Optimal Klackers

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Introduction

Klackers is a game where players roll dice and then flip down tiles numbered 1 through 9 depending on the sum of the dice. The object of the game is to flip down as many tiles as possible in order to achieve the minimum score. In this paper, we consider the one player game and two player game. For the one player game, we derive the optimal strategy under which the odds of flipping down all the tiles ("klacking") is about 10% with an expected score of about 11. We also develop a simple heuristic strategy that performs near-optimally. For the two player game, we show that the second player has a limited advantage over the first player.

Rules of the Game

At the beginning of the game, nine tiles numbered 1 through 9 are face up. The player rolls two six-sided dice. The player then flips down *any* combination of tiles that has the same sum as the two dice. Once a tile has been flipped down, it remains down for the duration of the game. Play continues with the player again rolling the dice and flipping down some combination of the remaining face up tiles with the same sum. The game continues until either the player "klacks" (flips down all the tiles) or no combination of the remaining tiles add up to the sum of the dice, at which point the player's score is the sum of the remaining tiles. The goal of the game is get the lowest score possible. In the two player game, the goal of each player is to get a lower score than the other player.

The author first encountered Klackers at The Olde Schoolhouse Cafe near Durango, Colorado, where the game is commonly played at most local taverns. Visit http://www.klackers.com to learn more about Klackers or to play the game online. Klackers is similar to the game Klappenspiel, which was analyzed by Benjamin and Stanford [Benjamin and Stanford 1995]. However, our analysis differs in two minor and three substantial ways.

<u>Minor Differences</u>

- Klappenspiel has 10 tiles, Klackers has 9 tiles. Although this clearly affects numerical results, it does not affect any of the "interesting" results involving the optimal strategy.

- According to Benjamin and Stanford, the objective of Klappenspiel is to maximize the probability of flipping down all the tiles; the objective of Klackers is to minimize the expected score. We show that the optimal strategies and results under both objectives are very similar.

Substantial Differences

- In Klappenspiel, the player has only two choices. She can flip down two tiles that correspond to the two individual die values rolled or she can flip down one tile that corresponds to the sum of the two dice. In Klackers, the player can flip down *any* combination of tiles that add up to the sum of the two dice. Consider the beginning of the game when all nine tiles are face up and assume that the player has rolled a 3 and a 6. In Klappenspiel, the player has *two* choices: she can flip down both the 3 and 6 tiles or just the 9 tile. In Klackers, the player has *eight* choices: she can flip down the 9 tile; she can flip down the 1 and 8, the 2 and 7, the 3 and 6, or the 4 and 5 tiles; or, she can flip down the 1, 2 and 6, the 1, 3 and 6, or the 2, 3, and 4 tiles. These additional choices add complexity to the game and to the optimal strategy.

- In Klappenspiel, the player always rolls two dice. In Klackers, the player has the option of switching from two dice to one, which we show is optimal when only small tiles remain.

- Benjamin and Stanford's analysis considers only the one player game of Klappenspiel. Typically, Klackers is played with two (or more) players. Our analysis assumes two players. The first player has a turn and tries to minimize his score. The second player has a turn, but now her goal is not necessarily to minimize her score, but simply to beat the score of the first player. We show that the second player has a slight advantage over the first player while playing optimally.

The rest of the paper is organized as follows: We first consider the one player game. After discussing our methodology, we discuss the optimal results and strategy. Next, we consider a simple heuristic and show that it's performance is nearly optimal. Finally, we discuss the two player game.

Methodology

Similar to Benjamin and Stanford, we use dynamic programming to solve for the optimal strategy. However, we have more choices to consider. For each possible set of tiles, we have two decisions to make. First, we must decide whether to roll one or two dice. Second, given the outcome of the roll, we must decide which combination of tiles to flip down from a potentially large list of possibilities. To solve for the optimal strategy, we wrote a C++ program the determines the best play for each potential combination of tiles (or "boards".)

Since there are 9 tiles in Klackers, there are $2^9 = 512$ possible boards. Following the example of Benjamin and Stanford, we represent each board with binary notation, with each face up tile as a 1 and each face down tile as a 0. Examples: Let S = the sum of the dice after the player rolls (regardless of whether the player chooses to roll one or two dice). If a combination of the remaining tiles sums up to S, then the player

Board	Binary	Notes
0	000000000	All tiles are face down.
1	000000001	Only Tile 1 is face up.
2	000000011	Tiles 1 and 2 are face up.
149	001011010	Tiles 2, 4, 5 and 7 are face up.
511	111111111	All tiles are face up.

Table 1: Examples of Different Boards

can continue playing. If there is more than one combination that equals S, the player must make a decision of which tiles to flip down. If there is no combination that equals S, the player's turn is done and his final score is the sum of the remaining face up tiles.

Consider the example listed above of board 149. In this case the player has the 2, 4, 5, and 7 tiles remaining with a current score of 18. The player's turn is done if he rolls a sum of 1, 3, 8, or 10. The player's turn is forced if he rolls a sum of 2, 4, 5, 6, or 12. The player has to decide which tiles to klack down if he rolls a sum of 7, 9, or 11 (keeping in mind that his goal is to minimize his expected score). Note that the only feasible plays are where the tiles on the new board add up to S less than the current board, and additional face up tiles may be turned down (but face down tiles must remain down). For example, board 149 in binary is 001011010; if the player rolls $S = 7$, he can continue to either board 000011010 (flipping down the 7 tile) or 001001010 (flipping down the 2 and 5 tiles).

Given that the player's objective is to minimize his expected score, we solve for the optimal policy using a dynamic program, starting with board 0 where all tiles are face down and recursively solving for the best play (number of dice and best choice given the player's roll). Define $E^*[B]$ = the minimum expected score from board B under the optimal strategy. Therefore, $E^*[0] = 0$. Further, define $E_1[B]$ and $E_2[B]$ as the minimum expected scores from board B by rolling 1 or 2 dice, respectively. Finally, define $E^*[B|S]$ = the minimum expected score from board B under the optimal strategy given the sum of the dice is S. If no combination of tiles sums up to S from the current board B, then $E^*[B|S]$ = the sum of the current tiles. Then,

$$\begin{aligned} E^*[1] &= \min\{E_1[1], E_2[1]\} \\ &= \min\left\{\sum_{i=1}^{6}\left(\frac{1}{6}\right)E^*[1|i], \sum_{j=2}^{12}p(j)E^*[1|j]\right\} \\ &= \min\left\{\left(\frac{1}{6}\right)0 + \left(\frac{5}{6}\right)1, 1\right\} \\ &= \left(\frac{5}{6}\right) \end{aligned}$$

where $p(j)$ is the probability of rolling a sum of j on two dice. Similarly, we can define the minimum expected score for any board B as:

$$
\begin{aligned}
E^*[B] &= \min\{E_1[B], E_2[B]\} \\
&= \min\left\{\sum_{i=1}^{6}\left(\frac{1}{6}\right)E^*[B|i], \sum_{j=2}^{12}p(j)E^*[B|j]\right\}
\end{aligned}
$$

Now, we solve the dynamic program for each board B, starting and 0 and eventually solving for $E^*[511]$, the initial position with all tiles face up. Along the way, we keep track of the optimal number of dice to roll and the optimal strategy from each board given the sum rolled.

One Player Game

Under the optimal strategy, the expected score in Klackers, $E^*[511]$, is 11.0 (noting that the initial sum of the tiles is 45). Secondly, the probability of "klacking" down all the tiles, $P(0)$, is 9.7%. Third, the distribution of final scores under the optimal strategy is shown in Figure 1.

The optimal strategy itself may be summed up in three simple rules, with a mere 178 exceptions!

- Rule 1: Only roll one die if the sum of the remaining tiles is six or less.
- Rule 2: Flip down the minimum number of tiles possible.
- Rule 3: Choose the highest remaining tile(s).

Rule 1 states that for most of the game, the player should roll two dice. He should only switch to one die if he close to klacking with the sum of the remaining tiles equal to six or less. Rule 2 states that it is best to flip down the minimum number of tiles possible, choosing a single tile over a pair of tiles, or a pair of tiles over a triplet. Rule three states that if there is a choice, choose the pair (or triplet) that contains the highest available tile. For example, if the player rolls a sum of eight, the best play is to flip down the 8 tile if possible; if the 8 tile is already down, the next best play is to flip down the 1 and the 7 tiles (or, 1,7) followed by 2,6 then 3,5. If no singleton or pair is possible, then 1,2,5 is preferred over 1,3,4.

The player must decide whether to roll one or two dice for 511 different boards. Rule 1 is correct for $\frac{510}{511} = 99.8\%$ of these decisions. The only exception is that it is also optimal to roll one die when the 1, 2 and 4 tiles remain, which may make intuitive sense since any sum from 1 to 6 can be attained with these tiles. The player must also know the optimal play for twelve possible sums of the dice on 511 boards, a total of 6132 situations. It turns out that 1848 of the situations are dead ends where the player's turn is over and 2522 of these situations have only one feasible play, which leaves 1774 situations where the player

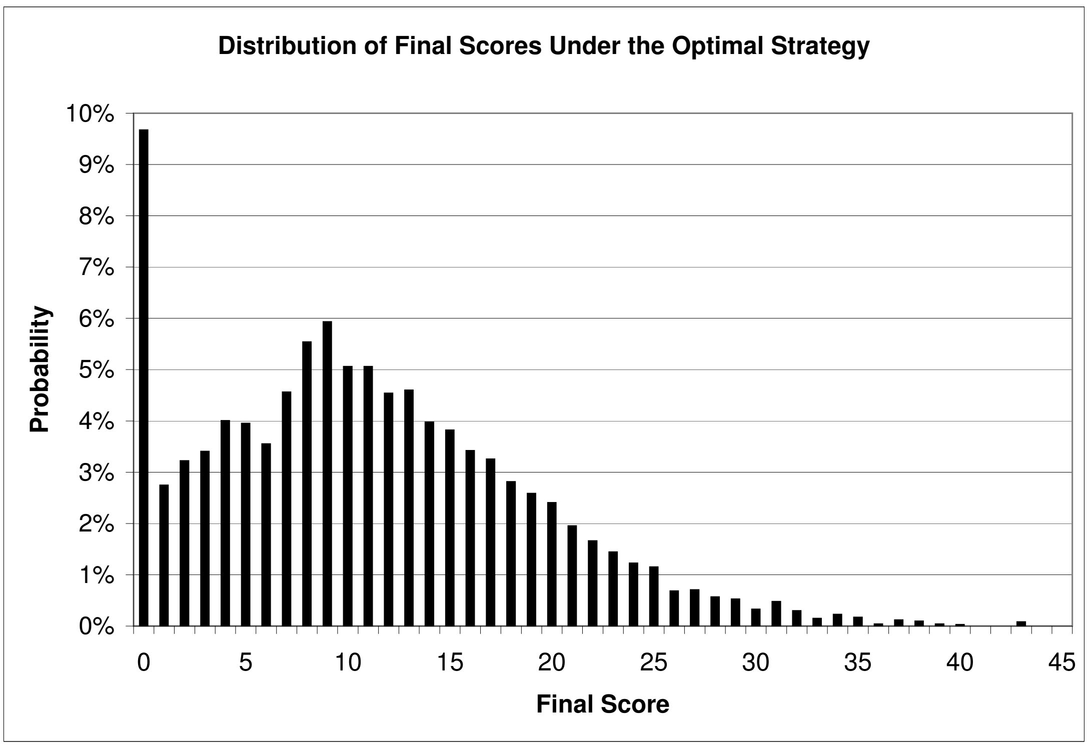

Figure 1: Distribution of Final Scores

must decide which tiles to flip down. Rule 2 is correct 100% of the time; it is always optimal to flip down the minimum number of tiles. Of the 1774 situations where the player has a choice, there are 528 where the player must apply Rule 3 (e.g., the player rolls $S = 10$ and must choose either the 3,7 or 4,6). Rule 3 is correct only $\frac{351}{528} = 66.5\%$ of the time, with 177 exceptions to the rule. However, these exception occur on only $\frac{177}{6132}$ of the possible plays, or less than 3% of the time.

Interestingly, these exceptions have no discernible pattern that the authors can determine. Consider four different boards where high tiles have already been flipped down. Specifically, consider the board with the 1 through 5 tiles remaining, that is 12345, and 123456, 1234567, and 12345678. The optimal plays for each board and each sum of dice are listed in Table 2, with exceptions to Rule 3 in bold print. For board 12345678,

Board vs. S	2	3	4	5	6	7	8	9	10	11	12
12345	2	3	4	5	5,1	**4,3**	5,3	5,4	5,4,1	5,4,2	5,4,3
123456	2	3	4	5	6	6,1	**5,3**	**5,4**	6,4	6,5	**5,4,3**
1234567	2	3	4	5	6	7	7,1	**6,3**	7,3	7,4	7,5
12345678	2	3	4	5	6	7	8	8,1	8,2	8,3	8,4

Table 2: Optimal Plays for Four Boards

there are no exceptions; i.e., always follow Rule 3. Board 1234567 has one exception: when the sum of the dice is 9, flip down the 6,3 instead of the 7,2 (or 5,4). Boards 12345 and 123456 also have exceptions for some rolls but not for others. All 177 exceptions are listed in Appendix 1 and we challenge a clever reader to develop a simple rule for all of them!

So, how can a player realistically expect to win at Klackers? The answer is simple: Follow the Rules. Although this heuristic ignores the exceptions discussed above, it performs nearly as well as the optimal strategy and requires the player to learn only two simple rules. In Table 3, we compare this heuristic (HEU) to the optimal results obtained by minimizing the expected score (MES) and maximizing the probability of flipping down all the tiles for a score of 0 (MP0). The heuristic performs very well against both strategies,

	$E^*[511]$	$P(O)$
MES	11.0	9.7%
MP0	11.1	9.8%
HEU	11.2	9.5%

Table 3: Comparison of Strategies

with only a slightly higher expected score and slightly lower chance of klacking. Also note that the two optimal strategies have very similar results. In fact, the MES and MP0 strategies differ by only 70 plays (out of 1774), where the MP0 strategy basically follows Rules 1, 2 and 3 with a similar number (163) of exceptions.

Two Player Game

In the previous section, we discussed the optimal strategy for a single player. However, Klackers is usually played in groups of two or more people, all competing for the lowest score (and perhaps a cash wager.) Here, we restrict our attention to the two player game: Player 1 goes first, hoping to klack down all the tiles or at least achieve a low score; Player 2 goes second, with the ultimate goal of beating Player 1's score. Later we will show that Player 2 can earn an advantage over player 1, but first we compare the three previously discussed strategies.

The three strategies discussed above are to minimize the expected score (MES), maximize the probability of flipping down all the tiles for a score of 0 (MP0), and the heuristic (HEU) that simply follows Rules 1, 2 and 3. When the players follow these strategies, the order of play doesn't matter, so we compare Player A versus Player B in Table 4. Table 4 lists the probability that Player A wins ($P(A)$), the probability that Player B wins ($P(B)$), the probability of a tie ($P(T)$), and Player A's expected winnings per game $E[W] = 1P(A) + 0P(T) - 1P(B)$ assuming that each player has bet one unit on the game. When both players employ the same strategy, they tie about 4.3% of the time and split the rest of the games. In the last three cases, Player A employs a superior strategy, wins more frequently and has a positive expectation of winning.

Player A	Player B	$P(A)$	$P(B)$	$P(T)$	$E[W]$
MEV	MEV	47.83%	47.83%	4.33%	0
MP0	MP0	47.84%	47.84%	4.31%	0
HEU	HEU	47.87%	47.87%	4.26%	0
MEV	MPO	47.91%	47.76%	4.33%	0.0015
MEV	HEU	48.66%	47.04%	4.29%	0.0162
MP0	HEU	48.58%	47.13%	4.29%	0.0145

Table 4: Two Player Comparison of Strategies

Now let us consider the order in which the players play the game. Player 1 goes first, Player 2 goes second. Regardless of the strategy employed by Player 1, Player 2's strategy will be to beat whatever final score Player 1 achieves. Let M be Player 1's final score and N be Player 2's final score. We assume that Player 2's goal is to maximize the probability that $N < M$, so for each possible outcome of Player 1, Player 2 will have an optimal strategy to beat that outcome.

As an aside, Player 2's actual goal may not be to maximize $P(N < M)$, but rather to maximize $E[W] = 1P(N < M) + E[T]P(N = M) - 1P(N > M)$, where $E[T]$ is Player 2's expectation from a tie, which is not necessarily 0. Usually when a tie occurs, the players both have another turn to determine the winner, in the same order. If Player 2 has an advantage over Player 1 from the outset and knows she will again play second, then $E[T] > 0$. Furthermore, in some instances when a tie occurs, each player must double their original bet, in which case Player 2 would value ties even more. However, in our analysis below, we assume that Player 2 simply maximizes her probability of winning, which surely correlates with the strategy of maximizing her expectation.

We also assume that Player 1 continues to follow the strategies listed earlier (MES, MP0, or HEU). Technically, knowing how Player 2 will react to any score M, Player 1 would either want to maximize his probability of winning given Player 2's strategy or maximize his expectation given Player 2's strategy. However, for the sake of comparison to our previous results, we assume that Player 1 follows one of the previous strategies.

So, the question at this point is how does Player 2's strategy change? Player 1 will score $M \in [0, 45]$. Player 2 must know the optimal strategy to maximize $P(N < M)$ for each possible M! We solved dynamic programs for each value of M to determine the optimal strategies for Player 2 conditional on M. Overall, Player 2 continues to follow the Rules from the previous section. Rule 2 remains accurate, but for Rule 3 the exceptions change somewhat for each possible value of M. There are too many differing exceptions to Rule 3 to come up with a revised simple strategy for Player 2 that takes advantage of going second; however, some insight can be gained by examining the exceptions to Rule 1.

Rule 1 initially stated that the player should only roll one die when the sum of the remaining tiles is six or less. For Player 2, the revised Rule 1 may roughly be written as:

- Revised Rule 1 for Player 2: Consider rolling one die if the sum of your remaining tiles is within six of M, Player 1's final score.

It is not always optimal to switch to one die when Player 2's current sum exceeds M by six or less, but many such instances exist. For example, assume that Player 1 scored $M = 10$. If Player 2 currently has board 1234 (with the 1 through 4 tiles remaining and a current sum of 10), rolling one die guarantees victory while rolling two dice has a $\frac{3}{36} = 8.3\%$ chance of losing. Similarly, if Player 2 currently has board 1239, it is optimal to switch to one die if $M = 12, 13, 14$ or 15. A complete list of these one/two dice exceptions is given in Appendix 2. Under the assumption that Player 2 can play optimally (P2O), her advantage over Player 1 is listed in Table 5. Clearly, Player 2 has a significant advantage over Player 1, although it would be extremely difficult for Player 2 to play optimally. In reality, Player 2 has a slight advantage over Player 1 by switching to one die at the right times.

Player 2	Player 1	$P(2)$	$P(1)$	$P(T)$	$E[W]$
P2O	MEV	48.32%	47.63%	4.05%	0.0069
P2O	MP0	48.40%	47.55%	4.05%	0.0084
P2O	HEU	49.15%	46.84%	4.01%	0.0230

Table 5: Player 2's Advantage

Conclusion

In conclusion, we have optimized the game Klackers using dynamic programming. In the one player game, the player should follow three basic rules to achieve a minimum score, although the exact strategy is quite complicated. In the two player game, the second player has a theoretically significant advantage over the first player, but in reality the advantage is limited.

References

Benjamin, Arthur T. and Derek Stanford. 1995. Optimal Klappenspiel. *The UMAP Journal* 16(1): 11-20.

www.klackers.com, accessed 06/05/07.

About the Author

Eric Huggins received a B.S. in mathematics from Harvey Mudd College (where he studied under Art Benjamin) and M.S. and Ph.D. degrees in industrial and operations engineering from the University of Michigan. He currently teaches operations management at Fort Lewis College in Durango, CO where he lives and frequently plays Klackers.

Table 6: Appendix 1: Exceptions to Rule 3

Exception	Boards
2,3	12346 12347 12348 12349 123468 123469 123479 1234679 1234689 12346789
2,4	12457 12458 12459 123457 123459 124579 1234579
3,4	2345 1346 12345 12346 23456 23458 13468 123458 23459 13469 123459 123469 134568 134569 234569 234589 134689 1234589 1345689
2,5	1256 12356 12568 12569 123568 124569 1235689
3,5	2356 1357 12356 12357 123456 123457 23567 23569 13579 123569 234567 234569 134579 1234569 1234579 135679 2345679
2,6	1267 12467 123567 12679 123679 124679 125679 1235679
4,5	3456 2457 1458 13456 12457 123456 123457 24567 34567 34568 24578 234578 345678
3,6	2367 1368 123467 23678 1234567 235678
2,7	12578 124678
4,6	3467 23467 34567 134567 124568 24678 14679 34678 34679 24689 234678 134679 1345678 345679 245689 346789 2346789 3456789
3,7	23478 123478 123579 234578 23789 234789 236789 2345789
2,8	12689 123589 124689 125789 1235689 1235789 1246789 12356789
2,3,5	12356
1,4,5	12457
5,6	4567 14567 124567 35678 135678 45678 125679 25689 45679 35689 1235678 245678 145679 1245678 2345678 345689 1345689 356789 456789
4,7	134678 34789
3,8	123489 23589 123589 235689 1234789
2,4,5	12458
2,3,6	12367
5,7	24578 124578 234578 145678 1245678 2345678 245789 1356789 1456789 12456789
4,8	134589 234689 134789 1345789 2346789 13456789
3,4,5	13456 23456 123456
2,4,6	12467 23467 124679
2,3,7	123467

Table 7: Appendix 2: Revised Rule 1 for Player 2

M	Boards Where Player 2 Should Roll One Die
0	1, 2, 12, 3, 13, 4, 23, 14, 5, 123, 24, 15, 6
1	1, 2, 12, 3, 13, 4, 23, 14, 5, 123, 24, 15, 6
2	2, 12, 3, 13, 4, 23, 14, 5, 123, 24, 15, 6, 124
3	12, 3, 13, 4, 23, 14, 5, 123, 24, 15, 6, 124
4	13, 4, 23, 14, 5, 123, 24, 15, 6, 124
5	23, 14, 5, 123, 24, 15, 6, 124, 34
6	123, 24, 15, 6, 124, 34, 25, 125, 35, 135, 45
7	124, 34, 25, 16, 125, 35, 26, 234, 135, 45, 36, 145, 46, 56
8	134, 125, 35, 26, 234, 135, 45, 36, 1234, 145, 46, 56
9	234, 135, 45, 126, 36, 1234, 145, 46, 56
10	1234, 145, 136, 46, 236, 56
11	1235, 236, 146, 56, 128, 129, 139
12	138, 129, 238, 139, 1238, 1239
13	238, 148, 139, 1238, 239, 1239
14	1238, 239, 149, 1239, 249
15	1239, 249, 159, 1249, 1358, 459
16	1249, 1358, 459, 1459
17	1358, 2349, 459, 1459
18	2349, 1359, 459, 1459
19	1459
21	12389
22	12389
23	12389
26	13589