overtime production, stage 2 is paying the overtime cost of 10 per unit very often. In this case, both Contract A and Contract B set a system base-stock level of 35. Under either contract, the two stages can negotiate the transfer payment in order to share the 4.06% savings, with $15.44 < W < 28.75$.

5.4 Conclusion

In this chapter, we described various numerical experiments. In Section 5.1, we showed that in general, the centralized optimal policies result in reasonable savings over the decentralized optimal policies, somewhere in the neighborhood of 2% or 3%. The centralized optimal policies also tend to (but do not always) decrease the system inventory. We discussed three factors that affect these savings: demand variation, the setup cost for overtime production, and holding costs. As demand variation increases, so do the savings. Similarly, as the setup cost K_o increases, savings are increased as the centralized model allows for stage 1 to underorder to avoid forcing overtime production at stage 2. When we assume that the holding costs at both stages are high, savings are increased under centralized control, as well. In Section 5.2, we showed that in general, Contract B produces results that are very near optimal. The discount factor α has a strong effect on these results, and as $\alpha \uparrow 1$, Contract B is more likely to determine the appropriate value for the system base-stock level. Finally, in the last section, we discussed two numerical examples. The first example is typical, showing that the savings of the centralized policy come from lower inventories and decreasing the likelihood of overtime production. The second example was less typical, where the inventory actually increases under decentralized control, but a significant savings still exists on the total cost.

CHAPTER VI

CONCLUSION

In this thesis, we have studied a two-stage supply chain where the upstream stage always meets demand from the downstream stage, using two different forms of expediting when necessary. Our work models the interaction between two stages of an automotive supply chain, and although our models are simplified versions of reality, we feel that our theoretical results have practical value. In Chapter II, we showed that the choice between overtime production and premium freight depends on the size of the shortage and the relative costs of the two expediting methods, and we showed that standard inventory control policies are optimal for regular production. In Chapter III, we showed that under centralized control, the downstream stage needs to be sensitive to the amount of inventory available upstream and order accordingly, and the upstream stage must ensure the system inventory level is optimal. In Chapter IV, we discussed contracts that coordinate the system and allow both stages to share in the centralized savings. Finally, in Chapter V, we showed that the centralized system affects reasonable savings over the decentralized system, and hence it may be worthwhile for the two stages to work together.

We have analyzed the supply chain under three forms of control: decentralized, centralized, and coordinated. Under decentralized control, both stages minimize their own costs and follow base-stock policies (or an (s, S) policy at stage 2 when we include a production setup cost). Under centralized control, the optimal policies for both stages depend on the amount of system inventory available, and the optimal policy for the system is a base-stock policy. Under coordinated control, the two stages achieve system optimality following a contract with a multi-tiered wholesale cost and a linear transfer payment. We compared the different forms of control at the end of the thesis using numerical analysis.

Throughout this thesis, we have used various techniques to prove our results. In some cases, we simply applied standard techniques from the inventory control literature. However, in other cases, we had to develop novel ways to prove our results. In Section 2.3, we faced a problem with two separate inventory control decisions, one occurring before random demand was realized, and one afterwards. We solved the problem by defining overtime- and regular-periods, then showing how the two periods were related. We were then able to optimize the overtime production decision and the regular production decision, in that order. In Chapter III, we were unable to decouple our two-stage problem, a standard technique. Instead, we solved the problem by first substituting system variables for stage 2 variables and then relaxing two of our constraints on inventory levels, making the stage 1 decision a myopic problem. Later, we showed that the solution to the relaxed problem also solved the fully constrained problem. Lastly, due to our multi-tiered inventory control policies in the centralized model, in Chapter IV we developed contracts with multi-tiered wholesale costs that allowed us to coordinate the two stages.

Numerous extensions to our model exist and are worthy of further study. These include, but are not limited to, positive lead times, capacity constraints, demand forecasting, the N-stage supply chain, multiple suppliers, multiple assemblers, and substitutable products. Clearly, our model would be much more robust if we included the possibility of positive lead times for either regular or premium freight shipments. Unfortunately, a review of the literature (see Section 1.2) shows that analytical results would be difficult, if not impossible, to determine under this assumption. However, we may be able to develop reasonably efficient heuristic policies based on our current results and to test these policies using simulation. Another realistic assumption would be to include capacity constraints on either regular production, overtime production, or premium freight shipments. It would be interesting to see how a capacity constraint on just one of the expediting methods affects how the two different methods are utilized. Note that if both methods of expediting have capacities, we can not guarantee the 100% fill rate at stage 2. For examples of papers with capacity constraints, see [22] and [26].

In reality, both stages of the supply chain will probably have demand forecasts that may or may not be accurate. When we originally spoke to two managers at Visteon, they explained that they receive demand forecasts from Ford varying in length from six months to two weeks. However, the forecasts rarely match the actual demand and hence we include stochastic demand in our model. It would be interesting to study how the changing forecasts affect production, and to see how much better forecasting would save the supply chain. For a recent review of demand forecasting, see [6]. In this thesis, we have limited our scope to a two-stage supply chain. We could expand our model to an N-stage supply chain, but we would have to carefully reconsider our assumptions about overtime production and premium freight. For example, we assume that products shipped by premium freight are produced very early in the day and shipped instantaneously, arriving downstream to be used the same day. Is it realistic to assume these shipments can occur twice or more in the same day and still be on time?

It may be worthwhile to consider our model with multiple suppliers, multiple assemblers, or both. Originally, Visteon made nearly all its parts for Ford, but Visteon has since branched out and now sells parts to several different companies. How multiple assemblers would affect the overtime production decision is an interesting question. See [1], [9], and [44] for examples of systems with one supplier and multiple retailers. Finally, the automotive supply chain clearly involves more than one part, and in fact, many of the parts may be substitutable. A study of our model with multiple, substitutable parts may yield interesting results concerning when to substitute and when to expedite.

APPENDIX

APPENDIX

Case 2 from Section 2.3

In this case, we have the following stationary policy μ_2, which happens to be a $(-1, \tilde{S})$ policy for overtime production:

$$\mu_2 = \begin{cases} \tilde{x} \geq 0 \quad \rightarrow \quad \tilde{y} = \tilde{x}, y = y^*_{\tilde{x}} \\ \tilde{x} < 0 \quad \rightarrow \quad \tilde{y} = \tilde{S}, y = \begin{cases} y^*_0 & \text{if } \tilde{S} = 0 \\ \tilde{y}^*_+ & \text{if } \tilde{S} = \tilde{y}^*_+. \end{cases} \end{cases}$$

From equations (2.8) and (2.9), we have also that $\tilde{f}^*(\tilde{x}) =$

$$\begin{cases} (h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y^*_{\tilde{x}} - \tilde{x}) + \alpha c_2 y^*_{\tilde{x}} + \alpha E_D[\tilde{f}^*(y^*_{\tilde{x}} - D)] & \text{if } \tilde{x} \geq 0 \\ \begin{cases} K_o - c_o\tilde{x} + \alpha K_2 \delta(y^*_0) + \alpha c_2 y^*_0 + \alpha E_D[\tilde{f}^*(y^*_0 - D)] & \text{if } \tilde{S} = 0 \\ K_o - c_o\tilde{x} + (c_o + h_2)\tilde{y}^*_+ + \alpha E_D[\tilde{f}^*(\tilde{y}^*_+ - D)] & \text{if } \tilde{S} = \tilde{y}^*_+ \end{cases} & \text{if } \tilde{x} < 0. \end{cases}$$

Also, plugging μ_2 into equation (2.6) we get:

$$\tilde{f}^*(\tilde{x}) = \begin{cases} (h_2 - \alpha c_2)\tilde{x} + \alpha f^*_m(\tilde{x}) & \text{if } \tilde{x} \geq 0 \\ K_o - c_o\tilde{x} + (c_o + h_2 - \alpha c_2)\tilde{S} + \alpha f^*_m(\tilde{S}) & \text{if } \tilde{x} < 0. \end{cases}$$

We get $G_2(y) \equiv c_2 y + \sum_{d=0}^{y}(h_2 - \alpha c_2)(y - d)p_d + \sum_{d=y+1}^{\infty}(K_{all} - c_o(y - d))\, p_d$ and following the proof for case 1, we show that $G_2(y) \rightarrow \infty$ as $|y| \rightarrow \infty$, $-G_2(y)$ is unimodal and the minimum point $y^0 \geq 0$.

$$\begin{aligned}
\Delta G_2(y) &= c_2(y+1) - c_2 y + \sum_{d=0}^{y+1}(h_2 - \alpha c_2)(y+1-d)p_d - \sum_{d=0}^{y}(h_2 - \alpha c_2)(y-d)p_d \\
&\quad + \sum_{d=y+2}^{\infty} \left(K_{all} - c_o(y+1-d)\right) p_d - \sum_{d=y+1}^{\infty} \left(K_{all} - c_o(y-d)\right) p_d \\
&= c_2 + (h_2 - \alpha c_2)\sum_{d=0}^{y} p_d - (K_{all} + c_o)p_{y+1} - c_o \sum_{d=y+2}^{\infty} p_d \\
&= c_2 + (h_2 - \alpha c_2)F(y) - c_o(1 - F(y)) - K_{all}p_{y+1} \\
&= c_2 - c_o + (c_o + h_2 - \alpha c_2)F(y) - K_{all}p_{y+1}.
\end{aligned}$$

At this point, note that as $y \to -\infty$, $\Delta G_2(y) \to c_2 - c_o < 0$ and as $y \to \infty$, $\Delta G_2(y) \to h_2 + c_2(1-\alpha) > 0$. Thus, as $y \to -\infty$, $G_2(y) \to \infty$ and as $y \to \infty$, $G_2(y) \to \infty$. Also note that for $y < 0$,

$$\Delta G_2(y) = c_2 - c_o - K_{all}p_{y+1} < 0$$

by Lemma 6 and so any minimum point of $G_2(y)$, y^0, will be non-negative. Finally, for $y \geq d_0$, rewrite $\Delta G_2(y)$ as:

$$\Delta G_2(y) = c_2 - c_o + F(y)\left[(c_o + h_2 + \alpha c_p) - K_{all}\frac{p_{y+1}}{F(y)}\right].$$

Again, the terms in the brackets are all nondecreasing due to logconcavity. By the same argument as for case 1, $\Delta G_2(y)$ changes signs exactly once and $-G_2(y)$ is unimodal.

Case 3 from Section 2.3

In this case, we have the following stationary policy μ_3, which can be considered an $(\tilde{s},\tilde{S})$ policy for overtime production with $\tilde{s}=-\infty$:

$$\mu_3 = \begin{cases} \tilde{x} \geq 0 & \rightarrow \quad \tilde{y}=\tilde{x}, y=y^*_{\tilde{x}} \\ \tilde{x} < 0 & \rightarrow \quad \tilde{y}=\tilde{x}, y=y^*_p. \end{cases}$$

From equations (2.8) and (2.9), we have also that:

$$\tilde{f}^*(\tilde{x}) = \begin{cases} (h_2-\alpha c_2)\tilde{x} + \alpha K_2 \delta(y^*_{\tilde{x}}-\tilde{x}) + \alpha c_2 y^*_{\tilde{x}} + \alpha E_D[\tilde{f}^*(y^*_{\tilde{x}}-D)] & \text{if } \tilde{x} \geq 0 \\ \alpha K_p - \alpha(c_p+c_2)\tilde{x} + \alpha K_2 + \alpha c_2 y^*_p + \alpha E_D[\tilde{f}^*(y^*_p - D)] & \text{if } \tilde{x} < 0. \end{cases}$$

Also, plugging μ_3 into equation (2.6) we get:

$$\tilde{f}^*(\tilde{x}) = \begin{cases} (h_2-\alpha c_2)\tilde{x} + \alpha f^*_m(\tilde{x}) & \text{if } \tilde{x} \geq 0 \\ \alpha K_p - \alpha(c_p+c_2)\tilde{x} + \alpha f^*_m(\tilde{x}) & \text{if } \tilde{x} < 0. \end{cases}$$

For this case we get $G_3(y) \equiv c_2 y + \sum_{d=0}^{y}(h_2-\alpha c_2)(y-d)p_d + \sum_{d=y+1}^{\infty}(\alpha K_p - \alpha(c_p + c_2)(y-d))p_d$ and following the proof for case 1, we show that $G_3(y) \rightarrow \infty$ as $|y| \rightarrow \infty$, $-G_3(y)$ is unimodal and the minimum point $y^0 \geq 0$.

$$\begin{aligned}
\Delta G_3(y) &= c_2(y+1) - c_2 y + \sum_{d=0}^{y+1}(h_2-\alpha c_2)(y+1-d)p_d - \sum_{d=0}^{y}(h_2-\alpha c_2)(y-d)p_d \\
&\quad + \sum_{d=y+2}^{\infty}\left(\alpha K_p - \alpha(c_p+c_2)(y+1-d)\right)p_d \\
&\quad - \sum_{d=y+1}^{\infty}\left(\alpha K_p - \alpha(c_p+c_2)(y-d)\right)p_d \\
&= c_2 + (h_2-\alpha c_2)\sum_{d=0}^{y} p_d - (\alpha K_p + \alpha(c_p+c_2))p_{y+1} - \alpha(c_p+c_2)\sum_{d=y+2}^{\infty} p_d \\
&= c_2 + (h_2-\alpha c_2)F(y) - \alpha(c_p+c_2)(1-F(y)) - \alpha K_p p_{y+1} \\
&= c_2 - \alpha(c_p+c_2) + (h_2+\alpha c_p)F(y) - \alpha K_p p_{y+1}.
\end{aligned}$$

At this point, note that as $y \to -\infty$, $\Delta G_3(y) \to c_2 - \alpha(c_p + c_2) < 0$ by assumption (A7) and as $y \to \infty$, $\Delta G_3(y) \to h_2 + c_2(1-\alpha) > 0$. Thus, as $y \to -\infty$, $G_3(y) \to \infty$ and as $y \to \infty$, $G_3(y) \to \infty$. Also note that for $y < 0$,

$$\Delta G_3(y) = c_2 - \alpha(c_p + c_2) - \alpha K_p p_{y+1} < 0$$

and so any minimum point of $G_3(y)$, y^0, will be non-negative. Finally, for $y \geq d_0$, rewrite $\Delta G_3(y)$ as:

$$\Delta G_3(y) = c_2 - \alpha(c_p + c_2) + F(y)\left[(h_2 + \alpha c_p) - \alpha K_p \frac{p_{y+1}}{F(y)}\right].$$

Again, the terms in the brackets are all nondecreasing since demand is logconcave. By the same argument as for case 1, $\Delta G_3(y)$ changes signs only once and $-G_3(y)$ is unimodal.

Case 4 from Section 2.3

In this case, we have the following stationary policy μ_4, which is not an $(\tilde{s}, \tilde{S})$ policy for overtime production:

$$\mu_4 = \begin{cases} \tilde{x} \geq 0 & \rightarrow \quad \tilde{y} = \tilde{x}, y = y^*_{\tilde{x}} \\ \tilde{s} < \tilde{x} < 0 & \rightarrow \quad \tilde{y} = \tilde{S}, y = \begin{cases} y^*_0 & \text{if } \tilde{S} = 0 \\ \tilde{y}^*_+ & \text{if } \tilde{S} = \tilde{y}^*_+ \end{cases} \\ \tilde{x} < \tilde{s} & \rightarrow \quad \tilde{y} = \tilde{x}, y = y^*_p. \end{cases}$$

From equations (2.8) and (2.9), we have also that $\tilde{f}^*(\tilde{x}) =$

$$\begin{cases} (h_2 - \alpha c_2)\tilde{x} + \alpha K_2 \delta(y^*_{\tilde{x}} - \tilde{x}) + \alpha c_2 y^*_{\tilde{x}} + \alpha E_D[\tilde{f}^*(y^*_{\tilde{x}} - D)] & \text{if } \tilde{x} \geq 0 \\ \begin{cases} K_o - c_o\tilde{x} + \alpha K_2 \delta(y^*_0) + \alpha c_2 y^*_0 + \alpha E_D[\tilde{f}^*(y^*_0 - D)] & \text{if } \tilde{S} = 0 \\ K_o - c_o\tilde{x} + (c_o + h_2)\tilde{y}^*_+ + \alpha E_D[\tilde{f}^*(\tilde{y}^*_+ - D)] & \text{if } \tilde{S} = \tilde{y}^*_+ \end{cases} & \text{if } \tilde{s} < \tilde{x} < 0 \\ \alpha K_p - \alpha(c_p + c_2)\tilde{x} + \alpha K_2 + \alpha c_2 y^*_p + \alpha E_D[\tilde{f}^*(y^*_p - D)] & \text{if } \tilde{x} \leq \tilde{s}. \end{cases}$$

For this case, we get

$$\begin{aligned} G_4(y) \equiv \; & c_2 y + \sum_{d=0}^{y} (h_2 - \alpha c_2)(y-d)p_d + \sum_{d=y+1}^{y-\tilde{s}-1} \left(K_{all} - c_o(y-d)\right) p_d + \\ & \sum_{d=y-\tilde{s}}^{\infty} \left(\alpha K_p - \alpha(c_p + c_2)(y-d)\right) p_d \end{aligned}$$

and following the proof for case 1, we show that under the conditions of case 4 ($c_o > \alpha(c_p + c_2)$ and $C^*_{OT} < C^*_{PF}$), $G_4(y) \to \infty$ as $|y| \to \infty$, $-G_4(y)$ is unimodal and the minimum point $y^0 \geq 0$.

$$\begin{aligned} & \Delta G_4(y) \\ = \; & c_2(y+1) - c_2 y + \sum_{d=0}^{y+1} (h_2 - \alpha c_2)(y+1-d)p_d - \sum_{d=0}^{y} (h_2 - \alpha c_2)(y-d)p_d + \\ & \sum_{d=y+2}^{y-\tilde{s}} \left(K_{all} - c_o(y+1-d)\right) p_d - \sum_{d=y+1}^{y-\tilde{s}-1} \left(K_{all} - c_o(y-d)\right) p_d + \\ & \sum_{d=y-\tilde{s}+1}^{\infty} \left(\alpha K_p - \alpha(c_p + c_2)(y+1-d)\right) p_d - \sum_{d=y-\tilde{s}}^{\infty} \left(\alpha K_p - \alpha(c_p + c_2)(y-d)\right) p_d \\ = \; & c_2 + (h_2 - \alpha c_2) \sum_{d=0}^{y} p_d - (K_{all} + c_o)p_{y+1} + (K_{all} - c_o(\tilde{s}+1))p_{y-\tilde{s}} \\ & - c_o \sum_{d=y+2}^{y-\tilde{s}-1} p_d - (\alpha K_p - \alpha(c_p + c_2)\tilde{s})p_{y-\tilde{s}} - \alpha(c_p + c_2) \sum_{d=y-\tilde{s}+1}^{\infty} p_d \\ = \; & c_2 + (h_2 - \alpha c_2)F(y) - c_o(F(y-\tilde{s}) - F(y)) - \alpha(c_p + c_2)(1 - F(y-\tilde{s})) \\ & - K_{all} p_{y+1} + \left(K_{all} - \alpha K_p - (c_o - \alpha(c_p + c_2))\tilde{s}\right) p_{y-\tilde{s}}. \end{aligned}$$

Now by Lemma 6,

$$\begin{aligned}
\Delta G_4(y) &= c_2 - \alpha(c_p + c_2) + (c_o + h_2 - \alpha c_2)F(y) + (\alpha(c_p + c_2) - c_o)F(y - \tilde{s}) \\
&\quad -K_{all}p_{y+1} + (C^*_{OT} - C^*_{PF} - (c_o - \alpha(c_p + c_2))\tilde{s})\, p_{y-\tilde{s}} \\
&= c_2 - \alpha(c_p + c_2) + (c_o + h_2 - \alpha c_2)F(y) + (\alpha(c_p + c_2) - c_o)F(y - \tilde{s} - 1) \\
&\quad -\alpha K_p p_{y+1} + (1-\beta)(\alpha(c_p + c_2) - c_o))p_{y-\tilde{s}}
\end{aligned}$$

by Lemma 7 where $0 \leq \beta < 1$. At this point, note that as $y \to -\infty$, $\Delta G_4(y) \to c_2 - \alpha(c_p + c_2) < 0$ by assumption (A7) and as $y \to \infty$, $\Delta G_4(y) \to h_2 + c_2(1-\alpha) > 0$. Thus, as $y \to -\infty$, $G_4(y) \to \infty$ and as $y \to \infty$, $G_4(y) \to \infty$. Also note that for $y < 0$,

$$\begin{aligned}
\Delta G_4(y) &= c_2 - \alpha(c_p + c_2) + (\alpha(c_p + c_2) - c_o)F(y - \tilde{s} - 1) \\
&\quad -\alpha K_p p_{y+1} + (1-\beta)(\alpha(c_p + c_2) - c_o))p_{y-\tilde{s}} \\
&< 0
\end{aligned}$$

and so any minimum point of $G_4(y)$, y^0, will be non-negative. Finally, for $y \geq d_0$, we have that

$$\Delta G_4(y) = c_2 - \alpha(c_p + c_2) + F(y)F_4(y)$$

where

$$\begin{aligned}
F_4(y) &= (c_o + h_2 - \alpha c_2) + (\alpha(c_p + c_2) - c_o)\frac{F(y - \tilde{s} - 1)}{F(y)} + \\
&\quad (1-\beta)(\alpha(c_p + c_2) - c_o)\frac{p_{y-\tilde{s}}}{F(y)} - \alpha K_p \frac{p_{y+1}}{F(y)}.
\end{aligned}$$

Again, $F_4(y)$ is nondecreasing. By the same argument as for case 1, $\Delta G_4 y)$ changes signs exactly once and $-G_4(y)$ is unimodal.

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ABSTRACT

SUPPLY CHAIN MANAGEMENT WITH

OVERTIME AND PREMIUM FREIGHT

by

Eric Logan Huggins

Chairperson: Tava Maryanne Olsen

This thesis models a two-stage supply chain where the upstream stage (stage 2) **always** meets demand from the downstream stage (stage 1). Demand is stochastic; hence, shortages will occasionally occur at stage 2. Stage 2 must fill these shortages by expediting, using overtime production and/or premium freight shipments. We derive optimal inventory control policies under decentralized, centralized, and coordinated control and perform numerical analysis to compare the results.

Under decentralized control, stage 1 ignores stage 2 and follows a simple base-stock policy; stage 2 also follows a simple base-stock policy if there is no setup cost for regular production. When we include this setup cost at stage 2, two decisions must be made: how much to produce during regular production and how much to

produce during overtime production. We show that the optimal regular production policy is an (s, S) policy and that the optimal overtime production policy depends on the cost parameters.

Under centralized control, the two stages work together to minimize system costs. By substituting system variables for stage 2 variables and relaxing some constraints, we show that the optimal policy at stage 1 has two order-up-to levels and depends on the available system inventory. We also show that the optimal policy for the system is a base-stock policy and the optimal policy for stage 2 is to ensure the system base-stock level is achieved.

To coordinate the two stages, we develop two contracts. Both contracts depend on a two-tiered wholesale cost and a linear transfer payment. Contract A achieves system optimality, but requires the two stages to share cost information. Without sharing cost information, Contract B achieves near-optimality for the system (optimality for the average cost case). Under both contracts, an appropriate transfer payment may be negotiated that benefits both stages.

We perform numerical analysis to compare the supply chain under different forms of control. We show that centralized control may affect significant savings over decentralized control, particularly if the demand variation, holding costs, or expediting costs are high. We also show that Contract B yields nearly optimal results, particularly if the discount factor is close to one.